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PRECONDITIONED AOR ITERATIVE METHOD FOR Z-MATRICES†

GUANGBIN WANG*, NING ZHANG AND FUPING TAN

Abstract. In this paper, we present a preconditioned iterative method for solving linear systems $Ax = b$, where A is a Z-matrix. We give some comparison theorems to show that the rate of convergence of the new preconditioned iterative method is faster than the rate of convergence of the previous preconditioned iterative method. Finally, we give one numerical example to show that our results are true.

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1. Introduction

For solving linear systems

$$Ax = b, \tag{1}$$

where A is an $n \times n$ square matrix, and x and b are n-dimensional vectors, the basic iterative method is

$$Mx^{k+1} = Nx^k + b, \ k = 0, 1 \cdots \tag{2}$$

where $A = M - N$ and M is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c, \ k = 0, 1 \cdots$$

where $T = M^{-1}N$, $c = M^{-1}b$.

Assuming that A has unit diagonal entries and let $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular parts of A, respectively. Then,

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(i) the iteration matrix of the classical Gauss-Seidel-type method is given by

$$T = (I - L)^{-1}U \tag{3}$$

(ii) the iteration matrix of the classical SOR-type method is given by

$$L_r = (I - rL)^{-1}[(1 - r)I + rU] \tag{4}$$

where $r \neq 0$ is a real parameter called the relaxation parameter.

(iii) the iteration matrix of the classical AOR-type method is given by

$$L_{r,w} = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU] \tag{5}$$

where w and r are real parameters and $w \neq 0$.

Transform the original system (1) into the preconditioned form $PAx = Pb$.

Then, we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb, k = 0, 1 \cdots$$

where $PA = M_p - N_p$ and M_p is nonsingular. Thus the equation above can also be written as

$$x^{k+1} = Tx^k + c, \ \ k = 0, 1 \cdots$$

where $T = M_p^{-1}N_p$, $c = M_p^{-1}Pb$.

The preconditioned matrix of the following general form was introduced by [1]:

$$P = I + S = \begin{pmatrix} 1 & 0 & \cdots & -a_{1k_1} & 0 & 0 \\ 0 & 1 & \cdots & 0 & -a_{2k_2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -a_{nk_n} & \cdots & 0 & 1 \end{pmatrix}$$

where $i \neq k_i, \ k_i \in \{1, 2, \cdots, i-1, i+1, \cdots, n\}, \ i = 1, 2, \cdots, n$.

Assuming that

$$\hat{A} = PA = (I + S)A = \hat{D} - \hat{L} - \hat{U},$$

with

$$\hat{D} = I + D_1, \ \hat{L} = L + L_1, \ \hat{U} = U + U_1,$$

where D_1, $-L_1$ and $-U_1$ are diagonal, strictly lower and strictly upper triangular parts of SA, respectively.

We consider these splittings for $\hat{A}$:

$$\hat{A} = \begin{cases} (\hat{D} - \hat{L}) - \hat{U} \\ \dfrac{1}{r}(\hat{D} - r\hat{L}) - \dfrac{1}{r}[(1 - r)\hat{D} + r\hat{U}] \\ \dfrac{1}{w}(\hat{D} - r\hat{L}) - \dfrac{1}{w}[(1 - w)\hat{D} + (w - r)\hat{L} + w\hat{U}] \end{cases} \tag{6}$$

In view of (6), the iteration matrices associated with $\hat{A}$ are:

$$\hat{T} = (\hat{D} - \hat{L})^{-1}\hat{U} \tag{7}$$

$$\hat{L}_r = (\hat{D} - r\hat{L})^{-1}[(1-r)\hat{D} + r\hat{U}] \tag{8}$$

$$\hat{L}_{r,w} = (\hat{D} - r\hat{L})^{-1}[(1-w)\hat{D} + (w-r)\hat{L} + w\hat{U}] \tag{9}$$

In paper [2], Xue-Zhong Wang et al presented the preconditioned SOR-type iterative method with the preconditioned matrix P, and gave some comparison theorems. The main theorem in [2] is given as follows.

Theorem ([2]). Let L_r and $\hat{L}_r$ be the iteration matrices of the SOR-type methods given by (4) and (8). If A is a Z-matrix with $0 < a_{ik_i}a_{k_ii} < 1$, $(i = 1, 2\cdots, n)$, and $0 < r < 1$, then

$$(1)\ \rho(\hat{L_r}) > \rho(L_r),\ if\ \rho(L_r) > 1;$$
$$(2)\ \rho(\hat{L_r}) = \rho(L_r),\ if\ \rho(L_r) = 1;$$
$$(3)\ \rho(\hat{L_r}) < \rho(L_r),\ if\ \rho(L_r) < 1.$$

Now, we extend the preconditioned matrix, and consider a more general case:

$$(P')_{ij} = (I + S')_{ij} = \begin{cases} -a_{ij}, j \neq i; \\ \quad 1, j = i \end{cases} \tag{10}$$

where

$$S' = \begin{pmatrix} 0 & -a_{12} & \cdots & -a_{1,n-1} & -a_{1n} \\ -a_{21} & 0 & \cdots & -a_{2,n-1} & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{n,n-1} & 0 \end{pmatrix}.$$

In this paper, we present the preconditioned AOR-type iterative method for solving linear systems $Ax = b$, where A is a Z-matrix. We give some comparison theorems to show that the rate of convergence of the new preconditioned iterative method is faster than the rate of convergence of the previous preconditioned iterative method. Finally, we give one numerical example to show that our results are true.

Let $\bar{A} = P'A = (I + S')A = \bar{D} - \bar{L} - \bar{U}$, with

$$\bar{D} = I + \bar{D}_1, \bar{L} = L + \bar{L}_1, \bar{U} = U + \bar{U}_1,$$

where $\bar{D}_1$, $-\bar{L}_1$ and $-\bar{U}_1$ are diagonal, strictly lower and strictly upper triangular parts of $S'A$, respectively.

By (7)-(9), we can get:

$$\bar{T} = (\bar{D} - \bar{L})^{-1}\bar{U} \tag{11}$$

$$\bar{L}_r = (\bar{D} - r\bar{L})^{-1}[(1-r)\bar{D} + r\bar{U}] \tag{12}$$

$$\bar{L}_{r,w} = (\bar{D} - r\bar{L})^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}] \tag{13}$$

Remark 1: It is well known that for certain values of the parameters w and r, we obtain the successive over relaxation (SOR) and Gauss-Seidel methods, whose iteration matrices are denoted by L_r and T, respectively.

2.Comparison Results of the preconditioned AOR-type method with preconditioner P' and the classical AOR-type method

We need the following definitions and results.

Lemma 2.1 (Young [3]). A matrix A is a Z-matrix if $a_{ij} \leq 0$, $\forall i, j = 1, 2, \cdots, n$, $i \neq j$.

Lemma 2.2 (Young [3]). Let $A \geq 0$ be an irreducible matrix. Then
(1) A has a positive real eigenvalue equals to its spectral radius;
(2) To $\rho(A)$ there corresponds an eigenvector $x > 0$;
(3) $\rho(A)$ is a simple eigenvalue of A .

Lemma 2.3 (Varga [4]). Let A be a nonnegative matrix. Then
(1) If $\alpha x \leq Ax$ for some nonnegative vector x, $x \neq 0$, then $\alpha \leq \rho(A)$;
(2) If $Ax \leq \beta x$ for some positive vector x , then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Now we give the main results as follows.

Theorem 2.1. Let A be a Z-matrix with $0 < \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1$, $i = 1, 2, \cdots, n$, then $\bar{A}$ is a Z-matrix.

Proof. From the definition of $\bar{A}$, we can obtain:

$$\begin{cases} a_{ij} - \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} \leq 0, \ j \neq i; \\ 1 - \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} > 0, \ j = i, \end{cases}$$

where $i = 1, 2, \cdots, n$. Then, we complete the proof. □

Theorem 2.2. Let T, $\bar{T}$, L_r, $\bar{L}_r$, $L_{r,w}$, $\bar{L}_{r,w}$ be the iteration matrices of the methods given by (3)-(5) and (11)-(13). If A is a Z-matrix with $0 < \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1$, $i = 1, 2, \cdots, n$, then T, $\bar{T}$, L_r, $\bar{L}_r$, $L_{r,w}$, and $\bar{L}_{r,w}$ are non-negative and irreducible matrices.

Proof. By Theorem 2.1 in paper [2], A is an irreducible Z-matrix. And from $A = I - L - U$, we have $L + U$ is a nonnegative and irreducible matrix. Further, since

$$\bar{L}_{r,w} = [\bar{D} - r\bar{L}]^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}]$$

and $\bar{L}, \bar{U} \geq 0$, we have

$$\begin{aligned}\bar{L}_{r,w} &= [\bar{D} + r\bar{L} + (r\bar{L})^2 + \cdots][(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}] \\ &= (1-w)\bar{D}^2 + (w-r)\bar{D}\bar{L} + w\bar{D}\bar{U} + r(1-w)\bar{D}\bar{L} + r(w-r)\bar{L}^2 + rw\bar{L}\bar{U} \\ &+ \ nonnegativeterms.\end{aligned}$$

From the definition of $\bar{A}$, we can have, $\bar{D} \geq 0, \bar{L} \geq 0, \bar{U} \geq 0$. Therefore, $\bar{L}_{r,w}$ is a nonnegative and irreducible matrix. Similarly, we can prove T, $\bar{T}$, L_r, $\bar{L}_r$, and $L_{r,w}$ are nonnegative and irreducible matrices.

Some results for the AOR-type, SOR-type and Gauss-Seidel-type methods are given below:

Theorem 2.3. Let $L_{r,w}$ and $\bar{L}_{r,w}$ be the iteration matrices of the AOR-type methods given by (5) and (13). If A is a Z-matrix with $0 < \sum_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1$, $i = 1, 2, \cdots, n, 0 \leq r \leq w \leq 1, w \neq 0$, then

$$\begin{aligned}&(1)\ \rho(\bar{L}_{r,w}) > \rho(L_{r,w}),\ if\ \rho(L_{r,w}) > 1; \\ &(2)\ \rho(\bar{L}_{r,w}) = \rho(L_{r,w}),\ if\ \rho(L_{r,w}) = 1; \\ &(3)\ \rho(\bar{L}_{r,w}) < \rho(L_{r,w}),\ if\ \rho(L_{r,w}) < 1.\end{aligned}$$

Proof. From Theorem 2.1, it is clear that $L_{r,w}$ and $\bar{L}_{r,w}$ are nonnegative and irreducible matrices. Thus, from Lemma 2.2, we know that there exists a positive vector $x = (x_1, x_2, \cdots, x_n)^T$ such that

$$L_{r,w}x = \lambda x,$$

where $\lambda = \rho(L_{r,w})$, or, equivalently,

$$[(1-w)I + (w-r)L + wU] = \lambda(I - rL)x.$$

Now we consider

$$
\begin{aligned}
&\bar{L}_{r,w}x - \lambda x \\
&= [\bar{D} - r\bar{L}]^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}]x - \lambda x \\
&= [\bar{D} - r\bar{L}]^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U} - \lambda(\bar{D} - r\bar{L})]x \\
&= [\bar{D} - r\bar{L}]^{-1}[\bar{D} - w\bar{D} + w\bar{L} - r\bar{L} + w\bar{U_1} + wU - \lambda\bar{D} + \lambda r\bar{L}]x \\
&= [\bar{D} - r\bar{L}]^{-1}[w(-\bar{D} + \bar{L} + \bar{U_1} + U) - r\bar{L} + (1-\lambda)\bar{D} + \lambda r\bar{L}]x \\
&= [\bar{D} - r\bar{L}]^{-1}[w(\bar{U} - \bar{D} + \bar{L}) - r(1-\lambda)\bar{L} + (1-\lambda)\bar{D}]x \\
&= [\bar{D} - r\bar{L}]^{-1}[w(\bar{U_1} - \bar{D_1} + \bar{L_1}) + r(\lambda-1)\bar{L} + (1-\lambda)\bar{D_1}]x \\
&= [\bar{D} - r\bar{L}]^{-1}[S^{'}[(1-w)I + (w-r)L + wU] \\
&\ + S^{'}(1 - wL + wL - rL) + r(\lambda-1)\bar{L_1} - (\lambda-1)\bar{D_1}]x \\
&= [\bar{D} - r\bar{L}]^{-1}[\lambda S^{'}(I - rL) - S^{'}(I - rL) + r(\lambda-1)\bar{L_1} - (\lambda-1)\bar{D_1}]x \\
&= [\bar{D} - r\bar{L}]^{-1}[(\lambda-1)S^{'}\frac{(1-w)I + (w-r)L + wU}{\lambda} \\
&+ r(\lambda-1)\bar{L_1} - (\lambda-1)\bar{D_1}]x \\
&= \frac{\lambda-1}{\lambda}[\bar{D} - r\bar{L}]^{-1}[(1-w)S^{'} + (w-r)S^{'}L + wS^{'}U + \lambda r\bar{L_1} - \lambda\bar{D_1}]x.
\end{aligned}
\tag{14}
$$

Let

$$y = [\bar{D} - r\bar{L}]^{-1}[(1-w)S^{'} + (w-r)S^{'}L + wS^{'}U + \lambda r\bar{L_1} - \lambda\bar{D_1}]x,$$

then $y > 0$.

(1) If $\lambda > 1$, then $\bar{L}_{r,w}x - \lambda x \geq 0$, but not equal to 0. Therefore $\bar{L}_{r,w}x \geq \lambda x$. By Lemma 2.3, we get $\rho(\bar{L}_{r,w}) > \lambda = \rho(L_{r,w})$.

(2) If $\lambda = 1$, then $\bar{L}_{r,w}x - \lambda x = 0$, but not equal to 0. Therefore $\bar{L}_{r,w}x = \lambda x$. By Lemma 2.3, we get $\rho(\bar{L}_{r,w}) = \lambda = \rho(L_{r,w})$.

(3) If $\lambda < 1$, then $\bar{L}_{r,w}x - \lambda x < 0$, but not equal to 0. Therefore $\bar{L}_{r,w}x \leq \lambda x$. By Lemma 2.3, we get $\rho(\bar{L}_{r,w}) < \lambda = \rho(L_{r,w})$.

Similar to the proof of Theorem 2.3, we can have the following corollaries:

Corollary 2.1. Let L_r and $\bar{L}_r$ be the iteration matrices of SOR-type methods given by (4) and (12). If A is a Z-matrix with $0 < \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1,\ i = 1, 2\cdots n,\ 0 \leq r \leq 1$, then

$$
\begin{aligned}
&(1)\ \rho(\bar{L}_r) > \rho(L_r),\ if\ \rho(L_r) > 1; \\
&(2)\ \rho(\bar{L}_r) = \rho(L_r),\ if\ \rho(L_r) = 1; \\
&(3)\ \rho(\bar{L}_r) < \rho(L_r),\ if\ \rho(L_r) < 1.
\end{aligned}
$$

Corollary 2.2. Let T and $\bar{T}$ be the iteration matrices of Gauss-Seidel-type methods given by (3) and (11). If A is a Z-matrix with$0 < \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1$,

$i = 1, 2 \cdots n$, then

$$(1)\ \rho(\bar{T}) > \rho(T),\ if\ \rho(T) > 1;$$
$$(2)\ \rho(\bar{T}) = \rho(T),\ if\ \rho(T) = 1;$$
$$(3)\ \rho(\bar{T}) < \rho(T),\ if\ \rho(T) < 1.$$

3. Comparison results of preconditioned AOR-type methods with preconditioners P' and P

Now, we give some comparison results of preconditioned AOR-type methods with preconditioners P' and P.

Theorem 3.1. Let $\bar{L}_{r,w}$ and $\hat{L}_{r,w}$ be the iteration matrices of the preconditioned AOR-type methods with the preconditioned matrix P' and the preconditioned AOR-type methods with the preconditioned matrix P, respectively. If A is a Z-matrix with $0 < \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1,\ i = 1, 2 \cdots n,\ 0 \leq r \leq w \leq 1,\ w \neq 0$, then

$$(1)\ \rho(\bar{L}_{r,w}) \geq \rho(\hat{L}_{r,w}),\ if\ \rho(L_{r,w}) \geq 1;$$
$$(2)\ \rho(\bar{L}_{r,w}) < \rho(\hat{L}_{r,w}),\ if\ \rho(L_{r,w}) < 1.$$

Proof. First, from the definition of $\bar{A}$ and $\hat{A}$, we have $\bar{D} - r\bar{L} \leq \hat{D} - r\hat{L}$. Then, since $\bar{D} - r\bar{L}$ and $\hat{D} - r\hat{L}$ are two lower triangular L-matrices with $\bar{D} - r\bar{L} \leq \hat{D} - r\hat{L}$, we can obtain that $(\bar{D} - r\bar{L})^{-1} \geq (\hat{D} - r\hat{L})^{-1}$. Now consider

$$\bar{L}_{r,w}x - \hat{L}_{r,w}x = (\bar{L}_{r,w}x - \lambda x) - (\hat{L}_{r,w}x - \lambda x)$$

In view of (14), we know that $\bar{L}_{r,w}x - \hat{L}_{r,w}x$ equals

$$\begin{aligned} &\frac{\lambda - 1}{\lambda}(\bar{D} - r\bar{L})^{-1}[(1 - w)S'L + (w - r)S'L + wS'U + \lambda r\bar{L}_1 - \lambda\bar{D}_1]x \\ &- \frac{\lambda - 1}{\lambda}(\hat{D} - r\hat{L})^{-1}[(1 - w)SL + (w - r)SL + wSU + \lambda r\bar{L}_1 - \lambda\bar{D}_1]x \\ &\geq \frac{\lambda - 1}{\lambda}(\hat{D} - r\hat{L})^{-1}[(1 - w)K + (w - r)KL + wKU]x, \end{aligned} \tag{15}$$

where

$$(K)_{ij} = \begin{cases} -a_{ij} > 0,\ j \neq i\, and\, j \neq k_i; \\ 0, \qquad otherwise. \end{cases}$$

Obviously,

$$(\hat{D} - r\hat{L})^{-1}[(1 - w)K + (w - r)KL + wKU] \geq 0.$$

Therefore,

(1) if $\lambda \geq 1$, the right hand side of the inequality (15) is greater than zero, and we can get $\rho(\bar{L}_{r,w}) \geq \rho(\hat{L}_{r,w})$ by Lemma 2.3.

(2) if $\lambda < 1$, the right hand side of the inequality (15) is less than zero, and we can get $\rho(\bar{L}_{r,w}) < \rho(\hat{L}_{r,w})$ by Lemma 2.3. □

Corollary 3.1. Let $\bar{L}_r$ and $\hat{L}_r$ be the iteration matrices of the preconditioned SOR-type methods with the preconditioned matrices P' and P, respectively . If A is a Z-matrix with $0 < \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1$, $i = 1, 2 \cdots n, 0 < r \leq 1$, then

$$(1)\ \rho(\bar{L}_r) \geq \rho(\hat{L}_r),\ if\ \rho(L_r) \geq 1;$$
$$(2)\ \rho(\bar{L}_r) < \rho(\hat{L}_r),\ if\ \rho(L_r) < 1.$$

Corollary 3.2. Let $\bar{T}$ and $\hat{T}$ be the iteration matrices of the preconditioned Gauss-Seidel-type methods with the preconditioned matrices P' and P , respectively. If A is a Z-matrix with $0 < \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1$, $i = 1, 2 \cdots n$, then

$$(1)\ \rho(\bar{T}) \geq \rho(\hat{T}),\ if\ \rho(T) \geq 1;$$
$$(2)\ \rho(\bar{T}) < \rho(\hat{T}),\ if\ \rho(T) < 1.$$

Similar to the proof of Theorem 3.1, we can get the following two theorems.

Theorem 3.2. Let $L_{r,w}$and $\bar{L}_{r,w}$ be the iteration matrices of the classical AOR-type methods and the preconditioned AOR-type methods with the preconditioned matrix P' , respectively. If A is a Z-matrix with $0 < \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1$, $i = 1, 2 \cdots n$, and $0 \leq r_2 < r_1 \leq w \leq 1$, then

$$(1)\ \rho(\bar{L}_{r_1,w}) \geq \rho(\bar{L}_{r_2,w}),\ if\ \rho(L_{r,w}) \geq 1;$$
$$(2)\ \rho(\bar{L}_{r_1,w}) < \rho(\bar{L}_{r_2,w}),\ if\ \rho(L_{r,w}) < 1.$$

Theorem 3.3. Let $L_{r,w}$and $\hat{L}_{r,w}$ be the iteration matrices of the classical AOR-type methods and the preconditioned AOR-type methods with the preconditioned matrix P , respectively. If A is a Z-matrix with $0 < \sum\limits_{k=1,k\neq i}^{n} a_{ik}a_{ki} < 1$, $i = 1, 2 \cdots n$, and $0 \leq r_2 < r_1 \leq w \leq 1$, then

$$(1)\ \rho(\hat{L}_{r_1,w}) \geq \rho(\hat{L}_{r_2,w}),\ if\ \rho(L_{r,w}) \geq 1;$$
$$(2)\ \rho(\hat{L}_{r_1,w}) < \rho(\hat{L}_{r_2,w}),\ if\ \rho(L_{r,w}) < 1.$$

4. Example

We consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & -0.1 & -0.1 & 0 & -0.2 & -0.4 \\ -0.3 & 1 & -0.2 & 0 & -0.3 & -0.2 \\ 0 & -0.2 & 1 & -0.5 & -0.1 & 0 \\ -0.1 & -0.3 & -0.1 & 1 & -0.2 & -0.1 \\ -0.2 & -0.3 & -0.2 & -0.1 & 1 & -0.1 \\ -0.3 & -0.1 & -0.1 & -0.2 & -0.1 & 1 \end{pmatrix}$$

We choose $w = 0.6$, then we can obtain the following results by Theorem 2.3, Theorem 3.1-3.3:

Table 1. Spectral radius for different r of AOR-type methods

r	$\rho(L_{r,w})$	$\rho(\hat{L}_{r,w})$	$\rho(\bar{L}_{r,w})$
0	0.9131	0.8730	0.8087
0.0500	0.9112	0.8704	0.8053
0.1000	0.9091	0.8676	0.8019
0.1500	0.9070	0.8647	0.7982
0.2000	0.9047	0.8616	0.7945
0.2500	0.9024	0.8584	0.7905
0.3000	0.8999	0.8550	0.7863
0.3500	0.8972	0.8514	0.7820
0.4000	0.8944	0.8476	0.7773
0.4500	0.8915	0.8436	0.7725
0.5000	0.8883	0.8393	0.7673
0.5500	0.8849	0.8348	0.7618
0.6000	0.8813	0.8299	0.7560

From Table 1 , we can conclude that the rate of convergence of the preconditioned AOR-type method is faster than the rate of convergence of the classical AOR-type iterative method.

Remark 2: From the example, we can easily obtain that the preconditioned AOR-type method with preconditioner P' is better than the method with preconditioner P . If we apply it to the SOR-type method, we can get that the preconditioned SOR-type method with preconditioner P' is better than the method with preconditioner P, which is proposed by paper [2]. If we apply it to the Gauss-Seidel-type method, we can also get that the preconditioned Gauss-Seidel-type method with preconditioner P' is better than the method with preconditioner P.

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Guangbin Wang received his Ph.D from Shanghai University, China in 2004. Since 2004 he has been at Qingdao University of Science and Technology, China. His research interests include numerical linear algebra, matrix theory.

Department of Mathematics, Qingdao University of Science and Technology, Qingdao, 266061, China.

e-mail: wguangbin750828@sina.com

Ning Zhang received his BS from Qingdao University of Science and Technology, China in 2006. Since 2007 he has started his MS work at Qingdao University of Science and Technology, China. His research interests include numerical linear algebra, matrix theory.

Department of Mathematics, Qingdao University of Science and Technology, Qingdao, 266061, China.
e-mail: zhangningqust@163.com

Fuping Tan received his MS from Shanghai University, China in 1997. Since 1997 he has been at Shanghai University, China. His research interests include numerical linear algebra, matrix theory.

Department of Mathematics, Shanghai University, Shanghai, 200444, China.
e-mail: fptan@shu.edu.cn