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MULTIOBJECTIVE SECOND-ORDER NONDIFFERENTIABLE SYMMETRIC DUALITY INVOLVING (F, α, ρ, d)-CONVEX FUNCTIONS

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Abstract. In this paper, a pair of Wolfe type second-order nondifferentiable multiobjective symmetric dual program over arbitrary cones is formulated. Weak, strong and converse duality theorems are established under second-order (F, α, ρ, d)-convexity assumptions. An illustration is given to show that second-order (F, α, ρ, d)-convex functions are generalization of second-order F-convex functions. Several known results including many recent works are obtained as special cases.

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1. Introduction

A mathematical programming problem with single objective is called scalar programming problem. However, just considering one criterion at a time usually does not represent real life problems well because in most of the cases, two or more objectives are associated with a problem. Such a mathematical optimization model with two or more objectives is called a multiobjective programming problem. Often several objectives are conflicting in nature. For example, it may be impossible to select an alternative to a problem which would maximize both profit and market share for a company. Therefore, various terms like efficiency, weak efficiency and proper efficiency are there in the literature to describe the optimality of multiobjective problems.

Convexity plays an important role in nonlinear programming. Many mathematical models used in economics, management sciences, applied mathematics

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and engineering involve non convex functions. So, it is possible to generalize the notion of convexity and to explore the extent of the validity of results to larger class of multiobjective optimization problems. Duality has been extended to multiobjective optimization since late 1970's. The concept of (F, ρ)-convexity was introduced by Preda [7]. Ahmad [1] established optimality conditions and duality results for multiobjective programming problems involving (F, ρ)-convexity assumptions.

The study of second-order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used. Motivated by the concept of second-order duality in nonlinear problems, introduced by Mangasarian [6], several researchers [2-4, 9-14] have worked in this field. Recently, Yang et al. [12] presented a new pair of Mond-Weir type second-order symmetric models for a class of non-differentiable multiobjective programs.

Zhang and Mond [14] extended the class of (F, ρ)-convex functions to second-order (F, ρ)-convex functions and obtained duality results for second-order Mangasarian type, Mond-Weir type and generalized Mond-Weir type multiobjective dual problems. Ahmad and Husain [3] introduced second-order (F, α, ρ, d)-convex functions and their generalizations and developed weak, strong and strict converse duality theorems for second-order Mond-Weir type multiobjective dual.

The purpose of this paper is to study a pair of nondifferentiable multiobjective second-order symmetric dual programs of Wolfe type model. Weak, strong and converse duality theorems are proved under second-order (F, α, ρ, d)-convexity assumptions. An example which is second-order (F, α, ρ, d)-convex but not second-order F-convex is exemplified. Our study extends some of the known results in [2, 4, 11, 13].

2. Notations and preliminaries

Consider the following multiobjective programming problem:

$$\begin{aligned} (\mathbf{P}) \quad & \text{minimize} \quad \phi(x) \\ & \text{subject to} \quad -g(x) \in Q, \; x \in C, \end{aligned}$$

where $C \subseteq R^n$, $\phi : R^n \to R^k$, $g : R^n \to R^m$, Q is closed convex cone with non-empty interior in R^m. Let $X^0 = \{x \in C : -g(x) \in Q\}$, be the set of all feasible solutions of (P).

Definition 1. *A point $\bar{x} \in X^0$ is a weak efficient solution of (P) if there exist no other $x \in X^0$ such that $\phi(x) < \phi(\bar{x})$.*

Definition 2. *A point $\bar{x} \in X^0$ is an efficient solution of (P) if there exist no other $x \in X^0$ such that $\phi(x) \leq \phi(\bar{x})$.*

Let C_1 and C_2 be closed convex cones with non-empty interiors in R^n and R^m, respectively.

Definition 3. *The positive polar cone* C_i^* *of* C_i $(i = 1, 2)$ *is defined as*

$$C_i^* = \{z : x^T z \geq 0, \text{ for all } x \in C_i\}.$$

Definition 4. *Let* C *be a compact convex set in* R^n*. The support function* $S(x \mid C)$ *of* C *is defined by*

$$S(x \mid C) = \max\{x^T y : y \in C\}.$$

The support function $S(x \mid C)$, being convex and everywhere finite, has a subdifferential at every x in the sense of Rockafellar, that is, there exists $z \in R^n$ such that

$$S(y \mid C) \geq S(x \mid C) + z^T(y - x) \text{ for all } y \in C.$$

The subdifferential of $S(x \mid C)$ is given by

$$\partial S(x \mid C) = \{z \in C : z^T x = S(x \mid C)\}.$$

For any set $S \subset R^n$ the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leq 0 \text{ for all } z \in S\}.$$

It can be easily seen that for a compact convex set C, y is in $N_C(x)$ if and only if $S(y \mid C) = x^T y$, or equivalently, x is in $\partial S(y \mid C)$.

Definition 5 ([11, 12])**.** *A functional* $F : X \times X \times R^n \mapsto R$ *(where* $X \subseteq R^n$*) is sublinear with respect to the third variable if for all* $(x, u) \in X \times X$,

(*i*) $F(x, u; a_1 + a_2) \leq F(x, u; a_1) + F(x, u; a_2)$ *for all* $a_1, a_2 \in R^n$,
(*ii*) $F(x, u; \alpha a) = \alpha F(x, u; a)$, *for all* $\alpha \in R_+$ *and for all* $a \in R^n$.

For notational convenience, we write $F_{x,u}(a) = F(x, u; a)$.

Now we consider a function $f = (f_1, f_2, \ldots, f_k) : X \mapsto R^k$ differentiable at $u \in X$, $\rho = (\rho_1, \rho_2, \ldots, \rho_k) \in R^k$ and $d = (d_1, d_2, \ldots, d_k) \in R^k$.

Definition 6. *A twice differentiable function* f_i *over* X *is said to be second-order* F*-convex at* $u \in X$ *with respect to* $q \in R^n$, *if* $\forall x \in X$,

$$f_i(x) - f_i(u) + \frac{1}{2} q^T \nabla_{xx} f_i(u) q \geq F_{x,u}(\nabla_x f_i(u) + \nabla_{xx} f_i(u) q).$$

A twice differentiable vector function $f : X \to R^k$ *is said to be second-order* F*-convex at* $u \in X$ *with respect to* $q \in R^n$, *if each of its components* f_i *is second-order* F*-convex at* $u \in X$ *with respect to* $q \in R^n$.

Definition 7. *A twice differentiable function* f_i *over* X *is said to be second-order* (F, α, ρ_i, d_i)*-convex at* u *on* X, *if* $\forall x \in X$, *there exist vector* $q \in R^n$, *a real valued function* $\alpha : X \times X \to R_+ \backslash \{0\}$, *a real valued function* $d_i(\cdot, \cdot) : X \times X \to R$ *and a real number* ρ_i *such that*

$$f_i(x) - f_i(u) + \frac{1}{2} q^T \nabla_{xx} f_i(u) q \geq F_{x,u}[\alpha(x, u)(\nabla_x f_i(u) + \nabla_{xx} f_i(u) q)] + \rho_i d_i^2(x, u).$$

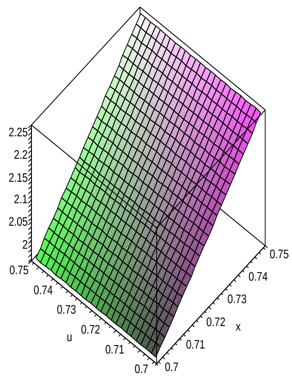

Fig. 1. $f_1 = sin^2x - sin^2u - sin2u + 4x^2$

A twice differentiable vector function $f : X \mapsto R^k$ is said to be second-order (F, α, ρ, d)-convex at u if each of its components f_i is second-order (F, α, ρ_i, d_i)-convex at u.

Remark 1.

- (i) If $k = 1$ and $q = 0$, the above definition become that of (F, α, ρ, d)-convex functions introduced by Liang et al. [5].
- (ii) For single objective programming problem and $\alpha(x, u) = 1$, the definition of second-order (F, α, ρ_i, d_i)-convexity reduces to second-order (F, ρ)-convexity given by Srivastava and Bhatia [10].

Example 1. *Let $X = [0.7, 0.75] \subset R$. Let the function $f : X \to R$ be defined by $f(x) = \sin^2 x$ and $\alpha : X \times X \to R_+\backslash\{0\}$ be identified by $\alpha(x, u) = (x + u + 1)$. Let the functional $F : X \times X \times R \to R$ be defined by*

$$F_{x,u}(a) = \frac{a}{(x + u + 1)}$$

and $d : X \times X \to R$ be given by

$$d(x, u) = \sqrt{x^2 + u^2}.$$

For $\rho = -4$, we have

$$\begin{aligned}
L &= f(x) - f(u) + \frac{1}{2}q^T \nabla_{xx} f(u)q - F_{x,u}[\alpha(x, u)(\nabla_x f(u) + \nabla_{xx} f(u)q)] - \rho d^2(x, u) \\
&= \sin^2 x - \sin^2 u + q^2 \cos 2u - F_{x,u}[(x + u + 1)(\sin 2u + 2q \cos 2u)] - (-4)(\sqrt{x^2 + u^2})^2 \\
&= \sin^2 x - \sin^2 u - \sin 2u + 4x^2 + q^2 \cos 2u - 2q \cos 2u + 4u^2 \\
&= f_1 + f_2 \ (say)
\end{aligned}$$

where $f_1 = \sin^2 x - \sin^2 u - \sin 2u + 4x^2 \geq 0 \ \forall \ x, u \in X$ as can be seen from Figure 1, and $f_2 = q^2 \cos 2u - 2q \cos 2u + 4u^2 \geq 0 \ \forall \ u \in X$ and $q \in (-10^{18}, 10^{18})$ as can

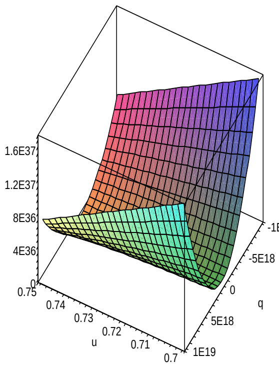

Fig. 2. $f_2 = q^2 cos2u - 2qcos2u + 4u^2$

be seen from Figure 2. Hence $L \geq 0$. *Therefore* f *is second-order* (F, α, ρ, d)-*convex. But* f *is not second-order* F-*convex since for* $q = 1$, *we have*

$$\begin{aligned} M &= f(x) - f(u) + \frac{1}{2} q^T \nabla_{xx} f(u) q - F_{x,u}[\nabla_x f(u) + \nabla_{xx} f(u) q] \\ &= \sin^2 x - \sin^2 u + \cos 2u - F_{x,u}[\sin 2u + 2\cos 2u] \\ &= \sin^2 x - \sin^2 u + \cos 2u - \frac{1}{(x+u+1)}(\sin 2u + 2\cos 2u) \\ &< 0 \ \forall \ x, u \in X \end{aligned}$$

as be can seen from Figure 3.

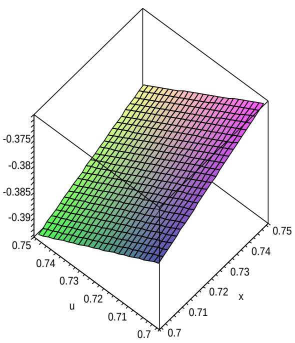

Fig. 3. $sin^2 x - sin^2 u + cos2u - \frac{1}{(x+u+1)}(sin2u + 2cos2u)$

Hence the function f *is second-order* (F, α, ρ, d)-*convex but is not second-order* F-*convex.*

3. Wolfe type symmetric duality

We now consider the following pair of second-order Wolfe type nondifferentiable multiobjective programming problems with k-objectives:

Primal problem (PP).

$$\begin{aligned}\text{minimize}\quad & G(x,y,\lambda,p)=\{G_1(x,y,\lambda,p),G_2(x,y,\lambda,p),\cdots,G_k(x,y,\lambda,p)\}\\ \text{subject to}\quad & -\nabla_y(\lambda^T f)(x,y)+z-\nabla_{yy}(\lambda^T f)(x,y)p\in C_2^*,\\ & z\in E,\\ & \lambda^T e_k=1,\\ & \lambda>0,\ x\in C_1.\end{aligned}$$

Dual problem (DP).

$$\begin{aligned}\text{maximize}\ & H(u,v,\lambda,q)=\{H_1(u,v,\lambda,q),H_2(u,v,\lambda,q),\cdots,H_k(u,v,\lambda,q)\}\\ \text{subject to}\ & \nabla_x(\lambda^T f)(u,v)+w+\nabla_{xx}(\lambda^T f)(u,v)q\in C_1^*,\\ & w\in D,\\ & \lambda^T e_k=1,\\ & \lambda>0,\ v\in C_2\ ,\end{aligned}$$

where for $i=1,2,\cdots,k$,

$$\begin{aligned}G_i(x,y,\lambda,p)&=f_i(x,y)+S(x\mid D)-y^T\nabla_y(\lambda^T f)(x,y)\\ &\quad -y^T\left(\nabla_{yy}(\lambda^T f)(x,y)\right)-\frac{1}{2}p^T\left(\nabla_{yy}(\lambda^T f)(x,y)p\right),\\ H_i(u,v,\lambda,q)&=f_i(u,v)-S(v\mid E)-u^T\nabla_x(\lambda^T f)(u,v)\\ &\quad -u^T\left(\nabla_{xx}(\lambda^T f)(u,v)q\right)-\frac{1}{2}q^T\left(\nabla_{xx}(\lambda^T f)(u,v)q\right).\end{aligned}$$

The above problem (PP) and (DP) can be further expressed as:

Primal problem (PP).

$$\begin{aligned}\text{Minimize}\quad G(x,y,\lambda,p)&=f(x,y)+S(x\mid D)e_k-y^T\nabla_y(\lambda^T f)(x,y)e_k\\ &\quad -y^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k-\frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k,\end{aligned}$$

$$\text{subject to}\quad -\nabla_y(\lambda^T f)(x,y)+z-\nabla_{yy}(\lambda^T f)(x,y)p\in C_2^*, \tag{1}$$

$$z\in E, \tag{2}$$

$$\lambda^T e_k=1, \tag{3}$$

$$\lambda>0,\ x\in C_1. \tag{4}$$

Dual problem (DP).

$$\begin{aligned}\text{Maximize}\quad H(u,v,\lambda,q)&=f(u,v)-S(v\mid E)e_k-u^T\nabla_x(\lambda^T f)(u,v)e_k\\ &\quad -u^T(\nabla_{xx}(\lambda^T f)(u,v)q)e_k-\frac{1}{2}q^T(\nabla_{xx}(\lambda^T f)(u,v)q)e_k,\end{aligned}$$

$$\text{subject to} \quad \nabla_x(\lambda^T f)(u,v) + w + \nabla_{xx}(\lambda^T f)(u,v)q \in C_1^*, \tag{5}$$

$$w \in D, \tag{6}$$

$$\lambda^T e_k = 1, \tag{7}$$

$$\lambda > 0,\ v \in C_2\ , \tag{8}$$

where

(i) $f = \{f_1, f_2, \cdots, f_k\}$ is a differentiable function from $S_1 \times S_2 \to R^k$ ($S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets), $e_k = (1, \ldots, 1)^T \in R^k$,

(ii) q and p are vectors in R^n and R^m, respectively, $\lambda \in R^k$,

(iii) D and E are compact convex sets in R^n and R^m, respectively.

We prove the following duality results for the pair of problems (PP) and (DP).

Theorem 1 (**Weak Duality).** *Let (x, y, λ, z, p) be feasible for the primal problem (PP) and (u, v, λ, w, q) be feasible for the dual problem (DP). Let the sublinear functionals $F : S_1 \times S_1 \times R^n \mapsto R$ and $G : S_2 \times S_2 \times R^m \mapsto R$ satisfy the following conditions:*

$$F_{x,u}(a) + \alpha_1^{-1} a^T u \geq 0,\ \textit{for all } a \in C_1^*, \tag{A}$$

$$G_{v,y}(b) + \alpha_2^{-1} b^T y \geq 0,\ \textit{for all } b \in C_2^*. \tag{B}$$

Suppose that either (i) $\sum_{i=1}^{k} \lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2 + \rho_i^{(2)}(d_i^{(2)}(v,y))^2] \geq 0$ *or*
(ii) $\rho_i^{(1)} \geq 0$ *and* $\rho_i^{(2)} \geq 0$*, for all i. Furthermore, assume that*
$f_i(.,v) + (.)^T w$ $(1 \leq i \leq k)$ *is second-order* $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$*-convex at u,*
$f_i(x,.) - (.)^T z$ $(1 \leq i \leq k)$ *be second-order* $(G, \alpha_2, \rho_i^{(2)}, d_i^{(2)})$*-concave at y. Then*

$$G(x, y, \lambda, p) \nleq H(u, v, \lambda, q). \tag{9}$$

Proof. Since $f_i(.,v) + (.)^T w$ $(1 \leq i \leq k)$ is second-order $(F, \alpha_1, \rho_i^{(1)}, d_i^{(1)})$-convex, we have

$$\begin{aligned} &f_i(x,v) + x^T w - f_i(u,v) - u^T w + \frac{1}{2} q^T(\nabla_{xx} f_i(u,v)q) \\ &\geq F_{x,u}[\alpha_1(x,u)(\nabla_x f_i(u,v) + w + \nabla_{xx} f_i(u,v)q)] + \rho_i^{(1)}(d_i^{(1)}(x,u))^2. \end{aligned}$$

It follows from $\lambda > 0$, $\lambda^T e_k = 1$ and sublinearity of F that

$$\begin{aligned} &(\lambda^T f)(x,v) + x^T w - (\lambda^T f)(u,v) - u^T w + \frac{1}{2} q^T(\nabla_{xx}(\lambda^T f)(u,v)q) \\ &\geq F_{x,u}[\alpha_1(x,u)(\nabla_x(\lambda^T f)(u,v) + w + \nabla_{xx}(\lambda^T f)(u,v)q)] \\ &\quad + \sum_{i=1}^{k} \lambda_i \rho_i^{(1)}(d_i^{(1)}(x,u))^2. \end{aligned} \tag{10}$$

As $f_i(x,.)-(.)^T z$ $(1 \leq i \leq k)$ is second-order $(G, \alpha_2, \rho_i^{(2)}, d_i^{(2)})$-concave therefore we get

$$f_i(x,y) - y^T z - f_i(x,v) + v^T z - \frac{1}{2}p^T(\nabla_{yy} f_i(x,y)p)$$
$$\geq G_{v,y}\left[-\alpha_2(v,y)(\nabla_y f_i(x,y) - z + \nabla_{yy} f_i(x,y)p)\right] + \rho_i^{(2)}(d_i^{(2)}(v,y))^2.$$

Using $\lambda > 0$, $\lambda^T e_k = 1$ and sublinearity of G it follows that

$$\begin{aligned} &(\lambda^T f)(x,y) - y^T z - (\lambda^T f)(x,v) + v^T z - \frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p) \\ &\geq G_{v,y}[-\alpha_2(v,y)(\nabla_y(\lambda^T f)(x,y) - z + \nabla_{yy}(\lambda^T f)(x,y)p)] \\ &\quad + \sum_{i=1}^{k} \lambda_i \rho_i^{(2)}(d_i^{(2)}(v,y))^2. \end{aligned} \tag{11}$$

Adding the inequalities (10) and (11), we obtain

$$\begin{aligned} &(\lambda^T f)(x,y) - (\lambda^T f)(u,v) + x^T w - u^T w - y^T z + v^T z \\ &\quad + \frac{1}{2}q^T(\nabla_{xx}(\lambda^T f)(u,v)q) - \frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p) \\ &\geq F_{x,u}[\alpha_1(x,u)(\nabla_x(\lambda^T f)(u,v) + w + \nabla_{xx}(\lambda^T f)(u,v)q)] \\ &\quad + G_{v,y}[-\alpha_2(v,y)(\nabla_y(\lambda^T f)(x,y) - z + \nabla_{yy}(\lambda^T f)(x,y)p)] \\ &\quad + \sum_{i=1}^{k} \lambda_i[\rho_i^{(1)}(d_i^{(1)}(x,u))^2 + \rho_i^{(2)}(d_i^{(2)}(v,y))^2]. \end{aligned} \tag{12}$$

Since (x,y,λ,z,p) is feasible for the primal problem (PP) and (u,v,λ,w,q) is feasible for the dual problem (DP), $\alpha_1(x,u) > 0$, by the dual constraint (5), the vector $a = \alpha_1(x,u)(\nabla_x(\lambda^T f)(u,v) + w + \nabla_{xx}(\lambda^T f)(u,v)q) \in C_1^*$ and so from the hypothesis (A), we obtain

$$F_{x,u}(a) + \alpha_1^{-1} a^T u \geq 0. \tag{13}$$

Similarly,

$$G_{v,y}(b) + \alpha_2^{-1} b^T y \geq 0, \tag{14}$$

for the vector $b = -\alpha_2(v,y)\{\nabla_y(\lambda^T f)(x,y) - z + \nabla_{yy}(\lambda^T f)(x,y)p\} \in C_2^*$. Using (13), (14) and hypothesis (i) or (ii) in (12), we have

$$\begin{aligned} &(\lambda^T f)(x,y) - (\lambda^T f)(u,v) + x^T w - u^T w - y^T z + v^T z \\ &\quad + \frac{1}{2}q^T(\nabla_{xx}(\lambda^T f)(u,v)q) - \frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p) \\ &\geq -\alpha_1^{-1} u^T a - \alpha_2^{-1} y^T b. \end{aligned}$$

Substituting the values of a and b, we have

$$(\lambda^T f)(x,y) + x^T w - y^T \nabla_y(\lambda^T f)(x,y) - y^T(\nabla_{yy}(\lambda^T f)(x,y)p) - \frac{1}{2}p^T(\nabla_{yy}(\lambda^T f)(x,y)p)$$
$$\geq (\lambda^T f)(u,v) - v^T z - u^T \nabla_x(\lambda^T f)(u,v) - u^T(\nabla_{xx}(\lambda^T f)(u,v)q) - \frac{1}{2}q^T(\nabla_{xx}(\lambda^T f)(u,v)q).$$

In view of the fact that $x^T w \leq S(x \mid D)$, $v^T z \leq S(v \mid E)$ and $\lambda^T e_k = 1$, the last inequality yields

$$\begin{aligned}
&(\lambda^T f)(x,y) + S(x \mid D) - y^T \nabla_y (\lambda^T f)(x,y) \\
&\quad - y^T(\nabla_{yy}(\lambda^T f)(x,y)p) - \frac{1}{2} p^T(\nabla_{yy}(\lambda^T f)(x,y)p) \\
&\geq (\lambda^T f)(u,v) - S(v \mid E) - u^T \nabla_x (\lambda^T f)(u,v) \\
&\quad - u^T(\nabla_{xx}(\lambda^T f)(u,v)q) - \frac{1}{2} q^T(\nabla_{xx}(\lambda^T f)(u,v)q).
\end{aligned} \tag{15}$$

Now suppose contrary to the result that (9) not holds, that is ,

$$\begin{aligned}
&\{G_1(x,y,\lambda,p), G_2(x,y,\lambda,p), \ldots, G_k(x,y,\lambda,p)\} \\
&\leq \{H_1(u,v,\lambda,q), H_2(u,v,\lambda,q), \ldots, H_k(u,v,\lambda,q)\}
\end{aligned}$$

or

$$\begin{aligned}
&f(x,y) + S(x \mid D)e_k - y^T \nabla_y (\lambda^T f)(x,y) e_k \\
&\quad - y^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k - \frac{1}{2} p^T(\nabla_{yy}(\lambda^T f)(x,y)p)e_k \\
&\leq f(u,v) - S(v \mid E)e_k - u^T \nabla_x (\lambda^T f)(u,v) e_k \\
&\quad - u^T(\nabla_{xx}(\lambda^T f)(u,v)q)e_k - \frac{1}{2} q^T(\nabla_{xx}(\lambda^T f)(u,v)q)e_k.
\end{aligned}$$

Since $\lambda > 0$ and $\lambda^T e_k = 1$, we obtain

$$\begin{aligned}
&(\lambda^T f)(x,y) + S(x \mid D) - y^T \nabla_y (\lambda^T f)(x,y) \\
&\quad - y^T(\nabla_{yy}(\lambda^T f)(x,y)p) - \frac{1}{2} p^T(\nabla_{yy}(\lambda^T f)(x,y)p) \\
&< (\lambda^T f)(u,v) - S(v \mid E) - u^T \nabla_x (\lambda^T f)(u,v) \\
&\quad - u^T(\nabla_{xx}(\lambda^T f)(u,v)q) - \frac{1}{2} q^T(\nabla_{xx}(\lambda^T f)(u,v)q),
\end{aligned}$$

which contradicts (15). Hence (9) holds. □

We now state a weak duality theorem under second-order F-convexity assumptions. Its proof follows on the lines of Theorem 1 on taking

$$\alpha_1(x,u) = 1, \ \alpha_2(v,y) = 1,$$

and

$$\rho_1 = 0, \ \rho_2 = 0.$$

Theorem 2 (**Weak Duality**). *Let (x, y, λ, z, p) be feasible for the primal problem (PP) and (u, v, λ, w, q) be feasible for the dual problem (DP). Suppose that the sublinear functionals $F : S_1 \times S_1 \times R^n \mapsto R$ and $G : S_2 \times S_2 \times R^m \mapsto R$ satisfy the following conditions:*

$$F_{x,u}(a) + a^T u \geq 0, \ \textit{for all } a \in C_1^*, \tag{A}$$

$$G_{v,y}(b) + b^T y \geq 0, \ \textit{for all } b \in C_2^*. \tag{B}$$

Furthermore, assume that $f_i(.,v) + (.)^T w$ $(1 \leq i \leq k)$ is second-order F-convex at u and $f_i(x,.) - (.)^T z$ $(1 \leq i \leq k)$ is second-order G-concave at y. Then

$$G(x,y,\lambda,p) \nleq H(u,v,\lambda,q).$$

Theorem 3 (**Strong Duality**). *Let $f : S_1 \times S_2 \to R^k$ be thrice differentiable function and let $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ be a weak efficient solution of (PP). Suppose that*

- (i) *the matrix $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})$ is non singular,*
- (ii) *the vectors $\nabla_y f_1(\bar{x},\bar{y}),\ldots,\nabla_y f_k(\bar{x},\bar{y})$ are linearly independent,*
- (iii) *the vector $\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p})\bar{p} \notin span\{\nabla_y f_1(\bar{x},\bar{y}),\ldots,\nabla_y f_k(\bar{x},\bar{y})\}\backslash\{0\}$ and*
- (iv) *$\nabla_y(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p})\bar{p} = 0$ implies $\bar{p} = 0$, then*
- (I) *there exist $\bar{w} \in D$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q} = 0)$ is feasible for (DP), and*
- (II) *$G(\bar{x},\bar{y},\bar{\lambda},\bar{p}) = H(\bar{x},\bar{y},\bar{\lambda},\bar{q})$.*

Also, if the hypotheses of a weak duality theorem are satisfied for all feasible solutions of (PP) and (DP), then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q} = 0)$ is an efficient solution for (DP).

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is a weak efficient solution of (PP), there exist $\alpha \in R^k$, $\beta \in C_2$, $\eta \in R$, such that the following Fritz John optimality conditions ([8], Lemma 1) are satisfied at $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$:

$$\begin{aligned}&\Big\{\alpha^T(\nabla_x f(\bar{x},\bar{y}) + \gamma e_k) + \nabla_{xy}(\bar{\lambda}^T f)(\bar{x},\bar{y})(\beta - (\alpha^T e_k)\bar{y})\\ &+ \nabla_x[\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p}](\beta - (\alpha^T e_k)(\bar{y} + \frac{1}{2}\bar{p}))\Big\}(x - \bar{x})\\ &\geq 0, \text{ for all } x \in C_1,\end{aligned} \tag{16}$$

$$\begin{aligned}&\nabla_y f(\bar{x},\bar{y})[\alpha - (\alpha^T e_k)\bar{\lambda}] + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})(\beta - (\alpha^T e_k)\bar{y})\\ &+ \nabla_y[\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p}](\beta - (\alpha^T e_k)(\bar{y} + \frac{1}{2}\bar{p})) - (\alpha^T e_k)\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p}\\ &= 0,\end{aligned} \tag{17}$$

$$\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})(\beta - (\alpha^T e_k)(\bar{y} + \bar{p})) = 0, \tag{18}$$

$$\begin{aligned}&\nabla_y f(\bar{x},\bar{y})(\beta - (\alpha^T e_k)\bar{y}) + \eta e_k + [(\beta - (\alpha^T e_k)(\bar{y} + \frac{1}{2}\bar{p}))^T \nabla_{yy} f_1(\bar{x},\bar{y})\bar{p},\\ &\ldots,(\beta - (\alpha^T e_k)(\bar{y} + \frac{1}{2}\bar{p}))^T \nabla_{yy} f_k(\bar{x},\bar{y})\bar{p}] = 0,\end{aligned} \tag{19}$$

$$\beta^T[\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y}) - \bar{z} + \nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p}] = 0, \tag{20}$$

$$\beta \in N_E(\bar{z}), \tag{21}$$

$$\gamma \in D,\ \gamma^T \bar{x} = S(\bar{x} \mid D), \tag{22}$$

$$(\alpha,\beta,\eta) \neq 0. \tag{23}$$

From (18) and nonsingularity of $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})$, we have

$$\beta = (\alpha^T e_k)(\bar{y} + \bar{p}). \tag{24}$$

If $\alpha = 0$ then (24) yields $\beta = 0$. Further, equation (19) gives $\eta e_k = 0$ or $\eta = 0$. Consequently $(\alpha, \beta, \eta) = 0$, contradicting (23). Hence,

$$\alpha \geq 0 \quad \text{or} \quad \alpha^T e_k > 0. \tag{25}$$

Now, using (24) and (25) in (17), we get

$$\nabla_y [\nabla_{yy} (\bar{\lambda}^T f)(\bar{x}, \bar{y}) \bar{p}] \bar{p} = -\frac{2}{\alpha^T e_k} \nabla_y f(\bar{x}, \bar{y}) [\alpha - (\alpha^T e_k) \bar{\lambda}]. \tag{26}$$

which by hypothesis (iii) and (iv) implies

$$\bar{p} = 0. \tag{27}$$

Now, by (26) and (27), we obtain

$$\nabla_y f(\bar{x}, \bar{y}) [\alpha - (\alpha^T e_k) \bar{\lambda}] = 0.$$

Since the vectors $\{\nabla_y f_1(\bar{x}, \bar{y}), \ldots, \nabla_y f_k(\bar{x}, \bar{y})\}$ are linearly independent, therefore

$$\alpha = (\alpha^T e_k) \bar{\lambda}. \tag{28}$$

From (27) in (24), we get

$$\beta = (\alpha^T e_k) \bar{y}. \tag{29}$$

Using (25) and (27)-(29) in (16), we have

$$(\nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \gamma)(x - \bar{x}) \geq 0, \text{ for all } x \in C_1. \tag{30}$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$ and so (30) implies

$$x^T (\nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \gamma) \geq 0, \text{ for all } x \in C_1.$$

Therefore,

$$\nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \gamma \in C_1^*.$$

Also, from (25), (29) and $\beta \in C_2$, we obtain $\bar{y} \in C_2$. Thus $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w} = \gamma, \bar{q} = 0)$ satisfies the dual constraints from (5) to (8) in (DP) and so it is a feasible solution for the dual problem (DP). Now, letting $x = 0$ and $x = 2\bar{x}$ in (30), we get

$$\bar{x}^T (\nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \gamma) = 0 \text{ or } \bar{x}^T \nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) = -\bar{x}^T \gamma = -S(\bar{x} \mid D). \tag{31}$$

Moreover, since $\beta = (\alpha^T e_k) \bar{y}$ and $\alpha^T e_k > 0$, (21) implies $\bar{y} \in N_E(\bar{z})$ so that

$$\bar{y}^T \bar{z} = S(\bar{y} \mid E).$$

Further, from (20), (25), (27) and (29) and the above relation, we obtain

$$\bar{y}^T \nabla_y (\bar{\lambda}^T f)(\bar{x}, \bar{y}) = \bar{y}^T \bar{z} = S(\bar{y} \mid E). \tag{32}$$

Therefore, using (27), (31) and (32), we get

$$G(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0) = H(\bar{u}, \bar{v}, \bar{\lambda}, \bar{q} = 0)$$

that is, the two objective values are equal. Now, let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q} = 0)$ is not an efficient solution of (DP), then there exist $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{q} = 0)$ feasible for (DP), such that,

$$\begin{aligned}
&f(\bar{x}, \bar{y}) - S(\bar{y} \mid E)e_k - \frac{1}{2}\bar{q}^T(\nabla_{xx}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{q})e_k \\
&\quad - \bar{x}^T[\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y})e_k + (\nabla_{xx}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{q})e_k] \\
&\leq f(\bar{u}, \bar{v}) - S(\bar{v} \mid E)e_k - \frac{1}{2}\bar{q}^T(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{q})e_k \\
&\quad - \bar{u}^T[\nabla_x(\bar{\lambda}^T f)(\bar{u}, \bar{v})e_k + (\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{q})e_k].
\end{aligned}$$

Since $\bar{x}^T\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = -S(\bar{x} \mid D)$, $\bar{y}^T\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = S(\bar{y} \mid E)$ and $\bar{p} = 0$,

$$\begin{aligned}
&f(\bar{x}, \bar{y}) + S(\bar{x} \mid D)e_k - \frac{1}{2}\bar{p}^T(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})e_k \\
&\quad - \bar{y}^T[\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y})e_k + (\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p})e_k] \\
&\leq f(\bar{u}, \bar{v}) - S(\bar{v} \mid E)e_k - \frac{1}{2}\bar{q}^T(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{q})e_k \\
&\quad - \bar{u}^T[\nabla_x(\bar{\lambda}^T f)(\bar{u}, \bar{v})e_k + (\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{q})e_k],
\end{aligned}$$

which contradicts weak duality theorem. Hence $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q} = 0)$ is an efficient solution of (DP). □

Theorem 4 (Converse Duality). *Let $f : S_1 \times S_2 \to R^k$ be thrice differentiable function and let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{q})$ be a weak efficient solution of (DP). Suppose that*

(*i*) *the matrix $\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})$ is non singular,*
(*ii*) *the vectors $\nabla_x f_1(\bar{u}, \bar{v}), \ldots, \nabla_x f_k(\bar{u}, \bar{v})$ are linearly independent,*
(*iii*) *the vector $\nabla_x(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{q})\bar{q} \notin span\{\nabla_x f_1(\bar{u}, \bar{v}), \ldots, \nabla_x f_k(\bar{u}, \bar{v})\}\backslash\{0\}$ and*
(*iv*) *$\nabla_x(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{q})\bar{q} = 0$ implies $\bar{q} = 0$, then*
(*I*) *there exist $\bar{z} \in E$ such that $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ is feasible for (PP), and*
(*II*) *$G(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p}) = H(\bar{u}, \bar{v}, \bar{\lambda}, \bar{q})$.*

lso, if the hypotheses of a weak duality theorem are satisfied for all feasible solutions of (PP) and (DP), then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ is an efficient solution for (PP).

Proof. It follows on the lines of Theorem 3. □

4. Special cases

In this section, we consider some of the special cases of the problems studied in Section 3. In all these cases, $C_1 = R^n_+$ and $C_2 = R^m_+$.

(*i*) If $k = 1$, then our problems (PP) and (DP) reduces to the single objective nondifferentiable symmetric dual programs considered in Yang et al. [11].

(*ii*) If $k = 1$, and we take $D = \{Ay : y^T Ay \leq 1\}$, $E = \{Bx : x^T Bx \leq 1\}$, where A and B are positive semidefinite matrices, then $(x^T Ax)^{1/2} = S(x \mid D)$ and $(y^T By)^{1/2} = S(y \mid E)$. In this case (PP) and (DP) reduce to the problems considered in Ahmad and Husain [2].

(*iii*) If $D = 0$ and $E = 0$, then (PP) and (DP) reduce to (MP) and (MD) considered in Yang et al. [16] along with the nonnegativity restrictions $x \geq 0$ and $v \geq 0$. However, taking $F_{x,u}(a) = (x-u)^T a$ and $G_{v,y}(b) = (v-y)^T b$ along with the hypothesis (A) and (B) of Theorem 1 in [13] gives $x \geq 0$ and $v \geq 0$.

(*iv*) If $k = 1$, $D = 0$ and $E = 0$, then our problems (PP) and (DP) reduced to programs studied in Gulati et al. [4].

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