J. Appl. Math. & Informatics Vol. **28**(2010), No. 5 - 6, pp. 1439-1443
Website: http://www.kcam.biz

ON THE k-LUCAS NUMBERS VIA DETERMINENT†

GWANGYEON LEE* AND YUOHO LEE

Abstract. For a positive integer $k \geq 2$, the k-bonacci sequence $\{g_n^{(k)}\}$ is defined as: $g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0$, $g_{k-1}^{(k)} = g_k^{(k)} = 1$ and for $n > k \geq 2$, $g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}$. And the k-Lucas sequence $\{l_n^{(k)}\}$ is defined as $l_n^{(k)} = g_{n-1}^{(k)} + g_{n+k-1}^{(k)}$ for $n \geq 1$. In this paper, we give a representation of nth k-Lucas $l_n^{(k)}$ by using determinant.

AMS Mathematics Subject Classification : 05A19, 05B20, 11B39, 15A15.
Key word and phrases : Fibonacci sequence, k-bonacci number,k-Lucas number, determinant.

1. Introduction

In [1], [2] and [3], the authors have been introduced a generalization of Fibonacci sequence, which is called the k-bonacci sequence for positive integer $k \geq 2$. The *k-bonacci sequence* $\{g_n^{(k)}\}$ is defined as;

$$g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, \ \ g_{k-1}^{(k)} = g_k^{(k)} = 1$$

and for $n > k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

We call $g_n^{(k)}$ the nth *k-bonacci number*. By the definition of the k-bonacci sequence, we know that

$$\begin{aligned}
g_{k+1}^{(k)} &= g_k^{(k)} + g_{k-1}^{(k)} = 1 + 1 = 2,\\
g_{k+2}^{(k)} &= g_{k+1}^{(k)} + g_k^{(k)} + g_{k-1}^{(k)} = 2^2,\\
g_{k+3}^{(k)} &= g_{k+2}^{(k)} + g_{k+1}^{(k)} + g_k^{(k)} + g_{k-1}^{(k)} = 2^3,\\
&\ \ \vdots
\end{aligned}$$

Received April 12, 2010. Accepted June 25, 2010. *Corresponding author.
†This paper was supported by Hanseo University, 2010

$$g_{2k-2}^{(k)} = g_{2k-3}^{(k)} + \cdots + g_k^{(k)} + g_{k-1}^{(k)} = 2^{k-2},$$
$$g_{2k-1}^{(k)} = g_{2k-2}^{(k)} + \cdots + g_k^{(k)} + g_{k-1}^{(k)} = 2^{k-1}$$

Thus, we have $g_j^{(k)} = 2^{j-k}$ for $j = k, k+1, \ldots, 2k-1$. For example, if $k = 2$, then $\{g_n^{(2)}\}$ is the Fibonacci sequence and if $k = 5$, then $g_1^{(5)} = g_2^{(5)} = g_3^{(5)} = 0$, $g_4^{(5)} = g_5^{(5)} = 1$, and the 5-bonacci sequence is

$$0, 0, 0, 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, \cdots .$$

In [2], the authors gave interesting examples in combinatorics and probability related to the k-bonacci numbers.

We let L_n represent the nth Lucas number, that is, for $n \geq 1$, $L_n = F_{n-1} + F_{n+1}$ where $F_0 = 0$. In [1], the author also has been introduced a generalization of Lucas sequence, which is called the k-Lucas sequence for positive integer $k \geq 2$. Let $g_0^{(k)} = 0$. The *k-Lucas sequence* $\{l_n^{(k)}\}$ is defined by

$$l_n^{(k)} = g_{n-1}^{(k)} + g_{n+k-1}^{(k)}.$$

We call $l_n^{(k)}$ the nth *k-Lucas number*. Then we have $l_j^{(k)} = 2^{j-1}$, $j = 1, 2, \ldots, k-1$, $l_k^{(k)} = 1 + 2^{k-1}$, and $l_n^{(k)} = l_{n-1}^{(k)} + l_{n-2}^{(k)} + \cdots + l_{n-k}^{(k)}$ for $n > k$. If $k = 2$, then $l_n^{(2)} = L_n$. For example, if $k = 5$, then the 5-Lucas sequence is

$$1, 2, 4, 8, 17, 32, 63, 124, 244, 480, 943, 1854, \ldots .$$

In [3], the authors gave a representation of $g_n^{(k)}$ by using permanent and determinent for given matrix. In this paper, we give a representation of nth k-Lucas numbers via determinents of $(0, 1)$-matrices.

2. k-Lucas number

In [1], the author gave two matrices $S_n^{(k)}$ and $\mathfrak{C}_{(n,k)}$. Let $S_n^{(k)} = [s_{ij}]$ be the $n \times n$ (0,1)-matrix defined by $s_{ij} = 1$ if and only if $-1 \leq j - i \leq k-1$. For $k < n$, let $\mathfrak{C}^{(n,k)} = S_n^{(k)} - \sum_{j=2}^{k} E_{1j} + E_{1k+1}$ where E_{ij} denotes the $n \times n$ matrix with 1 in the (i, j) position and zeros elsewhere. If $k \geq n$, then the matrix E_{1j+1}, $j \geq k$, is not defined, and hence we let $\mathfrak{C}^{(n,k)} = S_n^{(k)} - \sum_{j=2}^{n} E_{1j}$ for $n \leq k$.

Let H_n be a $(1, -1)$-matrix of order n, defined by

$$H_n = \begin{bmatrix} 1 & (-1)^1 & (-1)^2 & (-1)^3 & \cdots & (-1)^{n-1} \\ 1 & 1 & (-1)^1 & (-1)^2 & \cdots & (-1)^{n-2} \\ 1 & 1 & 1 & (-1)^1 & \cdots & (-1)^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & (-1)^1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

In this paper, we consider the matrix $\mathfrak{C}^{(n,k)} \circ H_n$ for $n \geq 2$, where $\mathfrak{C}^{(n,k)} \circ H_n$ denotes the Hadamard product of $\mathfrak{C}^{(n,k)}$ and H_n.

First, we have the following lemma.

Lemma 2.1. *For $2 \leq n \leq k$, we have*

$$\det(\mathfrak{C}^{(n,k)} \circ H_n) = 2^{n-2} = l_{n-1}^{(k)}.$$

Proof. If $n = 2$, then $\det(\mathfrak{C}^{(2,k)} \circ H_2) = 1 = l_1^{(k)}$ and hence the lemma holds.

Now, we consider $n \geq 3$,

$$\det(\mathfrak{C}^{(n,k)} \circ H_n)$$

$$= \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & (-1)^1 & (-1)^2 & (-1)^3 & \cdots & (-1)^{n-3} & (-1)^{n-2} \\ 0 & 1 & 1 & (-1)^1 & (-1)^2 & \cdots & (-1)^{n-4} & (-1)^{n-3} \\ 0 & 0 & 1 & 1 & (-1)^1 & \cdots & (-1)^{n-5} & (-1)^{n-4} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 & (-1)^1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}_{n \times n}$$

$$= \det \begin{bmatrix} 1 & (-1)^1 & (-1)^2 & (-1)^3 & \cdots & (-1)^{n-3} & (-1)^{n-2} \\ 1 & 1 & (-1)^1 & (-1)^2 & \cdots & (-1)^{n-4} & (-1)^{n-3} \\ 0 & 1 & 1 & (-1)^1 & \cdots & (-1)^{n-5} & (-1)^{n-4} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 & (-1)^1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}_{(n-1) \times (n-1)}.$$

By induction on n and the expansion of determinent about the first column, we have $\det(\mathfrak{C}^{(n,k)} \circ H_n) = 2^{n-2} = l_{n-1}^{(k)}$. □

Let $\mathfrak{F}^{(n,k)} = [f_{ij}] = T_n + B_n$, where $T_n = [t_{ij}]$ is the $n \times n$ (0,1)-matrix defined by $t_{ij} = 1$ if and only if $|i - j| \leq 1$, and $B_n = [b_{ij}]$ is the $n \times n$ (0,1)-matrix defined by $b_{ij} = 1$ if and only if $2 \leq j - i \leq k - 1$. In [2], the following theorem gave a representation of the nth k-bonacci number $g_n^{(k)}$.

Theorem 2.2. *[2]. Let $\{g_n^{(k)}\}$ be the k-bonacci sequence. Then*

$$g_{n+k-2}^{(k)} = \det(\mathfrak{F}^{(n-1,k)} \circ H_{n-1}).$$

Since $l_n^{(k)} = g_{n+k-1}^{(k)} + g_{n-1}^{(k)}$, from the above theorem, we have

$$l_n^{(k)} = \det(\mathfrak{F}^{(n,k)} \circ H_n) + \det(\mathfrak{F}^{(n-k,k)} \circ H_{n-k}). \tag{2.1}$$

Now we have the following theorem.

Theorem 2.3. *Let k and n be positive integers. For $n \geq 2$, we have*

$$\det(\mathfrak{C}^{(n,k)} \circ H_n) = l_{n-1}^{(k)}.$$

Proof. If $n \leq k$, then we are done, by Lemma 2.1

Suppose that $n > k$. Then

$$\det(\mathfrak{C}^{(n,k)} \circ H_n)$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & (-1)^k & 0 & \cdots & 0 \\ 1 & 1 & (-1)^1 & (-1)^2 & \cdots & (-1)^{k-2} & (-1)^{k-1} & 0 & \cdots & 0 \\ 0 & 1 & 1 & (-1)^1 & \cdots & (-1)^{k-3} & (-1)^{k-2} & (-1)^{k-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \ddots & & (-1)^{k-1} \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & & & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & & & 1 & 1 & (-1)^1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{vmatrix}.$$

By the expansion of determinent about the first row and (2.1), we have

$$\begin{aligned} \det(\mathfrak{C}^{(n,k)} \circ H_n) &= \det(\mathfrak{F}^{(n-1,k)} \circ H_{n-1}) + \det(\mathfrak{F}^{(n-k-1,k)} \circ H_{n-k-1}) \\ &= l_{n-1}^{(k)}. \end{aligned}$$

Therefore, the proof is completed. □

In [1], the author gave a bipartite graph with bipartite adjacency matrix $A_n = T_n + E_{13} - E_{23} + E_{24} - E_{34}$. And the number of 1-factor of bipartite graph with bipartite adjacency matrix A_n is the $(n-1)$th Lucas number L_{n-1}. Also, in [1], the author proved that A_n is not permutation invariant to $\mathfrak{C}^{(n,2)}$, i.e., the matrix A_n is not similar to $\mathfrak{C}^{(n,2)}$. The next theorem shows that we can get the $(n-1)$th Lucas number L_{n-1} by using determinent of A_n.

Theorem 2.4. *For $n \geq 4$, the determinent of the matrix $A_n \circ H_n$ is the $(n-1)$th Lucas number L_{n-1}, i.e.,*

$$\det(A_n \circ H_n) = L_{n-1}.$$

Proof. If $n = 4$, then $\det(A_4 \circ H_4) = 4 = L_3$.

By induction on n, we assume that $\det A_n = L_{n-1}$ and consider $n+1$. By the expansion of determinent about the nth column of $A_{n+1} \circ H_{n+1}$, we have

$$\begin{aligned}
\det(A_{n+1} \circ H_{n+1}) &= \det \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & & 1 & 1 & -1 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & 1 & -1 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix} \\
&= \det(A_n \circ H_n) + \det(A_{n-1} \circ H_{n-1}) \\
&= L_{n-1} + L_{n-2} \\
&= L_n.
\end{aligned}$$

□

References

1. Gwang-Yeon Lee, k-Lucas numbers and associated bipartite graph, *Linear Algebra Appl.*, 320:51-61 (2000).
2. Gwang-Yeon Lee, S.-G. Lee and H.-G. Shin, On the k-generalized Fibonacci matrix Q_k, *Linear Algebra Appl.*, 251:73-88(1997).
3. Gwang-Yeon Lee, S.-G. Lee, J.-S. Kim and H.-K. Shin, The Binet formula and representations of k-generalized Fibonacci numbers, *The Fibonacci Quarterly*, 39(2):158-164(2001).

GwangYeon Lee received BS, MS, and Ph. D from SungKyunKwan University under the supervision of Professor Heo-Jae Lee. Since 1992 he has been at Hanseo University and became Professor in 2005. His research interests focus on combinatorial matrices and counting number theoreies.

Department of Mathematics, Hanseo University, Haemimyun Seosan, 356-706, Korea
e-mail: gylee@hanseo.ac.kr

Yuoho Lee received BS, MS, and Ph. D from SungKyunKwan University under the supervision of Professor Sang-Gu Lee. Since 2002 he has worked at Daegu Haany University. His research interests focus on linear preservers, combinatorial matrices and counting number theoreies.

Department of Internet Information, Daegu Haany University, Kyeongsan, 712-715, Korea
e-mail: gmjae@chungbuk.ac.kr