POSITIVE SOLUTIONS OF THE SECOND-ORDER SYSTEM OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. In this paper, a second-order system of multi-point boundary value problems in Banach spaces is investigated. Based on a specially constructed cone and the fixed point theorem of strict-set-contraction operators, the criterion of the existence and multiplicity of positive solutions are established. And two examples demonstrating the theoretic results are given.

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1. Introduction

The theory of ordinary differential equations in abstract spaces is an important new branch (see [1-6]). Recently, the existence and multiplicity of positive solutions for boundary value problems of ordinary differential equations have been of great interest in mathematics and engineering sciences (see [7, 8]). However, to the authors' knowledge, few paper has considered the existence of positive solutions for second-order system of multi-point boundary value problems, especially in abstract spaces. In scalar spaces, we refer the readers to [9-16].

Erbe and Wang [9] discussed the boundary value problem:

$$\begin{cases} -u''(t) = f(t, u(t)), \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{cases} \tag{1}$$

Using a Krasnosel˙skii fixed-point theorem, the existence of solutions of (1) is obtained with the assumption that f is superlinear or sublinear. Yang and Sun

[12] considered the boundary value problem of the following differential system:

$$\begin{cases} -u''(t) = f(t, v(t)), \\ -v''(t) = g(t, u(t)), \\ u(0) = u(1) = 0, \\ v(0) = v(1) = 0. \end{cases} \tag{2}$$

the existence of solutions of (2) is established by applying the degree theory. Hu [14] investigated the existence and multiplicity of positive solutions of boundary value problems:

$$\begin{cases} -u''(t) = f(t, v(t)), \\ -v''(t) = g(t, u(t)), \\ \alpha u(0) - \beta u'(0) = 0, \ \gamma u(1) + \delta u'(1) = 0, \\ \alpha v(0) - \beta v'(0) = 0, \ \gamma v(1) + \delta v'(1) = 0, \end{cases} \tag{3}$$

and establish some corresponding result by using a fixed point theorem due to Krasnosel'skii [17].

In this paper, Using the properties of Green function and the well-known fixed point theorem of strict-set-contraction [3, 4] stated in section 2, we investigate the existence and multiplicity of positive solutions of the following system of multi-point boundary value problems(BVPs)

$$\begin{cases} -u''(t) = f(t, v(t)), \ t \in (0, 1) \\ -v''(t) = g(t, u(t)), \ t \in (0, 1) \\ u(0) = \beta u'(0), \ \alpha u(\eta) = u(1), \\ v(0) = \beta v'(0), \ \alpha v(\eta) = v(1), \end{cases} \tag{4}$$

in Banach space E, where θ is zero element of E, $0 < \alpha < 1$, $\beta \geq 0$, $\eta \in (0, 1)$, $\rho = (1 - \alpha\eta) + \beta(1 - \alpha) \neq 0$, $f(t, \theta) \equiv \theta$, $g(t, \theta) \equiv \theta$.

This paper is organized as follows. In section 2, we present some preliminaries and lemmas, which are necessary to Sections 3 and 4. In section 3 the main results and the proofs concerning with the existence of positive solutions of BVPs (4) are given. The proofs concerning with the multiplicity of positive solutions of BVPs (4) are given in section 4. Finally, in section 5, we give some examples to illustrate our theoretic results.

2. Preliminaries

In this section, we provide some background material from the theory of cone in Banach space, and then state the fixed point theorem for a cone preserving operator and some lemmas about BVPs (4).

Let the real Banach space E with norm $\| \cdot \|$ be partially ordered by a cone P of E, i.e., $u \leq v$ if and only if $v - u \in P$, and P^* denotes the dual cone of P. $u < v$ if and only if $u \leq v$ and $u \neq v$, where $u, v \in E$.

A cone P is called normal if $\inf\{\|x + y\| : x, y \in P, \|x\| = \|y\| = 1\} > 0$. We denote the normal constant of P by N, i.e., $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\|$.

Set $I = [0, 1]$, then $C[I, E]$ is a Banach space with norm $\|u\|_C = \max_{0 \leq t \leq 1} \|u(t)\|$. $Q = \{u \in C[I, E] : u(t) \geq \theta \ for \ t \in I\}$ denotes a cone of the Banach space $C[I, E]$.

$(u(t), v(t)) \in C^2[I, E] \times C^2[I, E]$ is called a positive solution of BVPs (4), if $u(t), v(t) \in Q$, $u(t), v(t) \not\equiv \theta$ and satisfy BVPs (4).

For a bounded set V in Banach spaces, we denote $\alpha(V)$, $\alpha_C(V)$ the Kuratowski measure of noncompactness for a bounded set V in E and in $C[I, E]$, respectively. An operator $A : D \to E(D \subset E)$ is said to be a k-set contraction if A is continuous and bounded and there exists a constant $k \geq 0$ such that $\alpha(A(V)) \leq k\alpha(V)$ for any bounded $V \subset D$. A k-set contraction with $k < 1$ is called a strict-set-contraction. The closed balls in space E and $C[I, E]$ are denoted by $T_r = \{u \in E : \|u\| \leq r\}$ $(r > 0)$ and $B_r = \{u \in C[I, E] : \|u\|_C \leq r\}$ $(r > 0)$, respectively.

For application in what follows, we firstly state the following lemmas.

Lemma 2.1(Demling [2]). **Let $D \subset E$ and D is a bounded set, f is uniformly continuous and bounded from $I \times D$ into E. Then**

$$\alpha(f(I \times B)) = \max_{0 \leq t \leq 1} \alpha(f(t, B)), \ \forall \ B \subset D. \tag{5}$$

Lemma 2.2. If $H \subset C(J, E)$ is bounded and equicontinuous, then $\alpha_C(H) = \alpha(H(J)) = \max_{t \in J} \alpha(H(t))$, where $H(J) = \{x(t) : t \in J, x \in H\}, H(t) = \{x(t) : x \in H\}$.

Lemma 2.3(Cac and Gatica [3], Potter [4]). Let K be a cone of a real Banach space E and $K_{r,R} = \{u \in K : r \leq \|u\| \leq R\}$ with $0 < r < R$. Suppose that $A : K_{r,R} \to K$ is a strict-set-contraction such that one of the following two conditions is satisfied:
(i) $Au \not\leq u$ for any $u \in K, \|u\| = r$ and $Au \not\geq u$ for any $u \in K, \|u\| = R$;
(ii) $Au \not\geq u$ for any $u \in K, \|u\| = r$ and $Au \not\leq u$ for any $u \in K, \|u\| = R$.
Then the operator A has at least one fixed point $u \in K_{r,R}$ such that

$$r < \|u\| < R.$$

Besides the lemma 2.1, 2.2 and 2.3, we list some lemmas about the properties of the Green function and the solution of BVPs (4).

The Green function of the BVPs (4) can be explicitly given by

$$G(t, s) = \frac{1}{\rho} \begin{cases} (s + \beta)[(1 - \alpha\eta) - (1 - \alpha)t], \ 0 \leq s \leq \min\{t, \eta\}, \\ (s + \beta)(1 - t) + \alpha(t - s)(\eta + \beta), \ \eta \leq s \leq t, \\ (t + \beta)[(1 - \alpha\eta) - (1 - \alpha)s], \ t \leq s \leq \eta, \\ (t + \beta)(1 - s), \ \max\{t, \eta\} \leq s \leq 1. \end{cases} \tag{6}$$

Lemma 2.4. The Green function $G(t, s)$ satisfies
(i) $0 \leq G(t, s) \leq G(s, s) \leq M$, $(t, s) \in I \times I$;
(ii) $G(t, s) \geq \lambda G(s, s)$, $t \in [\frac{1}{4}, \frac{3}{4}], s \in I$,
where $\lambda = \min\{\frac{1}{4(1-\eta)}, \frac{3+4\beta}{4(1+\beta)}\} \leq 1$, $M = \frac{1}{\rho}(1 + \beta)(1 + \alpha)$.

Obviously, $(u(t), v(t))$ is the solution of BVPs (4), if and only if $(u(t), v(t)) \in C[I, E] \times C[I, E]$ is the solution of the system of integral equations

$$\begin{cases} u(t) = \int_0^1 G(t, s) f(s, v(s)) ds, \\ v(t) = \int_0^1 G(t, s) g(s, u(s)) ds. \end{cases} \tag{7}$$

where $G(t, s)$ is defined by (6).

Integral equations (7) can be transferred to the nonlinear integral equation

$$u(t) = \int_0^1 G(t, s) f(s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau) ds. \tag{8}$$

By (8), we can define an operator $A : C[I, E] \to C[I, E]$ as follows:

$$(Au)(t) = \int_0^1 G(t, s) f(s, \int_0^1 G(s, \tau) g(\tau, u(\tau)) d\tau) ds. \tag{9}$$

Lemma 2.5. The BVPs (4) has a solution $(u(t), v(t))$ if and only if $u(t)$ satisfy (8), i.e., u is a fixed point of the operator A defined by (9) and

$$v(t) = \int_0^1 G(t, s) g(s, u(s)) ds.$$

Lemma 2.6. Suppose $f, g \in C[I \times P, P]$, then the solution $u(t)$ of the nonlinear integral equation (8) satisfies $u(t) \geq \theta$, $t \in I$, that is, $u(t) \in Q$, $t \in I$, and $u(t) \geq \lambda u(s)$, $\forall t \in [\frac{1}{4}, \frac{3}{4}]$, $s \in I$.

Proof. In view of $f, g \in C[I \times P, P]$, Lemma 2.4 and (8), we have $u(t) \geq \theta$, $t \in I$.

Since $u(t)$ is the solution of the nonlinear integral equation (8), by (7), (8), and Lemma 2.4, we get

$$\begin{aligned} u(t) &= \int_0^1 G(t, s_1) f(s_1, \int_0^1 G(s_1, \tau) g(\tau, u(\tau)) d\tau) ds_1 \\ &= \int_0^1 \frac{G(t, s_1)}{G(s_1, s_1)} G(s_1, s_1) f(s_1, \int_0^1 G(s_1, \tau) g(\tau, u(\tau)) d\tau) ds_1 \\ &\geq \lambda \int_0^1 G(s_1, s_1) f(s_1, \int_0^1 G(s_1, \tau) g(\tau, u(\tau)) d\tau) ds_1 \\ &\geq \lambda \int_0^1 G(s, s_1) f(s_1, \int_0^1 G(s_1, \tau) g(\tau, u(\tau)) d\tau) ds_1 \\ &\geq \lambda u(s), \ t \in [\frac{1}{4}, \frac{3}{4}], \ s \in I. \end{aligned}$$

$\square$

To obtain the positive solution of BVPs (4), we should select a suitable sub-cone of $C[I, E]$. Set

$$K = \{u \in Q : u(t) \geq \lambda u(s), \ t \in [\frac{1}{4}, \frac{3}{4}], \ s \in I\},$$

where λ is stated in Lemma 2.4. It is clear that K is a cone of the Banach space $C[I, E]$ and $K \subset Q$. For any $u \in Q$, by Lemmas 2.5, we can obtain $A(u) \in K$. Then $A(Q) \subset K$, therefore

$$A(K) \subset K. \tag{10}$$

3. The existence of positive solutions

In this section, we study the existence of positive solutions for the BVPs (4). For convenience sake, we give the following hypotheses: (H_1) For any $r' > 0$, $r > 0$, f, g are uniformly continuous and bounded on $I \times P \cap T_{r'}$ and $I \times P \cap T_r$, respectively. Furthermore, there exist constants L_r, $L_{r'}$ such that

$$\alpha(g(t, x)) \leq L_r \alpha(D), \ t \in I, \ D \subset P \cap T_r,$$

$$\alpha(f(t, x)) \leq L_{r'} \alpha(D), \ t \in I, \ D \subset P \cap T_{r'},$$

where $L_{r'} \geq 0$, $L_r \geq 0$ satisfy

$$L_r L_{r'} < \frac{1}{4M^2},$$

and $M = \frac{1}{\rho}(1 + \beta)(1 + \alpha)$ stated in Lemma 2.4;
  (H_2)

$$\lim_{\|u\| \to 0} \sup_{t \in I} \frac{\|f(t, u)\|}{\|u\|} = 0, \ \lim_{\|u\| \to 0} \sup_{t \in I} \frac{\|g(t, u)\|}{\|u\|} = 0;$$

(H_3) There exists $\phi \in P^*$, such that $\phi(x) > 0$, for any $u > \theta$, and

$$\lim_{\|u\| \to \infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{\phi(f(t, u))}{\phi(u)} = \infty, \ \lim_{\|u\| \to \infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{\phi(g(t, u))}{\phi(u)} = \infty;$$

(H_4)

$$\lim_{\|u\| \to \infty} \sup_{t \in I} \frac{\|f(t, u)\|}{\|u\|} = 0, \ \lim_{\|u\| \to \infty} \sup_{t \in I} \frac{\|g(t, u)\|}{\|u\|} = 0;$$

(H_5) There exists $\phi \in P^*$, such that $\phi(x) > 0$, for any $u > \theta$, and

$$\lim_{\|u\| \to 0} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{\phi(f(t, u))}{\phi(u)} = \infty, \ \lim_{\|u\| \to 0} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{\phi(g(t, u))}{\phi(u)} = \infty;$$

We firstly prove the following lemma.

Lemma 3.1. Suppose that (H_1) hold, then operator A is a strict-set-contraction on $D \subset P \cap B_r$.

Proof. The proof is similar to the proof of Lemma 2 in [5]. By (H_1), and Lemma 2.1, we have

$$\alpha(f(I \times D)) = \max_{0 \leq t \leq 1} \alpha(f(t, D)) \leq L_{r'} \alpha(D), \ \forall D \subset P \cap T_{r'},$$
$$\alpha(g(I \times D)) = \max_{0 \leq t \leq 1} \alpha(g(t, D)) \leq L_r \alpha(D), \ \forall D \subset P \cap T_r. \tag{11}$$

Since f, g are uniformly continuous and bounded on $I \times P \cap T_{r'}$ and $I \times P \cap T_r$, respectively, we see that, from Lemma 2.4, the operator A defined by (9) is continuous and bounded on $Q \cap B_r$. For $S \subset Q \cap B_r$, the set $A(S) = \{Au :$

$u(t) \in S\}$ are uniformly bounded and equicontinuous. Therefore, by Lemma 2.2, we see

$$\alpha_C(A(S)) = \sup_{t \in I} \alpha(A(S(t))), \qquad (12)$$

where $A(S(t)) = \{Au(t) : u \in S, t \text{ is fixed}\} \subset P \cap T_r$, for any $t \in I$.

For any $u \in C[I, E]$, we firstly have

$$S_v = \{v(s) = \int_0^1 G(s, \tau)g(\tau, u(\tau))d\tau : \tau \in I, u \in S\} \subset P \cap T_{r'}.$$

Using the obvious formula $\int_0^1 u(t)dt \in \bar{c}o\{u(t) : t \in I\}$, we can get

$$
\begin{aligned}
\alpha(A(S(t))) \quad &= \alpha\{\int_0^1 G(t,s)f(s, \int_0^1 G(s,\tau)g(\tau, u(\tau))d\tau)ds : s \in I, u \in S\} \\
&= \alpha\{\int_0^1 G(t,s)f(s, v(s))ds : s \in I, v \in S_v\} \\
&\leq \alpha(\bar{c}o\{G(t,s)f(s, v(s)) : s \in I, v \in S_v\}) \\
&\leq M\alpha(\bar{c}o\{f(s, v(s)) : s \in I, v \in S_v\}) \\
&\leq M\alpha(f(I \times B_v)) \\
&\leq ML_{r'}\alpha(B_v),
\end{aligned} \qquad (13)
$$

where $B_v = \{v(s) : s \in I, v \in S_v\} \subset P \cap T_{r'}$.

From the fact obtained in the proof of Lemma 2(11) in [5], we know

$$\alpha(B_v) \leq 2\alpha(S_v). \qquad (14)$$

Similarly to (13), we have

$$
\begin{aligned}
\alpha(S_v) \quad &= \alpha\{\int_0^1 G(t,s)g(s, u(s))ds : s \in I, u \in S\} \\
&\leq \alpha(\bar{c}o\{G(t,s)g(s, u(s)) : s \in I, u \in S\}) \\
&\leq M\alpha(\bar{c}o\{g(s, u(s)) : s \in I, v \in S\}) \\
&\leq M\alpha(g(I \times B)) \\
&\leq ML_r\alpha(B),
\end{aligned} \qquad (15)
$$

where $B = \{u(s) : s \in I, u \in S\} \subset P \cap T_r$.
Similarly to (14), we have

$$\alpha(B) \leq 2\alpha(S). \qquad (16)$$

It follows from (12)-(16) that

$$\alpha_C A(S) \leq 4M^2 L_{r'} L_r \alpha_C(S), \ \forall S \subset Q \cap B_r.$$

Consequently, A is a strict-set-contraction on $S \subset Q \cap B_r$, because of

$$4M^2 L_{r'} L_r < 1.$$

$$\square$$

Now we consider the existence of positive solutions of BVPs (4).

Theorem 3.1. Let cone P be normal, and conditions (H_1), (H_2), (H_3) be satisfied. Then BVPs (4) has at least one positive solution.

Proof. By Lemma 2.5 and Lemma 2.6, we need to seek fixed points of A in the cone K. To the end, it suffices to show that the conditions of Lemma 2.3 hold with respect to A.

Firstly, from (H_2) and $f(t, \theta) \equiv \theta$, $g(t, \theta) \equiv \theta$, there exists a $\delta_1 > 0$ such that

$$\begin{aligned} \|f(t, u)\| &\leq \varepsilon_1 \|u\|, \ \forall u \in P, \ \|u\| < \delta_1, \ t \in I, \\ \|g(t, u)\| &\leq \varepsilon_1 \|u\|, \ \forall u \in P, \ \|u\| < \delta_1, \ t \in I, \end{aligned} \tag{17}$$

where $\varepsilon_1^2 \in (0, (NM^2)^{-1})$, that is,

$$0 < NM^2 \varepsilon_1^2 < 1. \tag{18}$$

For any $r \in (0, \min\{\delta_1, \frac{\delta_1}{M\varepsilon_1}\})$, we now prove that

$$Au \not\geq u \text{ for any } u \in K, \|u\|_C = r. \tag{19}$$

Indeed, suppose by contradiction that there exists $u_1 \in K$ with $\|u_1\|_C = r$, such that $Au_1 \geq u_1$. Together with (9) and Lemma 2.4, we have

$$\begin{aligned} \theta \leq u_1(t) \leq (Au_1)(t) \ &= \int_0^1 G(t, s) f(s, \int_0^1 G(s, \tau) g(\tau, u_1(\tau)) d\tau) ds \\ &\leq M \int_0^1 f(s, \int_0^1 G(s, \tau) g(\tau, u_1(\tau)) d\tau) ds, \end{aligned} \tag{20}$$

and

$$\begin{aligned} \| \int_0^1 G(s, \tau) g(\tau, u_1(\tau)) d\tau \| \ &\leq M \| \int_0^1 g(\tau, u_1(\tau)) d\tau \| \\ &\leq M \|g(\tau, u_1(\tau))\| \\ &\leq M\varepsilon_1 \|u_1(\tau)\| \\ &\leq M\varepsilon_1 r \\ &< \delta_1. \end{aligned} \tag{21}$$

By Lemma 2.4, (17), (18), (20),(21) and the cone P being normal, we get

$$\begin{aligned} \|u_1(t)\| \ &\leq NM \| \int_0^1 f(s, \int_0^1 G(s, \tau) g(\tau, u_1(\tau)) d\tau) ds \| \\ &\leq NM \int_0^1 \varepsilon_1 \| \int_0^1 G(s, \tau) g(\tau, u_1(\tau)) d\tau \| ds \\ &\leq NM\varepsilon_1 \| \int_0^1 G(s, \tau) g(\tau, u_1(\tau)) d\tau \| \\ &\leq NM^2 \varepsilon_1^2 r \\ &< r. \end{aligned}$$

So $\|u_1\|_C < r$, which contradicts $\|u_1\|_C = r$. Thus (19) is true.

Next, by (H_3), there exists $R_1 > 0$, such that

$$\begin{aligned} \phi(f(t, u)) &\geq M_1 \phi(u), \ \forall u \in P, \ \|u\| \geq R_1, \ t \in [\tfrac{1}{4}, \tfrac{3}{4}], \\ \phi(g(t, u)) &\geq M_1 \phi(u), \ \forall u \in P, \ \|u\| \geq R_1, \ t \in [\tfrac{1}{4}, \tfrac{3}{4}], \end{aligned} \tag{22}$$

where

$$M_1 > \max\{(\lambda^{\frac{3}{2}} \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s) ds)^{-1}, N(\lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(s, s) ds)^{-1}\}. \tag{23}$$

Now, for any

$$R > \frac{NR_1}{\lambda}, \tag{24}$$

we are going to verify that

$$Au \not\leq u \text{ for any } u \in K, \|u\|_C = R. \qquad (25)$$

Suppose by contradiction that there exists $u_2 \in K$ with $\|u_2\|_C = R$, such that $Au_2 \leq u_2$. Then $u_2(t) \geq \lambda u_2(s)$, $\lambda\|u_2(s)\| \leq N\|u_2(t)\|$, for $\forall\, t \in [\frac{1}{4}, \frac{3}{4}]$, $\forall\, s \in I$. And so, by (24)

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} \|u_2(t)\| \geq \frac{\lambda}{N} \cdot \max_{s \in I} \|u_2(s)\| = \frac{\lambda}{N} \cdot \|u_2\|_C = \frac{\lambda R}{N} > R_1. \qquad (26)$$

By (22), (26) and Lemma 2.4, we get

$$
\begin{aligned}
\phi(\int_0^1 G(s,\tau)g(\tau,u_2(\tau))d\tau) \;&\geq \int_0^1 G(s,\tau)\phi(g(\tau,u_2(\tau)))d\tau \\
&\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)\phi(g(\tau,u_2(\tau)))d\tau \\
&\geq \lambda M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)\phi(u_2(\tau))d\tau \\
&\geq \lambda M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)d\tau \cdot \phi(u_2(s)).
\end{aligned}
$$

Together with the property of ϕ, we imply

$$\int_0^1 G(s,\tau)g(\tau,u_2(\tau))d\tau \geq \lambda M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)d\tau \cdot u_2(s) \geq \theta.$$

Observing the cone P being normal, (23), (26), for any $s \in [\frac{1}{4}, \frac{3}{4}]$, we get

$$
\begin{aligned}
\|\int_0^1 G(s,\tau)g(\tau,u_2(\tau))d\tau\| \;&\geq \frac{1}{N}\lambda M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)d\tau \cdot \|u_2(s)\| \\
&\geq \frac{1}{N}\lambda M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)d\tau \cdot \min_{s \in [\frac{1}{4}, \frac{3}{4}]}\|u_2(s)\| \\
&\geq \frac{1}{N}\lambda M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)d\tau \cdot R_1 \qquad (27) \\
&\geq R_1.
\end{aligned}
$$

By (9),(22),(27), Lemma 2.4 and Lemma 2.6, we get

$$
\begin{aligned}
\phi(u_2(t_0)) \geq \phi(Au_2(t_0)) \;&= \int_0^1 G(t_0,s)\phi(f(s,\int_0^1 G(s,\tau)g(\tau,u_2(\tau))d\tau))ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)\phi(f(s,\int_0^1 G(s,\tau)g(\tau,u_2(\tau))d\tau))ds \\
&\geq M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)\phi(\int_0^1 G(s,\tau)g(\tau,u_2(\tau))d\tau)ds \\
&\geq M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s) \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,\tau)\phi(g(\tau,u_2(\tau)))d\tau ds \\
&\geq \lambda M_1^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s) \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)\phi(u_2(\tau))d\tau ds \\
&\geq \lambda M_1^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s) \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)d\tau \cdot \phi(u_2(s))ds \\
&\geq \lambda^2 M_1^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)d\tau \int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)ds \cdot \phi(u_2(t_0)) \\
&\geq \lambda(\lambda M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds)^2 \cdot \phi(u_2(t_0)),
\end{aligned}
$$

where $t_0 \in [\frac{1}{4}, \frac{3}{4}]$ is given. Observing $\phi(u_2(t_0)) > 0$, we can conclude

$$\lambda(\lambda M_1 \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds)^2 \leq 1,$$

which contradicts with (23). Therefore (25) is true. By (19) and (25), we showed that the condition (ii) of Lemma 2.3 is satisfied.

Finally, by Lemma 3.1 and (10), A is a strict-set-contraction on $K_{r,R} = \{x \in K : r \leq \|x\|_C \leq R\}$. From Lemma 2.3, we see that A has a fixed point u^* on $K_{r,R}$. And $(u^*, \int_0^1 G(t,s)g(s,u^*)ds)$ is a positive solution of BVPs (4). $\qquad \square$

Theorem 3.2. Let cone P be normal and conditions (H_1), (H_4), (H_5) be satisfied. Then BVPs (4) has at least one positive solution.

Proof. The proof is along the lines of that of Theorem 3.1.

Firstly, from (H_5), there exists $\delta_1 > 0$, such that

$$\begin{aligned}
\phi(f(t,u)) \geq M_2\phi(u), \ \forall u \in P, \ \|u\| \leq \delta_1, \ t \in [\tfrac{1}{4}, \tfrac{3}{4}], \\
\phi(g(t,u)) \geq M_2\phi(u), \ \forall u \in P, \ \|u\| \leq \delta_1, \ t \in [\tfrac{1}{4}, \tfrac{3}{4}],
\end{aligned} \tag{28}$$

where

$$M_2 > (\lambda^{\frac{3}{2}} \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds)^{-1}. \tag{29}$$

In view of that $g(t,\theta) \equiv \theta$ and g is continuous, we know that there exists a constant $\delta_1' \in (0, \delta_1)$, such that when $\|u\| \leq \delta_1'$, we have $\|g(t, u(t))\| \leq \frac{\delta_1}{M}$. Together with Lemma 2.4, we get

$$\|\int_0^1 G(t,s)g(s,u(s))ds\| \leq M\|g(t,u(t))\| \leq \delta_1. \tag{30}$$

For any $r \in (0, \delta_1')$, we are going to verify that

$$Au \nleq u \text{ for any } u \in K, \|u\|_C = r. \tag{31}$$

Indeed, suppose by contradiction that there exists $u_3 \in K$ with $\|u_3\|_C = r$, such that $Au_3 \leq u_3$. Then, by (9),(28),(30), Lemma 2.4 and Lemma 2.6, we have

$$\begin{aligned}
\phi(u_3(t_0)) \geq \phi(Au_3(t_0)) \ &= \int_0^1 G(t_0,s)\phi(f(s, \int_0^1 G(s,\tau)g(\tau, u_3(\tau))d\tau))ds \\
&\geq \lambda^2 M_2^2 \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)d\tau \int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)ds \cdot \phi(u_3(t_0)) \\
&\geq \lambda(\lambda M_2 \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds)^2 \cdot \phi(u_3(t_0)).
\end{aligned}$$

where $t_0 \in [\frac{1}{4}, \frac{3}{4}]$ is given. Observing $\phi(u_3(t_0)) > 0$, we can conclude

$$\lambda(\lambda M_2 \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds)^2 \leq 1,$$

which contradicts with (29). Therefore (31) is true.

Next, from (H_4) and $f(t, \theta) \equiv \theta$, $g(t, \theta) \equiv \theta$, there exists a $R_2 > 0$ such that

$$
\begin{aligned}
\|f(t, u)\| &\leq \varepsilon_2 \|u\|, \ \forall u \in P, \ \|u\| \geq R_2, \ t \in I, \\
\|g(t, u)\| &\leq \varepsilon_2 \|u\|, \ \forall u \in P, \ \|u\| \geq R_2, \ t \in I,
\end{aligned}
\tag{32}
$$

where $\varepsilon_2^2 \in (0, (NM^2)^{-1})$, that is

$$
0 < NM^2 \varepsilon_1^2 < 1. \tag{33}
$$

From that f, g are uniformly continuous and bounded on $I \times P \cap T_{R_2}$, we get

$$
\begin{aligned}
\sup_{t \in I, u \in P \cap T_{R_2}} \|f(t, u)\| &= b_1 < +\infty \\
\sup_{t \in I, u \in P \cap T_{R_2}} \|g(t, u)\| &= b_2 < +\infty.
\end{aligned}
\tag{34}
$$

It follows from (32) and (34) that

$$
\begin{aligned}
\|f(t, u)\| &\leq \varepsilon_2 \|u\| + b_1, \ \forall \ u \in P, \ t \in I \\
\|g(t, u)\| &\leq \varepsilon_2 \|u\| + b_2, \ \forall \ u \in P, \ t \in I.
\end{aligned}
\tag{35}
$$

Taking $R > \max\{R_2, \frac{NM^2 \varepsilon_2 b_2 + NM b_1}{1 - NM^2 \varepsilon_2^2}\}$, we now prove that

$$
Au \not\geq u, \ \text{for any } u \in K, \|u\|_C = R. \tag{36}
$$

Indeed, suppose by contradiction that there exists $u_4 \in K$ with $\|u_4\|_C = R$, such that $Au_4 \geq u_4$. From (9), Lemma 2.4, we get

$$
\begin{aligned}
\theta \leq u_4(t) \leq (Au_4)(t) \ &= \int_0^1 G(t, s) f(s, \int_0^1 G(s, \tau) g(\tau, u_4(\tau)) d\tau) ds \\
&\leq M \int_0^1 f(s, \int_0^1 G(s, \tau) g(\tau, u_4(\tau)) d\tau) ds.
\end{aligned}
\tag{37}
$$

Hence, by virtue of (33),(35),(37) and the cone P being normal, we obtain

$$
\begin{aligned}
\|u_4(t)\| \ &\leq NM \| \int_0^1 f(s, \int_0^1 G(s, \tau) g(\tau, u_4(\tau)) d\tau) ds \| \\
&\leq NM \int_0^1 \|f(s, \int_0^1 G(s, \tau) g(\tau, u_4(\tau)) d\tau) \| ds \\
&\leq NM \|f(s, \int_0^1 G(s, \tau) g(\tau, u_4(\tau)) d\tau) \| \\
&\leq NM (\varepsilon_2 \| \int_0^1 G(s, \tau) g(\tau, u_4(\tau)) d\tau \| + b_1) \\
&\leq NM (\varepsilon_2 M \|g(t, u_4(t))\| + b_1) \\
&\leq NM (\varepsilon_2 M (\varepsilon_2 \|u_4(t)\| + b_2) + b_1) \\
&\leq NM (\varepsilon_2 M (\varepsilon_2 \|u_4(t)\|_C + b_2) + b_1) \\
&\leq NM (\varepsilon_2 M (\varepsilon_2 R + b_2) + b_1) \\
&\leq NM^2 \varepsilon_2^2 R + NM^2 \varepsilon_2 b_2 + NM b_1) \\
&< R,
\end{aligned}
$$

which contradicts with $\|u_4\|_C = R$. Thus (36) is true.

By (31) and (36), we showed that the condition (i) of Lemma 2.3 is satisfied.

Finally, by Lemma 3.1 and (10), A is a strict-set-contraction on $K_{r,R} = \{x \in K : r \leq \|x\|_C \leq R\}$.

From Lemma 2.3, we see that A has a fixed point u^* on $K_{r,R}$. And $(u^*, \int_0^1 G(t, s) g(s, u^*) ds)$ is a positive solution of BVPs (4).   □

4. The multiplicity of positive solutions

(H_6) There exists $\eta_0,\ \eta_0' > 0$, such that

$$\begin{aligned}
\sup_{t \in I, u \in P \cap T_{\eta_0}} \|g(t,u)\| &\leq \eta_0', \\
\sup_{t \in I, u \in P \cap T_{M\eta_0'}} \|f(t,u)\| &< \tfrac{\eta_0}{NM};
\end{aligned} \tag{38}$$

(H_7) There exist constants $\eta_1 > 0$, and $\phi \in P^*, \phi(u) > 0$, for any $u > \theta$, such that

$$\begin{aligned}
\inf_{t \in [\frac{1}{4}, \frac{3}{4}], u \in K, \|u\|_C = \eta_1'} \tfrac{\phi(f(t,u))}{\phi(u)} &\geq M_0', \\
\inf_{t \in [\frac{1}{4}, \frac{3}{4}], u \in K, \|u\|_C = \eta_1} \tfrac{\phi(g(t,u))}{\phi(u)} &\geq M_0,
\end{aligned} \tag{39}$$

where $K = \{u \in Q : u(t) \geq \lambda u(s),\ t \in [\frac{1}{4}, \frac{3}{4}],\ s \in I\}$, $\eta_1' = \|\int_0^1 G(t,s)g(s,u(s))ds\|_C$, and

$$M_0 M_0' > [\lambda^3 (\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds)^2]^{-1}. \tag{40}$$

Theorem 4.1. Let cone P be normal and conditions (H_1), (H_3), (H_5), (H_6) be satisfied. Then BVPs (4) has at least two positive solutions.

Proof. As (25), (31) stated in the proof of Theorem 3.1 and Theorem 3.2, respectively. For the η_0 stated in the assumption (H_6) we can choose r, R with $R > \eta_0 > r > 0$ such that

$$Au \not\leq u \text{ for any } u \in K, \|u\|_C = R, \tag{41}$$

$$Au \not\leq u \text{ for any } x \in K, \|u\|_C = r. \tag{42}$$

Now, we are in position to prove that

$$Au \not\geq u \text{ for any } u \in K,\ \|u\|_C = \eta_0. \tag{43}$$

Indeed, suppose by contradiction that there exists $u_5 \in K$ with $\|u_5\|_C = \eta_0$, such that $Au_5 \geq u_5$. From (9), Lemma 2.4, we get

$$\begin{aligned}
\theta \leq u_5(t) \leq (Au_5)(t) &= \int_0^1 G(t,s)f(s,\int_0^1 G(s,\tau)g(\tau,u_5(\tau))d\tau)ds \\
&\leq M \int_0^1 f(s,\int_0^1 G(s,\tau)g(\tau,u_5(\tau))d\tau)ds.
\end{aligned} \tag{44}$$

From (38), Lemma 2.4, we can get

$$\|\int_0^1 G(t,s)g(s,u_5(s))ds\| \leq M\|g(t,u_5(t))\| \leq M\eta_0'. \tag{45}$$

Hence, by virtue of (38),(44),(45) and the cone P being normal, we have

$$\begin{aligned}
\|u_5(t)\| &\leq NM\|\int_0^1 f(s,\int_0^1 G(s,\tau)g(\tau,u_5(\tau))d\tau)ds\| \\
&\leq NM\|f(s,\int_0^1 G(s,\tau)g(\tau,u_5(\tau))d\tau)\| \\
&< \eta_0.
\end{aligned}$$

So, $\|u_5\|_C < \eta_0$, which contradicts with $\|u_5\|_C = \eta_0$. Thus (43) is true.

By Lemma 3.1 and (10), A is a strict-set-contraction on $K_{\eta_0,R} = \{u \in K : \eta_0 \leq \|u\|_C \leq R\}$, and on $K_{r,\eta_0} = \{u \in K : r \leq \|u\|_C \leq \eta_0\}$. Observing (41),(42),(43), and applying Lemma 2.3 to $A, K_{\eta_0,R}$ and A, K_{r,η_0}, respectively, we assert that there exist $u_1^* \in K_{\eta_0,R}$ and $u_2^* \in K_{r,\eta_0}$ such that $Au_1^* = u_1^*$ and $Au_2^* = u_2^*$. And $(u_1^*, \int_0^1 G(t,s)g(s,u_1^*)ds)$, $(u_2^*, \int_0^1 G(t,s)g(s,u_2^*)ds)$ are two positive solutions of BVPs (4). $\square$

Theorem 4.2. Let cone P be normal and conditions (H_1), (H_2), (H_4), (H_7) hold. Then BVPs (4) has at least two positive solutions.

Proof. As (19), (36) stated in the proof of Theorem 3.1 and Theorem 3.2, respectively. For the η_1 stated in the assumption (H_7) we can choose r, R with $R > \eta_1 > r > 0$ such that

$$Au \not\geq u \text{ for any } x \in K, \|u\|_C = r. \tag{46}$$

$$Au \not\geq u \text{ for any } u \in K, \|u\|_C = R, \tag{47}$$

Now, we are in position to prove that

$$Au \not\leq u \text{ for any } u \in K, \ \|u\|_C = \eta_1. \tag{48}$$

Indeed, suppose by contradiction that there exists $u_6 \in K$ with $\|u_6\|_C = \eta_1$, such that $Au_6 \leq u_6$.
For $v(t) = \int_0^1 G(t,s)g(s,u(s))ds$, similarly to Lemma 2.6, we can show that

$$v(t) \geq \lambda v(s), \ \forall t \in [\frac{1}{4}, \frac{3}{4}], \ s \in I,$$

that is, $v(t) = \int_0^1 G(t,s)g(s,u(s))ds \in K$. From (9), (39), Lemma 2.4, we get

$$
\begin{aligned}
\phi(u_6(t_0)) \geq \phi(Au_6(t_0)) \ &= \int_0^1 G(t_0,s)\phi(f(s,\int_0^1 G(s,\tau)g(\tau,u_6(\tau))d\tau))ds \\
&\geq \lambda^2 M_0 M_0' \int_{\frac{1}{4}}^{\frac{3}{4}} G(\tau,\tau)d\tau \int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)ds \cdot \phi(u_6(t_0)) \\
&\geq M_0 M_0' \lambda^3 (\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)d\tau)^2 \cdot \phi(u_6(t_0)).
\end{aligned}
\tag{49}
$$

where $t_0 \in [\frac{1}{4}, \frac{3}{4}]$ is given. Observing $\phi(u_6(t_0)) > 0$, we can conclude

$$M_0 M_0' \lambda^3 (\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)d\tau)^2 \leq 1,$$

which contradicts with (40). Thus (48) is true.

By Lemma 3.1 and (10), A is a strict-set-contraction on $K_{\eta_1,R} = \{u \in K : \eta_1 \leq \|u\|_C \leq R\}$ and on $K_{r,\eta_1} = \{u \in K : r \leq \|u\|_C \leq \eta_1\}$. From (10), (46), (47), (48), and Lemma 2.3, we assert that there exist $u_1^* \in K_{\eta_1,R}$ and $u_2^* \in K_{r,\eta_1}$ such that $Au_1^* = u_1^*$ and $Au_2^* = u_2^*$. And $(u_1^*, \int_0^1 G(t,s)g(s,u_1^*)ds)$, $(u_2^*, \int_0^1 G(t,s)g(s,u_2^*)ds)$ are two positive solutions of BVPs (4). $\square$

5. Two examples

Now, we consider two examples to illustrate our results.

Example 5.1. Averting the complex calculation of the measure of noncompactness, we consider the boundary value problems in $E = R^n$ (n-dimensional Euclidean space and $\|x\| = \sum_{i=1}^n x_i^2$)

$$
\begin{cases}
-u_i''(t) = f_i(t, v_1, v_2, \dots, v_n), \ t \in (0,1), \\
-v_i''(t) = g_i(t, u_1, u_2, \dots, u_n), \ t \in (0,1), \\
u_i(0) = \frac{1}{2}u_i'(0), \ \frac{1}{2}u_i(\frac{1}{2}) = u_i(1), \\
v_i(0) = \frac{1}{2}v_i'(0), \ \frac{1}{2}v_i(\frac{1}{2}) = v_i(1), \ i = 1, 2, \dots, n.
\end{cases}
\tag{50}
$$

where

$$
\begin{aligned}
&f_i(t, v_1, v_2, \dots, v_n) = \rho_1(\sqrt{v_{i+1}} \sin \pi t + [\exp(v_{i+2}^2) - 1]t^3), \ i = 1, 2, \dots, n-2, \\
&f_{n-1}(t, v_1, v_2, \dots, v_n) = \rho_1(\sqrt{v_n} \sin \pi t + [\exp(v_1^2) - 1]t^3), \\
&f_n(t, v_1, v_2, \dots, v_n) = \rho_1(\sqrt{v_1} \sin \pi t + [\exp(v_2^2) - 1]t^3), \\
&g_i(t, u_1, u_2, \dots, u_n) = \rho_2((2 - \sin \pi t)\sqrt{u_{i+1}} + u_{i+2}^2), \ i = 1, 2, \dots, n-2, \\
&g_{n-1}(t, u_1, u_2, \dots, u_n) = \rho_2((2 - \sin \pi t)\sqrt{u_n} + u_1^2), \\
&g_n(t, u_1, u_2, \dots, u_n) = \rho_2((2 - \sin \pi t)\sqrt{u_1} + u_2^2),
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_1^2 &= \frac{2}{45n} < \frac{2}{5ne^2}, \\
\rho_2^2 &= \frac{1}{23n}.
\end{aligned}
\tag{51}
$$

We can conclude that BVPs (50) has at least two positive solutions.

In fact, the BVPs (50) can be regarded as a BVPs of the form (4) in E. In this situation, $I = [0,1]$, $\theta = (0, 0, \dots, 0) \in R^n$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\eta = \frac{1}{2}$, $f = (f_1, f_2, \dots, f_n)$, $g = (g_1, g_2, \dots, g_n)$. Then $\rho = 1$, $M = \frac{5}{2}$, $\lambda = \frac{1}{3}$, $f : I \times P \to P$, $g : I \times P \to P$ are continuous and non-negative on I, where

$$
P = \{u = ((u_1, u_2, \dots, u_n)) \in R^n : u_i \geq 0, \ i = 1, 2, \dots, n\}.
\tag{52}
$$

Obviously P is a normal cone with normal constant $N = 1$ and $P^* = P$. We can easily prove that the conditions $(H_1), (H_3), (H_5)$ of Theorem 4.1 hold. Choosing $\phi = (1, 1, \dots, 1)$, we are going to prove that (H_6) hold. In fact, taking $\eta_0 = 1$, $\eta_0' = \frac{2}{5}$, we have

$$
\begin{aligned}
&\sup_{t \in I, u \in P \cap T_{\eta_0}} \|g(t, u)\| \\
&= \sup_{t \in I, u \in P, \|u\|_C = 1} [\sum_{i=1}^{n-1} (\rho_2((2 - \sin \pi t)\sqrt{u_i} + u_{i+1}^2))^2 \\
&+ (\rho_2((2 - \sin \pi t)\sqrt{u_n} + u_1^2))^2] \\
&\leq \rho_2^2 9n \leq \frac{2}{5} = \eta_0' \ (\text{ observing (51)}).
\end{aligned}
$$

Thus $M\eta_0' = 1$ and

$$
\begin{aligned}
&\sup_{t\in I, u\in P\cap T_{M\eta_0'}} \|f(t,u)\| \\
&= \sup_{t\in I, u\in P\cap T_{M\eta_0'}} \{\textstyle\sum_{i=1}^{n-2} \rho_1^2(\sqrt{v_{i+1}}\sin\pi t + [\exp{(v_{i+2}^2)} - 1]t^3)^2 \\
&\quad + \rho_1^2(\sqrt{v_n}\sin\pi t + [\exp{(v_1^2)} - 1]t^3)^2 \\
&\quad + \rho_1^2(\sqrt{v_1}\sin\pi t + [\exp{(v_2^2)} - 1]t^3)^2\} \\
&\leq \rho_1^2 n e^2 < \tfrac{2}{5} = \tfrac{\eta_0}{NM} \ (\text{ observing } (51)),
\end{aligned}
$$

which implies that condition (H_6) holds. By Theorem 4.1, the BVPs (50) has at least two positive solutions.

Example 5.2. Consider the boundary value problems still in $E = R^n$.

$$
\begin{cases}
-u_i''(t) = f_i(t, v_1, v_2, \ldots, v_n), \ t\in(0,1), \\
-v_i''(t) = g_i(t, u_1, u_2, \ldots, u_n), \ t\in(0,1), \\
u_i(0) = \frac{1}{2}u_i'(0), \ \frac{1}{2}u_i(\frac{1}{2}) = u_i(1), \\
v_i(0) = \frac{1}{2}v_i'(0), \ \frac{1}{2}v_i(\frac{1}{2}) = v_i(1), \ i = 1,2,\ldots,n.
\end{cases}
\tag{53}
$$

where

$$
\begin{aligned}
&f_i(t, v_1, v_2, \ldots, v_n) = \xi(2 - \sin\pi t)e^{-\max_{1\leq i\leq n} v_i}v_i^2, \ i = 1,2,\ldots,n-1, \\
&f_n(t, v_1, v_2, \ldots, v_n) = \xi(2 - \sin\pi t)e^{-\max_{1\leq i\leq n} v_i}v_1^2, \\
&g_i(t, u_1, u_2, \ldots, u_n) = (2 - t)e^{-\max_{1\leq i\leq n} u_i}u_i^2, \ i = 1,2,\ldots,n-1, \\
&g_n(t, u_1, u_2, \ldots, u_n) = (2 - t)e^{-\max_{1\leq i\leq n} u_i}u_1^2,
\end{aligned}
$$

and

$$
\xi = 1186 n^2 e^{10n+1} > \frac{527 \cdot 9 n^2 e^{10n+1}}{4}.
\tag{54}
$$

We can conclude that BVPs (53) has at least two positive solutions.

In fact, the BVPs (53) can be regarded as a BVPs of the form (4) in E. In this situation, $I = [0,1]$, $\theta = (0,0,\ldots,0) \in R^n$, $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $\eta = \frac{1}{2}$, $f = (f_1, f_2, \ldots, f_n)$, $g = (g_1, g_2, \ldots, g_n)$. Then $\rho = 1$, $M = \frac{5}{2}$, $\lambda = \frac{1}{3}$, $f : I \times P \to P$, $g : I \times P \to P$ are continuous and non-negative on I, where P is defined by (52). Moreover, we have

$$
[\lambda^3(\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds)^2]^{-1} \approx 526.
\tag{55}
$$

We can easily prove that the conditions $(H_1), (H_2), (H_4)$ of Theorem 4.2 hold. Choosing $\phi = (1,1,\ldots,1)$, we are in position to prove that (H_7) hold. As in the proof of Theorem 4.2, for $u \in K$, $v(t) = \int_0^1 G(t,s)g(s,u(s))ds \in K$, we can get

$$
\begin{aligned}
\min_{t\in[\frac{1}{4},\frac{3}{4}]} \|u(t)\| &\geq \tfrac{\lambda\eta_1}{N}, \\
\min_{t\in[\frac{1}{4},\frac{3}{4}]} \|v(t)\| &\geq \tfrac{\lambda\eta_1'}{N},
\end{aligned}
$$

therefore

$$\| \int_0^1 G(t,s)g(s,u(s))ds\| \quad \begin{aligned} &\leq M\|g(s,u(s))\| \\ &\leq M\sum_1^n (2-t)^2 e^{-\max_{1\leq i\leq n} u_i} u_i^4 \\ &\leq M\sum_1^n 4u_i^4 \\ &\leq M4n = 10n, \end{aligned}$$

that is,

$$\eta_1' \leq 10n. \tag{56}$$

By (56), we have

$$\begin{aligned} &\inf_{t\in[\frac{1}{4},\frac{3}{4}],u\in K,\|u\|_C=\eta_1'} \frac{\phi(f(t,u))}{\phi(u)} \\ &= \inf_{t\in[\frac{1}{4},\frac{3}{4}],u\in K,\|u\|_C=\eta_1'} \frac{\sum_{i=1}^n \xi(2-\sin\pi t)e^{-\max_{1\leq i\leq n} u_i} u_i^2}{\sum_{i=1}^n u_i} \\ &\geq \frac{2\xi e^{-10n}}{3n} := M_0'. \end{aligned} \tag{57}$$

Similarly, taking $\eta_1 = 1$, we get

$$\begin{aligned} &\inf_{t\in[\frac{1}{4},\frac{3}{4}],u\in K,\|u\|_C=1} \frac{\phi(g(t,u))}{\phi(u)} \\ &= \inf_{t\in[\frac{1}{4},\frac{3}{4}],u\in K,\|u\|_C=1} \frac{\sum_{i=1}^n (2-t)e^{-\max_{1\leq i\leq n} u_i} u_i^2}{\sum_{i=1}^n u_i} \\ &\geq \frac{2}{3en} := M_0. \end{aligned} \tag{58}$$

By (54), (55), (57), (58), we obtain

$$M_0 M_0' = \frac{2}{3en} \cdot \frac{2\xi e^{-10n}}{3n} = \frac{4\xi}{9n^2 e^{10n+1}} > 527 > [\lambda^3 (\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds)^2]^{-1},$$

which implies that condition (H_7) holds. By Theorem 4.2, the BVPs (53) has at least two positive solutions.

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