J. Appl. Math. & Informatics Vol. **28**(2010), No. 5 - 6, pp. 1431-1437
Website: http://www.kcam.biz

NEW TRAVELING WAVE SOLUTIONS TO THE SEVENTH-ORDER SAWADA-KOTERA EQUATION

JISHE FENG

Abstract. We use the (G'/G)-expansion method to seek the traveling wave solution of the Seventh-order Sawada-Kotera Equation. The solutions that we get are more general than the solutions given in literature. It is shown that the (G'/G)-expansion method provides a very effective and powerful mathematical tool for solving nonlinear equations in mathematical physics.

AMS Mathematics Subject Classification : 35K99, 35P05, 35P99.
Key words and phrases : (G′/G)-expansion method, Seventh-order Sawada-Kotera Equation, traveling wave solution.

1. Introduction

In this paper, we will extend the (G′/G)-expansion method to the following Seventh-order Sawada-Kotera equation (sSK) [1]

$$u_t + (63u^4 + 63(2u^2u_{2x} + uu_x^2) + 21(uu_{4x} + u_{2x}^2 + u_xu_{3x}) + u_{6x})_x = 0 \quad (1.1)$$

Recently, Wang et al. [2] proposed a new method called the (G′/G)-expansion method to find traveling wave solutions of NLEEs. Many literatures [3-19] show that the (G′/G)-expansion method is very effective; many nonlinear equations have been successfully solved. The rest of this paper is arranged as follows. In section 2, we simply provide the (G′/G)-expansion method and it algorithm. In section 3, we apply the method to the sSK equation, and some new traveling wave solutions are obtained. In addition, we conclude the paper in the last section.

Received March 5, 2010. Revised March 15, 2010. Accepted April 17, 2010.
© 2010 Korean SIGCAM and KSCAM.

2. The (G/G)-expansion method

Suppose that a nonlinear equation, say in two independent variable x and t, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0 \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the (G′/G)-expansion method.

Step 1. Combining the independent variables x and t into one variable $\xi = x - ct$, we suppose that

$$u(x,t) = u(\xi), \xi = x - ct \tag{2.2}$$

the traveling wave variable (2.2) permits us reducing (2.1) to an ODE for $u = u(\xi)$

$$P(u, -cu', u', c^2u'', -cu'', u'', \cdots) = 0 \tag{2.3}$$

Step 2. Suppose that the solution of ODE (2.3) can be expressed by a polynomial in (G′/G) as follows:

$$u(\xi) = \alpha_m \left(\frac{G'}{G}\right)^m + \alpha_{m-1} \left(\frac{G'}{G}\right)^{m-1} + \cdots \tag{2.4}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \gamma G = 0 \tag{2.5}$$

where $\alpha_m, \alpha_{m-1}, \cdots, \lambda$ and γ are constants to be determined later,$\alpha_m \neq 0$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.3). It is well known that exact solutions of Eq. (2.5) are as follows:

(i) When $\triangle = \lambda^2 - 4\gamma > 0$,

$$G(\xi) = \left(C_1 \sinh\left(\frac{\sqrt{\triangle}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\triangle}}{2}\xi\right)\right) e^{-\frac{\lambda}{2}\xi} \tag{2.6}$$

(ii) When $\triangle = \lambda^2 - 4\gamma < 0$,

$$G(\xi) = \left(C_1 \sin\left(\frac{\sqrt{-\triangle}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{-\triangle}}{2}\xi\right)\right) e^{-\frac{\lambda}{2}\xi} \tag{2.7}$$

(iii)When $\triangle = \lambda^2 - 4\gamma = 0$,

$$G(\xi) = (C_1 + C_2\xi)e^{-\frac{\lambda}{2}\xi} \tag{2.8}$$

where C_1 and C_2 are arbitrary constants.

Therefore, there are the follows:

(i) When $\triangle = \lambda^2 - 4\gamma > 0$,

$$\frac{G'}{G} = \frac{\sqrt{\triangle}}{2} \frac{C_1 \cosh\left(\frac{\sqrt{\triangle}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\triangle}}{2}\xi\right)}{C_1 \sinh\left(\frac{\sqrt{\triangle}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\triangle}}{2}\xi\right)} - \frac{\lambda}{2} \tag{2.9}$$

(ii) When $\triangle = \lambda^2 - 4\gamma < 0$,

$$\frac{G'}{G} = \frac{\sqrt{-\triangle}}{2} \frac{C_1 \cos\left(\frac{\sqrt{-\triangle}}{2}\xi\right) - C_2 \sin\left(\frac{\sqrt{-\triangle}}{2}\xi\right)}{C_1 \sin\left(\frac{\sqrt{-\triangle}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{-\triangle}}{2}\xi\right)} - \frac{\lambda}{2} \tag{2.10}$$

(iii) When $\triangle = \lambda^2 - 4\gamma = 0$,

$$\frac{G'}{G} = \frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2} \tag{2.11}$$

where C_1 and C_2 are arbitrary constants.

Step 3. By substituting (2.4) into (2.3) and using second order LODE (2.5), collecting all terms with the same order of (G′/G) together, left-hand side of (2.3) is converted into another polynomial in (G′/G). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_m, \alpha_{m-1}, \cdots, c, \lambda$ and γ.

Step 4. Assuming that the constants $\alpha_m, \alpha_{m-1}, \cdots, c, \lambda$ and γ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (2.5) have been well known for us, then substituting $\alpha_m, \alpha_{m-1}, \cdots, c, \lambda$ and γ the general solutions of Eq. (2.5) into (2.4), we have more traveling wave solutions of the NLEEs (2.1).

3. The The traveling wave solution of sSK equation

By considering the wave transformation Eq. (2.2), we change Eq. (2.5) to the form

$$-cu' + (63u^4 + 63(2u^2u'' + uu'^2) + 21(uu^{(4)} + u''^2 + u'u^{(3)}) + u^{(6)})' = 0 \tag{3.1}$$

integrating Eq.(3.1) with respect to ξ once yields

$$-cu + 63u^4 + 63(2u^2u'' + uu'^2) + 21(uu^{(4)} + u''^2 + u'u^{(3)}) + u^{(6)}) = \beta \tag{3.2}$$

where β is an integration constant.

Suppose that the solution of Eq. (3.2) can be expressed by a polynomial in (G'/G) as Eq. (2.4). By using Eq. (2.4) and Eq.(2.5) it is easily derived that

$$(\frac{G'}{G})' = \frac{G''G - G'^2}{G^2} = -(\frac{G'}{G})^2 - \lambda\frac{G'}{G} - \gamma$$

$$(\frac{G'}{G})'' = 2(\frac{G'}{G})^3 + 3\lambda(\frac{G'}{G})^2 + (2\gamma + \lambda^2)(\frac{G'}{G}) + \lambda\gamma \tag{3.3}$$

and

$$u^4 = \alpha_m^4(\frac{G'}{G})^{4m} + \cdots \tag{3.4}$$

$$u' = -m\alpha_m(\frac{G'}{G})^{m+1} + \cdots$$

$$u'' = m(m+1)\alpha_m(\frac{G'}{G})^{m+2} + \cdots$$

$$\cdots$$

$$u^{(6)} = m(m+1)\cdots(m+5)\alpha_m(\frac{G'}{G})^{m+6} + \cdots \tag{3.5}$$

Considering the homogeneous balance between u^4 and $u^{(6)}$ in Eq.(3.2), based on Eq.(3.4) and (3.5) we required that $4m = m + 6 \Rightarrow m = 2$, so we yields the solution of Eq. (3.2) as the following form

$$u(\xi) = \alpha_2(\frac{G'}{G})^2 + \alpha_1(\frac{G'}{G}) + \alpha_0, \alpha_2 \neq 0 \tag{3.6}$$

and therefore

$$u^2 = \alpha_2^2(\frac{G'}{G})^4 + 2\alpha_2\alpha_1(\frac{G'}{G})^3 + (2\alpha_2\alpha_0 + \alpha_1^2)(\frac{G'}{G})^2 + 2\alpha_1\alpha_0(\frac{G'}{G}) + (\alpha_0)^2, \tag{3.7}$$

$$u^4 = \alpha_2^4(\frac{G'}{G})^8 + 4\alpha_2^3\alpha_1(\frac{G'}{G})^7 + \cdots \tag{3.8}$$

$$u' = -2\alpha_2(\frac{G'}{G})^3 - (2\alpha_2\lambda + \alpha_1)(\frac{G'}{G})^2 - (2\alpha_2\gamma + \alpha_1\lambda)(\frac{G'}{G}) - \alpha_1\lambda, \tag{3.9}$$

$$u'' = 6\alpha_2(\frac{G'}{G})^4 + (10\alpha_2\lambda + 2\alpha_1)(\frac{G'}{G})^3 + \cdots, \tag{3.10}$$

$$u^{(3)} = -24\alpha_2(\frac{G'}{G})^5 - (54\alpha_2\lambda + 6\alpha_1)(\frac{G'}{G})^4 + \cdots, \tag{3.11}$$

$$u^{(4)} = 120\alpha_2(\frac{G'}{G})^6 + (336\alpha_2\lambda + 24\alpha_1)(\frac{G'}{G})^5 + \cdots, \tag{3.12}$$

$$u^{(6)} = 5040\alpha_2(\frac{G'}{G})^8 + (19440\alpha_2\lambda + 720\alpha_1)(\frac{G'}{G})^7 + \cdots, \tag{3.13}$$

By substituting Eq. (3.6)-(3.13) into Eq. (3.2) and collecting all terms with the same power of (G′/G) together, the left-hand side of Eq. (3.2) is converted into another polynomial in (G′/G). Equating each coefficient of this polynomial to zero, we obtain a set of simultaneous algebraic equations for $\alpha_1, \alpha_2, c, \lambda$ and λ. Solving the above system gives by Maple, we have the following solutions:

Case 1.

$$\alpha_2 = -4, \alpha_1 = \pm 4\sqrt{4\gamma + 3^{\frac{1}{4}}\beta^{\frac{1}{4}}}, \alpha_0 = -4\gamma - 3^{-\frac{3}{4}}\beta^{\frac{1}{4}}, c = -4 \times 3^{\frac{-1}{4}}\beta^{\frac{3}{4}}, \lambda = \mp\sqrt{4\gamma + 3^{\frac{1}{4}}\beta^{\frac{1}{4}}} \tag{3.14}$$

Case 2.

$$\alpha_2=-4,\alpha_1=\pm4\sqrt{4\gamma-3^{\frac{1}{4}}\beta^{\frac{1}{4}}},\alpha_0=-4\gamma+3^{-\frac{3}{4}}\beta^{\frac{1}{4}},c=4\times3^{\frac{-1}{4}}\beta^{\frac{3}{4}},\lambda=\mp\sqrt{4\gamma-3^{\frac{1}{4}}\beta^{\frac{1}{4}}} \tag{3.15}$$

Case 3.

$$\alpha_2=-2,\alpha_1=\pm4\sqrt{\gamma},\alpha_0=-2\gamma-\frac{\beta}{3^{\frac{3}{4}}(-7\beta^3)^{\frac{1}{4}}},c=4\times(\frac{7}{3}(-\beta^3))^{\frac{1}{4}},\lambda=\mp2\sqrt{\gamma} \tag{3.16}$$

Case 4.

$$\alpha_2=-2,\alpha_1=\pm4\sqrt{\gamma},\alpha_0=-2\gamma+\frac{\beta}{3^{\frac{3}{4}}(-7\beta^3)^{\frac{1}{4}}},c=-4\times(\frac{7}{3}(-\beta^3))^{\frac{1}{4}},\lambda=\mp2\sqrt{\gamma} \tag{3.17}$$

where γ is an arbitrary constant.

Substituting the solutions (3.14)-(3.17) into (3.6), we obtain the following traveling wave solutions: Substituting the solutions (3.14)-(3.17) into (3.6), we obtain the following traveling wave solutions:

$$u_1(x,t)=-3^{\frac{1}{4}}\beta^{\frac{1}{4}}(\frac{C_1cosh\frac{1}{2}\sqrt{\triangle}\xi+C_2sinh\frac{1}{2}\sqrt{\triangle}\xi}{C_1sinh\frac{1}{2}\sqrt{\triangle}\xi+C_2cosh\frac{1}{2}\sqrt{\triangle}\xi})^2+2\times3^{-\frac{3}{4}}\beta^{\frac{1}{4}}, \tag{3.18}$$

$$u_2(x,t)=-3^{\frac{1}{4}}\beta^{\frac{1}{4}}(\frac{C_1cos\frac{1}{2}\sqrt{-\triangle}\xi-C_2sin\frac{1}{2}\sqrt{-\triangle}\xi}{C_1sin\frac{1}{2}\sqrt{-\triangle}\xi+C_2cos\frac{1}{2}\sqrt{-\triangle}\xi})^2-2\times3^{-\frac{3}{4}}\beta^{\frac{1}{4}}, \tag{3.19}$$

where $\xi=x\pm4\times3^{-\frac{1}{4}}\beta^{\frac{3}{4}}t$, $\Delta=\lambda^2-4\gamma=\pm3^{\frac{1}{4}}\beta^{\frac{1}{4}}$, C_1 and C_2 are arbitrary constants.

$$u_3(x,t)=-2(\frac{C_2}{C_1+C_2(x-4\times(\frac{7}{3}(-\beta^3))^{\frac{1}{4}}t)})^2-\frac{\beta}{3^{\frac{3}{4}}(-7\beta^3)^{\frac{1}{4}}}, \tag{3.20}$$

$$u_4(x,t)=-2(\frac{C_2}{C_1+C_2(x+4\times(\frac{7}{3}(-\beta^3))^{\frac{1}{4}}t)})^2+\frac{\beta}{3^{\frac{3}{4}}(-7\beta^3)^{\frac{1}{4}}}, \tag{3.21}$$

where C_1 and C_2 are arbitrary constants.

If C_1 and C_2 are taken as special values, the various known results in the literature can be rediscovered, for instance, if $C_2=0$, then $u_1(x,t)$ can be written as

$$u_1(x,t)=-3^{\frac{1}{4}}\beta^{\frac{1}{4}}(\tanh(\frac{3^{\frac{1}{8}}\beta^{\frac{1}{8}}}{2}(x\pm4\times3^{-\frac{1}{4}}\beta^{\frac{3}{4}}t)))^2+2\times3^{-\frac{3}{4}}\beta^{\frac{1}{4}} \tag{3.18$'$}$$

Set $k=\frac{3^{\frac{1}{8}}\beta^{\frac{1}{8}}}{2}$, Eq. (3.18) becomes

$$u_1(x,t)=\frac{4}{3}k^2(2-3\tanh^2(k(x+\frac{256}{3}k^6t))) \tag{3.18$''$}$$

this agrees with the solution given in (6.6) of Ref.[20] and Ref.[1].

To our knowledge, the above solutions (3.18)-(3.21) have not been reported in literatures.

4. Conclusion

In this paper, we use the (G'/G)-expansion method to seek the traveling wave solution of the Seventh-order Sawada-Kotera Equation. It is a kind of more general than the solutions given in [1, 20]. It is shown that the (G'/G)-expansion method provides a very effective and powerful mathematical tool for solving nonlinear equations in mathematical physics.

References

1. Salah M. El-Sayed and Dogan Kaya, *An application of the ADM to seven-order Sawada-kotara equations*, Appl. Math. Comput.**157** (2004), 93-101.
2. M.L. Wang, X. Z. Li and J. L. Zhang, *The (G'/G)-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics*, Phys. Lett. A, **372** (2008), 417-423.
3. A. Bekir, *Application of the (G'/G)-expansion method for nonlinear evolution equations*, Phys. Lett. A, **372** (2008), 3400-3406.
4. J. Zhang, X. Wei and Y. Lu, *A generalized (G'/G)-expansion method and its applications*, Phys. Lett. A, **372** (2008), 3653-3658.
5. S. Zhang, J.L. Tong and W. Wang, *A generalized (G'/G)-expansion method for the mKdV equation with variable coefficients*, Phys. Lett. A, **372** (2008). 2254-2257.
6. S. Zhang, L. Dong, J.M. Ba and Y.N. Sun, *The (G'/G)-expansion method for nonlinear differential-difference equations*, Phys. Lett. A, **373** (2009), 905-910.
7. H. Q. Zhang, *New application of the (G'/G)-expansion method*, Commun. Nonlinear Sci. Numer. Simulat. **14**(2009), 3220-3225.
8. I.Aslan and T.Ozis,, *Analytic study on two nonlinear evolution equations by using the (G'/G)-expansion method*, Appl. Math. Comput. **209**(2009), 425-429.
9. I.Aslan and T.Ozis, *On the validity and reliability of the (G'/G)-expansion method by using higher-order nonlinear equations*, Appl. Math. Comput. doi:10.1016/j.amc.2009.01.075150.
10. I.Aslan, *Exact and explicit solutions to some nonlinear evolution equations by utilizing the (G'/G)-expansion method*, Appl. Math. Comput. doi:10.1016/j.amc.2009.05.038.
11. T. Ozis and I. Aslan, *Symbolic Computation and Construction of New Exact Travelling Wave Solutions to Fitzhugh-Nagumo and Klein-Gordon Equations*, Z. Naturforsch, **64a**(2009), 15-20.
12. T. Ozis and I. Aslan, *Symbolic computations and exact and explicit solutions of some nonlinear evolution equations in mathematical physics*, Commun. Theor. Phys. **51**(2009), 577-580.
13. H. Gao and R-X. Zhao, *New application of the (G'/G)-expansion method to higher-order nonlinear equations*, Appl. Math. Comput. doi:10.1016/j.amc.2009.08.041.
14. E. M. E. Zayed, *New traveling wave solutions for higher dimensional nonlinear evolution equations using a generalized (G'/G)-expansion method*, J. Phys. A: Math. Theor. **42** (2009), 195202-195215.
15. E. M. E. Zayed, *The (G'/G)-expansion method and its applications to some nonlinear evolution equations in the mathematical physics,*, J. Appl.Math. and Computing **30**(2009), 89-103.
16. Zhou Y. Bin and L. Chao, *Application of modified (G'/G)-expansion method to traveling wave solutions for Whitham-Broer-Kaup-Like equations*, Commu. Theore. Phys. **51**(2009), 664-670.
17. E. M. E. Zayed and Khaled A. Gepreel, *Some applications of the (G'/G)-expansion method to non-linear partial differential equations*, Appl. Math. Comput.**212**(2009), 1-13.

18. E. M. E. Zayed and K. A. Gepreel, *The (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics*, J. Math. Phys. **50** (2009), 013502-013513.
19. E. M. E. Zayed and S. Al-Joudi, *Applications of an Improved (G'/G)-expansion Method to Nonlinear PDEs in Mathematical Physics*, AIP Conf. Proc. Amer. Institute of Physics. **1168**(2009), 371-376.
20. E.J. Parkes and B.R, Duffy, *An automated tanh-function method for finding solitary wave solutions to non-linear evolution equations*, Comput. Phys. Commun. **98**(1996), 288-300.

Jishe Feng received his master's degree from Xi'an Jiaotong University under the direction of Software Engineering. Since 2002 he has been at the Longdong University, which named his an Associate Professor in 2007. His research interests focus on the methods to find the numerical solution of differential equations.

Department of Mathematics, Longdong University, Gansu 745000, P.R.China
e-mail: gsfjs6567@126.com