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EXISTENCE OF PERIODIC SOLUTION OF SOME ECO-EPIDEMIOLOGICAL SYSTEMS

ZHIJUN LIU AND SAHABUDDIN SARWARDI*

Abstract. The effect of impulse in the ecological models makes them more realistic. Recently, the eco-epidemiological models have become an important field of study from the both mathematical and ecological view points. In this article, we consider some eco-epidemiological systems under the influence of impulsive force. A set of sufficient conditions for the permanence of the system are derived. Stability of the trivial solution and at least one strictly positive periodic solution are obtained. Numerical examples are given in support to our analytical findings. Finally, a short discussion concludes the paper.

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1. Introduction

Mathematical ecology aims at the mathematical representation, treatment and modeling of complex ecosystems using a variety of applied mathematical techniques and tools to get a better understanding of the system. Predator-prey relationship plays an important role in regulating the numbers of prey and predators in complex ecosystems. A large portion of mathematical ecology is devoted to study the predator-prey interaction after the pioneering work of Alfred James Lotka and Vito Volterra in the mid-1920s. A lot of references are available in the literature. We mentioned here a very few, [25], [26], and references therein. Similarly, epidemiological models have also received great attention after the seminal model of KermackMcKendrick on SIRS (susceptible-infective-removed-susceptible) systems. Both theoretical and experimental investigations in these two fields namely ecology and epidemiology progressed independently along the

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years, until the late eighties and early nineties. There are so many references in this context; we are unable to cite all of them here, but an excellent one is [27].

Transmissible diseases can be an important factor in regulating animal population sizes. For instance, in 1988 the population of common seals in the Wash (the square-mouthed estuary on the northwest margin of East Anglia on the east coast of England) was the worst affected in the UK with almost 50 percent of the population died as a result of the epidemic. Both common seals and grey seals are found around the UK coast. The vast majority of both species are found around Scotland. There are approximately 119,000 grey seals and more than 36,000 common seals in Scottish waters. Another example would be the Myxomatosis, which became cause of enormous decreases in the rabbit population in Australia in the 1950s, [24]. Infectious disease can thus be a relevant factor in regulating animal population sizes and therefore cannot be ignored in ecological situation. Such idea has already been taken into account and became a new field of study named as "eco-epidemiology". Therefore, eco-epidemiology is a young field of research. However a good number of papers have been published in this direction for instance, [1]-[3], [16], [18]-[22]. Hadeler and Freedman [1] have modified the Rosenzweig prey-predator model in which the predators like to eat the infected prey compare to the uninfected prey, the predators get infection only by eating the diseased prey, and the prey obtains the disease from parasites spread into the environment by the predators. They also modeled a second situation in which the predators could only survive on the prey if some of the prey were more easily caught due to being infected. Beltrami and Carroll [2] have investigated the effect transmissible disease in phytoplankton-zooplankton system and observed that a small amount of infectious agent can destabilise the otherwise stable trophic configuration between a prey species and its grazer. Venturino [16] studied modifications of the classical LotkaVolterra prey-predator model in which an SI or SIS disease spreads among either the prey or the predator population. Greenhalgh and Haque [20] discussed the effect of transmission coefficient in a semi-ratio dependent predator-prey system with disease in prey species only. Haque and Venturino [18] studied the influence of communicable disease in a ratio-dependent predator-prey system with disease in predators. Comparisons of theses findings along with the results of some related investigations allow a general conclusion that an infection in either species, prey or predator, may act as a biological control.

On the other hand, the application of impulsive differential equation in the population dynamics is an important field of studies. The impulsive differential equations are suitable for the evolutionary process whose states are subject to sudden changes at certain moments. For instance, in the population ecology, the effect of impulse makes the models more practical from the ecological point of view since the ecological systems are often deeply perturbed by human exploit activities such as planting and harvesting. Therefore, we need to introduce the impulsive differential equations to have a more accurate descriptions to the system. A significant number of investigations have been performed in this direction; for example, Zhen et al. [28] studied the persistence in a LotkaVolterra competition systems with impulsive perturbations. The study of predator-prey system with defensive ability of prey and impulsive perturbations on the predator has been investigated in [29]. In [30], the authors have explored the existence of periodic solutions with strictly positive components of generalized ecological competition systems governed by impulsive differential equation with infinite delays. The other references in this context are [4]-[8]. Similarly a good number of studies have been performed to incorporate the impulsive effect on epidemiological models, for example, [10], [11] and the references therein. In [10], the authors have observed the effect of constant and pulse vaccination on SIR epidemic model with horizontal and vertical transmission. Donfrio [11] has considered the case of pulse vaccination strategy in a SIR Epidemic Model by introducing impulsive differential equation. We do not go through more details here but the reader could be able to find interesting results in these papers, [9]-[14].

In this study, we apply the impulsive differential equation on some important eco-epidemiological models since the application of impulsive differential equation on both ecological and epidemiological models are well established in the literature, [4]-[14]; but, so far we know, no attempt has been made to incorporate the impulsive differential equation with eco-epidemic problems, except from the papers of Haque and his collaborators, see [32] and [33]. This is one of earlier attempts to allow eco-epidemic situations under the influence of impulsive differential equations. Therefore, the present study will help to open a new window for research in this direction.

2. Our assumptions and the mathematical models

2.1. Disease on competing species

Let us first consider competitive system where two predators each other, or two species that survive in the same habitat on the same resources. For example, the letter case could be sheep and cows grazing on the same pasture. Therefore the classical model for the description of their interactions is

$$\begin{cases} \frac{dP}{dt} = P[a - bP - cQ] \\ \frac{dQ}{dt} = Q[d - eP - fQ]. \end{cases} \quad (1)$$

Now, assume that the disease spreads only among one of the competing species, let us say Q. Counting only the susceptible individual by P, another susceptible individual by Q, and infected individuals of the latter species, V. With the above assumptions Venturino [22] has considered the following model given below

$$\begin{cases} \frac{dP}{dt} = P[a - bP - cQ - \eta V] \\ \frac{dQ}{dt} = Q[d - eP - f(Q + V) - \delta V] + \nu V \\ \frac{dV}{dt} = V[\delta Q - gP - f(Q + V) - \nu] \end{cases} \quad (2)$$

where δ is the force of infection and ν the rate of infection recovery. For the shake of simplicity Venturino [22] assumed that $\nu = 0$ and then studied the mathematical properties of the system and its ecological meanings.

2.2. Predator-prey model with disease in the predator species only

Let us consider a predator-prey model where $R(t)$ represents the number of Rabbits and $F(t)$ denotes the number of foxes at time t. The classical model is given by

$$\begin{cases} \frac{dR}{dt} = R[a - bR - cF] \\ \frac{dR}{dt} = R[a - bR - cF] \end{cases} \tag{3}$$

Now consider the case when the disease spreads among the predators population. To formulate the model Venturino [22] has assumed that disease spreads only among the predators population. He denotes the sound preys by R, sound predators by F, infected predators by V and proposed the following model

$$\begin{cases} \frac{dR}{dt} = R[a - bR - cF - \eta V] \\ \frac{dF}{dt} = F[d + eR - f(F + V) - \delta V] + \nu V \\ \frac{dV}{dt} = V[\delta F + gR - f(F + V) - \nu] \end{cases} \tag{4}$$

where a and d represent the growth rate of the prey and the susceptible predator respectively. It has been assumed that the susceptible individual reproduces and infected are unable to do so. It was also assumed that the predators population have alternative sources of food. The parameters η means the death of the prey by disease. The parameters η and c differ for at least two reasons: (i) $\eta < c$ is used to model the situation in which predators are less able to catch prey; (ii) $\eta > c$ denotes the fact that the hunting abilities of sick predators may be unaffected, but prey surviving an attack may catch the disease and die of it. The more likely situation is (i) because infected predators would not able to run as healthy individuals do. Again $g \neq e$ allows the situation for which the food may be more valuable for the infected predators compared to the healthy individuals.

An alternative model for which Venturino [22] did not carry out the analysis is given by

$$\begin{cases} \frac{dR}{dt} = R[a - bR - cF - \eta V], \\ \frac{dF}{dt} = F[d + eR - f(F + V) - \delta V] + (\nu + h)V, \\ \frac{dV}{dt} = V[\delta F + gR - f(F + V) - \nu]. \end{cases} \tag{5}$$

Here the reproduction takes place also among diseased predators at a different net rate h than for sound ones, and the offsprings are assumed to be healthy.

Another model has been proposed in which the diseased predators give birth to susceptible offspring; for instance, it is well known that humans rubella during pregnancy may in some cases cause malformations in children, so they do belong to the susceptible class. Therefore the model (5) takes the form

$$\begin{cases} \frac{dR}{dt} = R[a - bR - cF - \eta V], \\ \frac{dF}{dt} = F[d + eR - f(F + V) - \delta V] + \nu V, \\ \frac{dV}{dt} = V[h + \delta F + gR - f(F + V) - \nu]. \end{cases} \tag{6}$$

Now all the above eco-epidemic models can be written as follows

$$\begin{cases} \frac{dy_1}{dt} = y_1[r_1 - a_{11}y_1 - a_{12}y_2 - a_{13}y_3], \\ \frac{dy_2}{dt} = y_2[r_2 - a_{21}y_1 - a_{22}y_2 - a_{23}y_3], \\ \frac{dy_3}{dt} = y_3[r_3 - a_{31}y_1 - a_{32}y_2 - a_{33}y_3]; \end{cases} \tag{7}$$

or simply

$$\frac{dy_i}{dt} = y_i[r_i - \sum_{j=1}^{3} a_{ij}y_j(t)], \quad i = 1, 2, 3, \tag{8}$$

where $r_1 = a$, $r_2 = d$, $r_3 = h$, $a_{11} = b$, $a_{12} = c$, $a_{13} = \eta$, $a_{21} = -e$, $a_{22} = f$, $a_{23} = f + \delta$, $a_{31} = -g$, $a_{32} = f - \delta$, $a_{33} = f$. Note that we have the following cases:

Case 1. When (a, b, c, η, d, e, f, δ, h, g)$>$ (0,0,0,0,0,0,0,0,0,0), then system (8) represents system (6) with $\nu = 0$.

Case 2. When h=0, then system (8) represents systems (5) and (4) with $\nu = 0$.

Case 3. When h=0, $e < 0$, $g < 0$, then system (8) represents system (2) with $\nu = 0$.

Let us consider a general non-autonomous dynamical system

$$\begin{cases} \frac{dy_i}{dt} = y_i[r_i(t) - \sum_{j=1}^{3} a_{ij}(t)y_j(t)], & t \neq \tau_k, k \in Z_+, \\ y_2(\tau_k^+) = (1 - h_{2k})y_2(\tau_k^-), & \\ y_3(\tau_k^+) = (1 - h_{3k})y_3(\tau_k^-), & t = \tau_k, \ i = 1, 2, 3, \end{cases} \tag{9}$$

where $h_{i(k+q)} = h_{ik}$, $h_{1k} \equiv 0$, $\tau_{k+q} = \omega + \tau_k$ and $Z_+ = \{1, 2, ...\}$, $r_i(t)$ are all continuous periodic functions with a common period ω. Therefore we consider some eco-epidemiological systems with impulsive forces in the predators/either species. Again notice that when we put $h_3 = 0$ in the above system we get the following eco-epidemic system with pulse vaccination for the susceptible predators:

$$\begin{cases} \frac{dy_i}{dt} = y_i[r_i(t) - \sum_{j=1}^{3} a_{ij}(t)y_j(t)], & t \neq \tau_k, k \in Z_+, \\ y_2(\tau_k^+) = (1 - h_{2k})y_2(\tau_k^-), & t = \tau_k, i = 1, 2, 3, \end{cases} \tag{10}$$

where $h_{i(k+q)} = h_{ik}$ and $\tau_{k+q} = \omega + \tau_k$. Here we study the dynamics of the system (9) which is a general model applicable to a variety of impulsive eco-epidemic situations as mentioned above.

3. Preliminary

In this section, we would like to recall few definitions and results from [23] for our feature use.

3.1. Notations, definitions and preliminary results

Let $J \subset R$. We denote $PC(J, R)$ as a set of functions $\psi : J \longrightarrow R$, which are continuous from left for $t \in J$, $t \neq \tau_k$, and have discontinuity of first kind at the point $\tau_k \in J$. Again we denote $PC^{'}(J, R)$ by the set of functions $\psi : J \longrightarrow R$

with a derivative $\frac{d\psi}{dt} \in PC(J,R)$. Throughout this article we deal with the following Banach Space of ω-periodic functions

$$PC_\omega = \{\psi \in PC([0,\omega],R)|\psi(0)=\psi(\omega)\}, \|\psi\|_{PC} = \sup\{\psi(t) : t \in [0,\omega]\}$$

and

$$PC_\omega^{'} = \{\psi \in PC^{'}([0,\omega],R)|\psi(0)=\psi(\omega)\}, \|\psi\|_{PC'} = \max\{\|\psi\|_{PC}, \|\psi'\|_{PC'}\}.$$

Moreover, for any $y \in C$ (or PC_ω) we denote $\bar{y} = \frac{1}{\omega}\int_0^\omega y(t)dt$.

Definition 1 ([23]). *The set $F \subseteq PC_\omega$ is said to quasiequicontinuous in $[0,\omega]$ if for any $\epsilon > 0$, there exist a $\delta > 0$ such that if $x \in F$, $k \in Z_+$, $t_1, t_2 \in [\tau_{k-1}, \tau_k] \cap [0,\omega]$ and $|t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \epsilon$.*

Lemma 1. *The set $F \subseteq PC_\omega$ is said to relatively compact iff (i) F is quasi-equicontinuous in J. (ii) F is bounded, that is, $\|\psi\|_{PC_\omega} = \sup\{|\psi(t)| : t \in J\} \le M$ for each $x \in F$ and some $M > 0$.*

The above Lemma 1 gives us the necessary and sufficient conditions for relative compactness in PC_ω.

Now for given $\alpha(t)$, $\beta(t) \in PC_\omega$, let us consider the impulsive logistic equation,

$$\begin{cases} \frac{dx}{dt} = x[\alpha(t) - \beta(t)x(t)], & t \neq \tau_k, \quad k \in Z_+, \\ x(\tau_k^+) = (1-h_k)x(\tau_k^-), & t = \tau_k, \end{cases} \tag{11}$$

where $h_{k+q} = h_k$, and we assume that $1 - h_k > 0$ $(k \in Z_+)$, then the following results can be obtained easily.

Lemma 2. *System (11) has a unique positive periodic solution $\theta_{[\alpha,\beta]}$ iff $\bar{\alpha} > \frac{1}{\omega}\ln(\sum_{k=1}^q \frac{1}{1-h_k})$.*

Now let us divide $\theta'_{[\alpha,\beta]} = \theta_{[\alpha,\beta]}[\alpha - \beta\theta_{[\alpha,\beta]}]$ by $\theta_{[\alpha,\beta]}$ and integrate over the intervals $(0,\tau_1), (\tau_k, \tau_{k+1})(k = 1,2,...q-1)$ and (τ_q, ω), we get,

$$\bar{\alpha} - \frac{1}{\omega}\int_0^\omega \beta\theta_{[\alpha,\beta]}dt = \frac{1}{\omega}\ln(\prod_{k=1}^q \frac{1}{1-h_i}).$$

Let $\phi_{[a,b]}(t, t_0^+, w_0)$ be the unique solution of the Cauchy problem

$$\begin{cases} \frac{dw}{dt} = w(t)[a(t) - b(t)w(t)], & t \ge t_0(t \neq \tau_k), \quad k \in Z_+, \\ w(\tau_k^+) = (1-h_k)w(\tau_k^-), & t = \tau_k, \\ w(t_0^+) = w_0, \end{cases} \tag{12}$$

then it would not be very difficult to prove the following two lemmas by the trick shown in [7].

Lemma 3. *Given $\alpha(t), \beta(t) \in PC_\omega$ with $\beta > 0$, for any w_0 we have*

$$\lim_{t\to\infty} |\phi_{[a,b]}(t, t_0^+, w_0) - \theta_{[\alpha,\beta]}| = 0$$

provided that $\bar{\alpha} - \frac{1}{\omega}\ln(\prod_{k=1}^q \frac{1}{1-h_k}) > 0$ and $1 - h_k > 0$ for $k \in Z_+$.

Lemma 4. *Let $x_0 \in R$ be given with $x_0 > 0$ and consider two functions $a(t), b(t) \in PC([t_0, \infty), R)$ with $b(t) > 0$. Assume that $x(t) \in PC'$ such that*

$$\begin{cases} \frac{dx}{dt} \geq x(t)[a(t) - b(t)x(t)], & t \geq t_0 (t \neq \tau_k), k \in Z_+, \\ x(\tau_k^+) \geq (1 - h_k)x(\tau_k^-), & t = \tau_k, \\ x(t_0^+) \geq x_0, \end{cases} \tag{13}$$

then $x(t) \geq \phi_{[a,b]}(t, t_0^+, x_0)$ for all $t \geq t_0$. Similarly we can show, $x(t) \leq \phi_{[a,b]}(t, t_0^+, x_0)$ for all $t > t_0$, by taking the reverse of all inequalities in (13).

4. Stability and persistence of system (9)

4.1. Stability of the trivial solution $(0, 0, 0)$

Let θ_{bi} be the unique ω-periodic solution of the following system

$$\begin{cases} \frac{dx_i}{dt} = x_i[r_i(t) - a_{ii}(t)x_i(t)], t \neq \tau_k, k \in Z_+, i = 1, 2, 3, \\ x_2(\tau_k^+) = (1 - h_{2k})x_2(\tau_k^-);\ x_3(\tau_k^+) = (1 - h_{3k})x_3(\tau_k^-), t = \tau_k. \end{cases} \tag{14}$$

If $(0, 0, 0)$ be a trivial solution of system (9), then we have the following conclusion.

Theorem 1. *The trivial solution $(0, 0, 0)$ of system (9) is linearly stable if and only if $\bar{r}_i - \frac{1}{\omega}\ln(\prod_{k=1}^{q} \frac{1}{1-h_{ik}}) < 0$ ($i = 1, 2, 3$ and $h_{1k} = 0$) and it will be unstable if and only if $\exists\ i_0$, $1 \leq i_0 \leq n$, $\bar{r}_{i_0} - \frac{1}{\omega}\ln(\prod_{k=1}^{q} \frac{1}{1-h_{i_0k}}) > 0$.*

Proof. The stability of an ω-periodic solution $\tilde{y}_i(t)$ $(i = 1, 2, 3)$ of system (9) can be determined by considering the behavior of small-amplitude perturbation of the solution. Let us define $y_i = \tilde{y}_i(t) + x_i(t)$ (i=1,2,3), then we can write

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix},$$

where $\Phi(t)$ satisfy

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} a_{11}^* - a_{11}\tilde{y}_1(t) & -a_{12}\tilde{y}_1(t) & -a_{13}\tilde{y}_1(t) \\ -a_{21}\tilde{y}_2(t) & a_{22}^* - a_{22}\tilde{y}_2(t) & -a_{23}\tilde{y}_2(t) \\ -a_{31}\tilde{y}_3(t) & -a_{32}\tilde{y}_3(t) & a_{33}^* - a_{33}\tilde{y}_3(t) \end{pmatrix} \Phi(t),$$

with $a_{ii}^* = r_i - \sum_{j=1}^{3} a_{ij}\tilde{y}_j(t), i = 1, 2, 3$. Here is $\Phi(0) = I$, the identity matrix. Now we impose the impulsive conditions to the system (9) and have

$$\begin{pmatrix} x_1(\tau_k^+) \\ x_2(\tau_k^+) \\ x_3(\tau_k^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - h_{2k} & 0 \\ 0 & 0 & 1 - h_{3k} \end{pmatrix} \begin{pmatrix} x_1(\tau_k) \\ x_2(\tau_k) \\ x_3(\tau_k) \end{pmatrix}.$$

Hence, if the absolute value of the eigenvalues of the following matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \prod_{k=1}^{q}(1 - h_{2k}) & \\ 0 & 0 & \prod_{k=1}^{q}(1 - h_{3k}) \end{pmatrix} \Phi(\omega)$$

are less than one, then the ω-periodic solution is locally stable.

The fundamental matrix for small amplitude solution about $(0,0,0)$ solves

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \Phi(t)$$

with $\phi(0) = I$, leads to

$$M = \begin{pmatrix} e^{\omega \bar{r}_1} & 0 & 0 \\ 0 & \prod_{k=1}^{q}(1-h_{2k})e^{\omega \bar{r}_2} & \\ 0 & 0 & \prod_{k=1}^{q}(1-h_{3k})e^{\omega \bar{r}_3} \end{pmatrix},$$

which implies that the Floquet of $(0,0,0)$ are $\prod_{k=1}^{q}(1-h_ik)e^{\omega r_i} < 1, i = 1,2,3$. This completes the proof. □

4.2. Persistence

Theorem 2. *The sufficient condition for system* (9) *to be persistent is* $\bar{r}_i > \frac{1}{\omega}[\prod_{k=1}^{q} \ln \frac{1}{(1-h_{ik})}] + \sum_{i\neq j}^{3} \bar{a}_{ij}\theta_{bj}$ $i = 1,2,3.$, *where* $h_{1k} \equiv 0$

Proof. From system (9), we have

$$\begin{cases} \frac{dy_i}{dt} \le y_i[r_i(t) - a_{ii}(t)y_i(t)], \; t \neq \tau_k, k \in Z_+, \\ y_2(\tau_k^+) = (1-h_{2k})y_2(\tau_k^-); \; y_3(\tau_k^+) = (1-h_{3k})y_3(\tau_k^-), t = \tau_k. \end{cases} \tag{15}$$

Combining the Lemma 3 and Lemma 4 we have $y_i(t) \le \theta_{bj}$ for very large t. Therefore, there exist a $\omega_1 > 0$, such that

$$y_i(t) \le \theta_{bj} \; i = 1,2,3, \; t \ge \omega_1.$$

Hence

$$\begin{cases} \frac{dy_i}{dt} \ge y_i[r_i(t) - \sum_{i\neq j}^{3} a_{ij}(t)\theta_{bj} - a_{ii}(t)y_i(t)], \; t \neq \tau_k, k \in Z_+, \\ y_2(\tau_k^+) = (1-h_{2k})y_2(\tau_k^-); \; y_3(\tau_k^+) = (1-h_{3k})y_3(\tau_k^-), t = \tau_k. \end{cases} \tag{16}$$

Again by the Lemmas 3 and Lemma 4, and with the conditions of Theorem 2, we have $y_i(t) \ge \theta_{[r_i - \sum_{i\neq j}^{3} a_{ij}\theta_{bj}, a_{ii}]}, \; i = 1,2,3, \; t \ge \omega$.

Let $\rho = \inf_{i\in(1,2,3)}\{\theta_{[r_i - \sum_{i\neq j}^{3} a_{ij}\theta_{bj}, a_{ii}]} | t \in [0,\omega]\}$, $\sigma = \sup_{i\in(1,2,3)}\{\theta_{bi} : |t \in [0,\omega]\}$, then we have

$$0 < \rho \le \lim_{t\to\infty} \inf \; y_i(t) \le \lim_{t\to\infty} \sup \; y_i(t) \le \sigma < \infty.$$

Hence the proof is completed. □

5. Existence and stability of the infection-free solution

Without any infection, the model (9) reduces to

$$\begin{cases} \frac{dy_1}{dt} = y_1[r_1(t) - a_{11}(t)y_1 - a_{12}(t)y_2 - a_{13}(t)y_3], & \\ \frac{dy_2}{dt} = y_2[r_2(t) - a_{21}(t)y_1 - a_{22}(t)y_2 - a_{23}(t)y_3], & t \neq \tau_k, k \in Z_+, \\ y_2(\tau_k^+) = (1-h_{2k})y_2(\tau_k^-), & t = \tau_k, \end{cases} \tag{17}$$

where $h_{i(k+q)} = h_{ik}$, $h_{1k} \equiv 0$ and $\tau_{k+q} = \omega + \tau_k$. The nature of the solution of the above system has been investigated in [7]. Here we mention their results in terms of our system parameters.

Theorem 3. *System* (17) *has a semi-trivial positive solution of the form* $(y_1(t), 0)$ *if and only if* $\bar{r}_1 > 0$. *The necessary and sufficient condition for which system* (17) *admits a semi-trivial positive solution of the form* $(0, y_2(t))$ *is* $\bar{r}_2 > \frac{1}{\omega}[\prod_{k=1}^{q} \ln \frac{1}{(1-h_{2k})}]$

Theorem 4. *(i) The trivial solution* $(0,0)$ *is unstable if and only if system (17) has a semi-trivial positive solution, that is, if and only if either* $\bar{r}_1 > 0$ *or* $\bar{r}_2 > \frac{1}{\omega}[\prod_{k=1}^{q} \ln \frac{1}{(1-h_{2k})}]$

Now for all case $r_1 = a > 0$, therefore system (17) has always a semi-trivial positive solution of the form $(y_1(t), 0)$, as a consequence the trivial solution $(0,0)$ is unstable. That is, extinction of the both the species can not possible in the infection-free system (17).

6. Existence and stability of the prey-free solution

In this section, we are interested to show the behaviors of the solution of system (9) in the absence of the prey population. Therefore here we deal with the following system of differential equations,

$$\begin{cases} \frac{dy_2}{dt} = y_2[r_2 - a_{22}y_2 - a_{23}y_3] \equiv F_1(y_2, y_3), & t \neq \tau_k, \\ \frac{dy_3}{dt} = y_3[r_3 - a_{32}y_2 - a_{33}y_3] \equiv F_2(y_2, y_3), & \tau_{k+1} = \tau_k + \omega, \end{cases} \tag{18}$$

$$\begin{cases} y_2(\tau_k^+) = (1-h_2)y_2(\tau_k^-) \equiv \Theta_1(y_2, y_3), & t = \tau_k, \\ y_3(\tau_k^+) = (1-h_3)y_3(\tau_k^-) \equiv \Theta_2(y_2, y_3), & k \in Z_+, \end{cases} \tag{19}$$

where $F_2(y_2, 0) = \Theta_2(y_2, 0) \equiv 0$, $\Theta_1 \neq 0$ for $y_2 \neq 0$ and $\Theta_2 \neq 0$ for $y_3 \neq 0$. All notations used in this section are the same as those in [17].

6.1. Stability of trivial periodic solution

Let us assume that Φ be the flow associated to (18), we have $X(t) = \phi(t, y_2(0), y_3(0))$, $0 < t \leq \tau$, where $X_0 = X(y_2(0), y_3(0))$ and $X = (y_2, y_3)^\omega$. We assume that the flow Φ applies to time τ. So, $X(\tau) = \phi(\tau, X_0)$. We also assume that system (18)-(19) with $y_3 = 0$, has a stable τ_0-periodic solution denoted by x_s. Denote $\zeta = (x_s, 0)$ as a τ_0-periodic solution of system (18)-(19) in the two dimensional space. We again denote $x_0 = x_s(0)$. $(x_0, 0)$ is the initial condition for ζ. Then, $\zeta(0) = (x_0, 0)$ and finally we arrive at the following conclusion in the form following theorems derived in [14].

Theorem 5. *If the following inequalities holds*

$$(i) \quad |\frac{\partial \Theta_1}{\partial y_2}(\Phi(\tau_0, (x_0, 0)))\frac{\partial \phi_1}{\partial y_2}(\tau_0, (x_0, 0))| < 1, \tag{20}$$

$$(ii) \quad |\frac{\partial \Theta_2}{\partial y_3}(\zeta(\tau_0))| \exp \int_0^{\tau_0} |\frac{\partial F_2(\zeta(r))}{\partial y_3} dr < 1, \tag{21}$$

then the trivial solution $\zeta = (x_s, 0)^\omega$ is exponentially stable.

Theorem 6. *If $\tau_0 > \frac{1}{r_2}\ln(\frac{1}{1-h_2})$, where $\tau_0 = \frac{\ln((1-h_2)^{\frac{a_{33}}{a_{22}}}(1-h_3)^{-1})}{r_3(1-\frac{r_2 a_{33}}{r_3 a_{22}})}$, then there exist a $\epsilon_0 > 0$, such that $|\frac{a_{33}}{r_3}| < \epsilon_0$, system (18)-(19) has a non trivial periodic solution.*

6.2. Subcritical case and bifurcation

Theorem 7. *If the equation* (20) *is satisfied along with $\acute{d_0} = 0$, where $\acute{d_0} = 1-(1-h_2)^{-\frac{a_{33}}{a_{22}}}(1-h_3)e^{\tau_0 r_3(1-\frac{r_2 a_{33}}{r_3 a_{22}})}$, then we have the following results:*
(a) If $BC \neq 0$, then we have a bifurcation. Moreover, we have a bifurcation of a nontrivial periodic solution of system $(18)-(19)$ provided $BC < 0$ and a subcritical case if $BC > 0$.
(b) If $BC = 0$, then we have an undetermined case, where

$$
\begin{aligned}
B = &-(1-h_2)r_3\left(1-\frac{a_{33}}{r_3}x_s(\tau_0)\exp(\int_0^{\tau_0}\tau_2(1-\frac{a_{33}}{r_3}x_s(\tau))d\tau)\right)\\
&+r_3\frac{a_{33}}{r_3}\left(\frac{(1-h_2)(1-h_3)\dot{x}_s(\tau_0)}{1-(1-h_2)^{-1}e-r_2\tau_0}\right)\times\int_0^{\tau_0}\left\{\exp\left(\int_u^{\tau_0}r_3\left(1-\frac{a_{33}}{r_3}x_s(\tau)\right)d\tau\right)\right.\\
&\left.\times\exp\left(\int_0^u r_3\left(1-\frac{a_{33}}{r_3}x_s(\tau)\right)d\tau\right)\right\}du,\\
C = &-\frac{2a_{33}(1-h_3)(1-h_2)^{(2-\frac{a_{23}}{a_{22}})}e^{-r_2\frac{a_{23}}{a_{22}}\tau_0}}{e^{r_2\tau_0(\frac{a_{23}}{a_{22}}-1)}-(1-h_2)e^{r_2\frac{a_{23}}{a_{22}}\tau_0}}\times\int_0^{\tau_0}\left\{\exp\left(\int_u^{\tau_0}r_3\left(1-\frac{a_{33}}{r_3}x_s(\tau)\right)d\tau\right)\right.\\
&\left.\times\exp\left(\int_0^u r_3\left(1-\frac{a_{33}}{r_3}x_s(\tau)\right)d\tau\right)\right\}du\\
&+\frac{2r_3(1-h_3)}{r_3/a_{32}}\int_0^{\tau_0}\left\{\exp\left(\int_u^{\tau_0}r_3\left(1-\frac{a_{33}}{r_3}x_s(\tau)\right)d\tau\right)\right.\\
&\left.\times\exp\left(\int_0^u r_3\left(1-\frac{a_{33}}{r_3}\times x_s(\tau)\right)d\tau\right)\right\}du\\
&-(1-h_3)a_{23}a_{33}\int_0^{\tau_0}\left[\exp\left(\int_u^{\tau_0}r_3\left(1-\frac{a_{33}}{r_3}x_s(\tau)\right)d\tau\right)\right.\\
&\left.\times\int_0^u\left\{\exp\left(\int_p^u r_3\left(1-\frac{2x_s(\tau)}{r_2/a_{22}}\right)d\tau\right)x_s(p)\exp\left(\int_0^p r_3\left(1-\frac{a_{33}}{r_3}x_s(\tau)\right)d\tau\right)\right\}dp\right]du.
\end{aligned}
$$

7. Existence of ω-periodic solution

In this section, the existence of at least one positive periodic solution has been shown by using the Mawhin's continuation theorem. To show our main results we need to go through some basic preparation. Let us make the following change of variables

$$y_i(t) = \exp\{x_i(t)\}, \quad i = 1, 2, 3,$$

then (9) reduces to

$$\begin{cases} x_i'(t) = r_i(t) - \sum\limits_{j=1}^{3} a_{ij} e^{x_j(t)}, \quad i = 1.2, 3, t \neq \tau_k, k \in Z_+, \\ \Delta x_2(t)|_{t=\tau_k} = x_2(\tau_k^+) - x_2(\tau_k^-) = \ln(1 - h_{2k}), \\ \Delta x_3(t)|_{t=\tau_k} = x_3(\tau_k^+) - x_3(\tau_k^-) = \ln(1 - h_{3k}). \end{cases} \tag{22}$$

Therefore, we consider the existence of periodic solutions of system (9); this is equivalent to finding solutions of the boundary-value problem consisting of (22) on $[0, \omega]$, with boundary conditions $x(0) = x(\omega)$.

$$\begin{cases} x_i'(t) = r_i(t) - \sum\limits_{j=1}^{n} a_{ij}(t) e^{x_j(t)}, t \neq \tau_k, t \in [0, \omega], \quad k = 1, 2, \cdots, q, \\ \Delta x_2(t)|_{t=\tau_k} = x_2(\tau_k^+) - x_2(\tau_k^-) = \ln(1 - h_{2k}), \\ \Delta x_3(t)|_{t=\tau_k} = x_3(\tau_k^+) - x_3(\tau_k^-) = \ln(1 - h_{3k}), \\ x_i(0) = x_i(\omega), \ i = 1, 2, 3. \end{cases} \tag{23}$$

Now let X, Z are real Banach spaces, $L : \mathrm{Dom}L \subset X \to Z$ be a Fredholm mapping of index zero (index $L = \dim \ker L -$ codim Im L) , and let $P : X \to X$, $Q : Z \to Z$ are continuous projectors such that Im $P =$ Ker L, Ker $Q =$ Im L and $X =$ Ker L $\bigoplus$ Ker$P$, $Z = \mathrm{Im}L \bigoplus \mathrm{Im}Q$. Denoting the restriction of L to $\mathrm{Dom}L \cap \mathrm{Ker}\ P$ by L_P, we have $K_P : \mathrm{Im}\ L \to \mathrm{Ker}\ P \cap \mathrm{Dom}\ L$, the inverse (to L_P); and $J : \mathrm{Im}\ Q \to \mathrm{Ker}\ L$, an isomorphism of Im Q onto Ker L.

Now the Mawhin's continuous theorem [15] can be written as follows:

Lemma 5. (*Continuation theorem*). *Let $\emptyset \subset X$ be an open bounded set and let, $N : X \to Z$ be a continuous operator which is L-compact on $\bar{\emptyset}$ (i.e.,$QN : \bar{\emptyset} \to Z$ and $K_P(I - Q)N : \bar{\emptyset} \to Z$ are compact), $L : Dom\ L \subset\ Z \to Z$ be a Fredholm mapping of index zero. Assume* (a) *for each $\lambda \in (0, 1)$, $x \in \partial\emptyset \cap Dom\ L$, $Lx \neq \lambda Nx$,* (b) *for each $x \in Ker\ L \cap \partial\emptyset$, $QNx \neq 0$, and*

$$deg\{JQN, \emptyset \cap Ker\ L, 0\} \neq 0.$$

Then $Lx = Nx$ has at least one solution in $Dom\ L \cap \bar{\emptyset}$.

To prove our main results by means of the continuation theorem explained before, we need to introduce some function spaces.

For any non-negative integer p, let
$C^{(p)}[0, \omega; t_1, \cdots, t_q] = \{x : [0, \omega] \to R^n | \, x^{(p)}(t)$ exists for $t \neq t_1, \cdots, t_q; x^{(p)}(t + 0), x^{(p)}(t - 0)$ exists at $t_1, \cdots, t_q$; and $x^{(j)}(t_k) = x^{(j)}(t_k - 0), k = 1, \cdots, q, j = 0, 1, 2, \cdots, p\}$ with the norm $\| x \|_p = \max\{\sup\limits_{t \in [0,\omega]} \| x^{(j)}(t) \|\}_{j=1}^p$, where $\| \cdot \|$ is any norm of R^n. It is trivial to see that $C^{(p)}[0, \omega; t_1, \cdots, t_q]$ is a Banach space.

Now, we prove the following theorem.

Theorem 8. *Suppose that $\bar{a}_{ij} > 0$, $\bar{R}_i > 0$, and*

$$\bar{R}_i > \sum_{\substack{j=1 \\ j \neq i}}^{3} \frac{\bar{a}_{ij}}{\bar{a}_{jj}} \bar{R}_j e^{2\bar{R}_i \omega}, \quad i, j = 1, 2, 3 \ \textit{and} \ \ i \neq j,$$

then system (9) *has at least one positive* ω*-periodic solution, where*

$$\bar{R}_i = \frac{1}{\omega}\sum_{k=1}^{q}(1-h_{ik}) + \bar{r}_i \text{ and } h_{1k} \equiv 0.$$

Proof. Note that the existence of solution of (23) is sufficient to proof our claim. In order to use the continuation theorem of coincidence degree theory, it is necessary to establish the existence of a solution of system (23), we take

$$X = \{x \in C[0,\omega; t_1, \cdots, t_q]\}, \qquad Z = X \times R^{3q}$$

Then X is a Banach space with the norm $\|\cdot\|_0$, and Z is also a Banach space with the norm $||z|| = ||x||_0 + ||y||, x \in X, y \in R^{3q}$. Let

$$\begin{aligned}
&\text{Dom}L = \{x \in C^{(1)}[0,\omega; t_1, \cdots, t_q]\}\\
&L : \text{Dom } L \to Z, x \to (x', \triangle x(t_1), \cdots, \triangle x(t_q)),\\
&N : X \to Z,
\end{aligned}$$

$$Nx = \left(\begin{pmatrix} r_1 - \sum\limits_{j=1}^{3} a_{1j}e^{x_j} \\ r_2 - \sum\limits_{j=1}^{3} a_{2j}e^{x_j} \\ r_3 - \sum\limits_{j=1}^{3} a_{3j}e^{x_j} \end{pmatrix}, \begin{pmatrix} 0 \\ \ln(1-h_{21}) \\ \ln(1-h_{31}) \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \ln(1-h_{2m}) \\ \ln(1-h_{3m}) \end{pmatrix} \right).$$

Obviously

$$\begin{aligned}
&\text{Ker}L = \{x : x = C \in R^3, t \in [0,\omega]\},\\
&\text{Im}L = \{z = (f, C_1, \cdots, C_q, d) \in Z : \textstyle\int_0^\omega f(s)ds + \sum\limits_{k=1}^{q} C_k = 0\}
\end{aligned}$$

and

$$\dim \text{Ker}L = 3 = \text{codim Im}L.$$

So, ImL is closed in Z, L is a Fredholm mapping of index zero. Take

$$\begin{aligned}
&Px = \tfrac{1}{\omega}\textstyle\int_0^\omega x(t)dt,\\
&Qz = Q(f, C_1, \cdots, C_q) = (\tfrac{1}{\omega}[\textstyle\int_0^\omega f(s)ds + \sum\limits_{k=1}^{q} C_k], 0, \cdots, 0).
\end{aligned}$$

It is easy to show that P and Q are continuous projectors such that

$$\text{Im}P = \text{Ker}L, \text{Im}L = \text{Ker}Q = \text{Im}(I-Q).$$

Furthermore, by an easy computation, we can find that the inverse $K_P : \text{Im}L \to \text{Ker}P \cap \text{Dom}L$ of L_p has the form

$$K_P z = \int_0^t f(s)ds + \sum_{t>t_k} C_k - \frac{1}{\omega}\int_0^\omega \int_0^t f(s)dsdt - \sum_{k=1}^{q} C_k. \tag{24}$$

Thus

$$QNx = \left(\left(\begin{array}{c} \frac{1}{\omega}\int_0^{\omega}[r_1(t) - \sum\limits_{j=1}^{3}\frac{1}{\omega}\int_0^{\omega} a_{1j}(t)e^{x_j(t)}]dt \\ \frac{1}{\omega}\int_0^{\omega}[r_2(t) - \sum\limits_{j=1}^{3}\frac{1}{\omega}\int_0^{\omega} a_{2j}(t)e^{x_j(t)}]dt + \frac{1}{\omega}\sum\limits_{k=1}^{q}\ln(1-h_{2k}) \\ \frac{1}{\omega}\int_0^{\omega}[r_3(t) - \sum\limits_{j=1}^{3}\frac{1}{\omega}\int_0^{\omega} a_{3j}(t)e^{x_j(t)}]dt + \frac{1}{\omega}\sum\limits_{k=1}^{q}\ln(1-h_{3k}) \end{array} \right), 0, \cdots, 0 \right),$$

$$\begin{aligned} K_P(I-Q)Nx \;&=\; \left(\begin{array}{l} \int_0^t[r_1(s) - \sum\limits_{j=1}^{3} a_{1j}(s)e^{x_j(s)}]ds \\ \int_0^t[r_2(s) - \sum\limits_{j=1}^{3} a_{2j}(s)e^{x_j(s)}]ds + \sum\limits_{t>t_k}\ln(1-h_{2k}) \\ \int_0^t[r_3(s) - \sum\limits_{j=1}^{3} a_{3j}(s)e^{x_j(s)}]ds + \sum\limits_{t>t_k}\ln(1-h_{3k}) \end{array} \right) \\ &- \left(\begin{array}{l} \frac{1}{\omega}\int_0^{\omega}\int_0^t[r_1(s) - \sum\limits_{j=1}^{3} a_{1j}(s)e^{x_j(s)}]dsdt \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t[r_2(s) - \sum\limits_{j=1}^{3} a_{2j}(s)e^{x_j(s)}]dsdt + \sum\limits_{k=1}^{q}\ln(1-h_{2k}) \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t[r_3(s) - \sum\limits_{j=1}^{3} a_{3j}(s)e^{x_j(s)}]dsdt + \sum\limits_{k=1}^{q}\ln(1-h_{3k}) \end{array} \right) \\ &- \left(\begin{array}{l} (\frac{t}{\omega}-\frac{1}{2})\int_0^{\omega}[r_1(s) - \sum\limits_{j=1}^{3} a_{1j}(s)e^{x_j(s)}]ds \\ (\frac{t}{\omega}-\frac{1}{2})\int_0^{\omega}[r_2(s) - \sum\limits_{j=1}^{3} a_{2j}(s)e^{x_j(s)}]ds + \sum\limits_{k=1}^{q}\ln(1-h_{2k}) \\ (\frac{t}{\omega}-\frac{1}{2})\int_0^{\omega}[r_3(s) - \sum\limits_{j=1}^{3} a_{3j}(s)e^{x_j(s)}]ds + \sum\limits_{k=1}^{q}\ln(1-h_{3k}) \end{array} \right). \end{aligned}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Using the Arzela-Ascoli theorem, we can easily show that $QN(\bar{\emptyset})$, $K_P(I-Q)N(\bar{\emptyset})$ are relatively compact for any open bounded set $\emptyset \subset X$. Therefore, N is L-compact on $\bar{\emptyset}$ for any open bounded set $\emptyset \subset X$.

Now we need to search for an appropriate open, bounded subset $\emptyset$ to use the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{cases} x_i'(t) = \lambda[r_i(t) - \sum_{j=1}^{3} a_{ij}(t)e^{x_j(t)}], t \neq \tau_k, k = 1, 2, \cdots, q, t \in [0,\omega], \\ \Delta x_2(t)|_{t=\tau_k} = \lambda\ln(1-h_2), \Delta x_3(t)|_{t=\tau_k} = \lambda\ln(1-h_3), \quad i = 1, 2, 3, x_i(0) = x_i(\omega). \end{cases} \tag{25}$$

Let us assume that $x(t) = (x_1(t), x_2(t), x_3(t)) \in X$ is a solution of system (25) for some $\lambda \in (0,1)$. Integrating (25) over the interval $[0,\omega]$, we obtain

$$\int_0^{\omega}[r_i(t) - \sum_{j=1}^{3} a_{ij}(t)e^{x_j(s)}]ds + \sum_{k=1}^{q}\ln(1+b_{ik}) = 0,$$

that is

$$\sum_{j=1}^{3}\int_0^{\omega} a_{ij}(t)e^{x_j(t)}dt = \bar{R}_i\omega, \tag{26}$$

where $\bar{R}_i = \frac{1}{\omega}\sum_{k=1}^{q}(1-h_{ik}) + \bar{r}_i$ and $h_{1k} \equiv 0$. It follows from (25) and (26) that

$$\int_0^{\omega}|x_i'(t)|dt \le \bar{r}_i\omega + \sum_{j=1}^{3}\int_0^{\omega} a_{ij}(t)e^{x_j(t)}dt + \sum_{k=1}^{q}\ln(1-h_{ik}) = 2\bar{R}_i\omega. \tag{27}$$

Since $x(t) \in X$, there exist $\xi_i \in [0,\omega]$ such that

$$x_i(\xi_i) = \min_{t\in[0,\omega]} x_i(t) \quad i = 1,2,3. \tag{28}$$

From (26) and (28), we observe that

$$\bar{a}_{ii}e^{x_i(\xi_i)} \le \bar{R}_i,$$

which implies

$$x_i(\xi_i) \le \ln\{\frac{\bar{R}_i}{\bar{a}_{ii}}\}. \tag{29}$$

From (27) and (29), we obtain

$$x_i(t) \le x_i(\xi_i) + \int_0^{\omega}|x_i'(t)|dt \le \ln\{\frac{\bar{R}_i}{\bar{a}_{ii}}\} + 2\bar{R}_i\omega \overset{def}{=} M_i^{+}. \tag{30}$$

On the other hand, there exist $\eta_i \in [0,\omega]$ such that

$$x_i(\eta_i^{+}) = \sup_{t\in[0,\omega]} x_i(t). \tag{31}$$

In the above formula, if η_i is not an impulse point, we have $x_i(\eta_i^{+}) = x_i(\eta_i)$; if $\eta_i = t_k$ and t_k is an impulse point, we have $x_i(\eta_i^{+}) = x_i(t_k^{+})$. From (26) and (30) we have

$$\bar{a}_{ii}e^{x_i(\eta_i^{+})} \ge \bar{R}_i - \sum_{\substack{j=1\\ j\ne i}}^{3}\bar{a}_{ij}e^{x_j(\eta_j^{+})},$$

which implies

$$x_i(\eta_i^{+}) \ge \ln\frac{\bar{R}_i - \sum\limits_{\substack{j=1\\ j\ne i}}^{3}\frac{\bar{a}_{ij}}{\bar{a}_{jj}}\bar{R}_j e^{2\bar{R}_i\omega}}{\bar{a}_{ii}} \overset{def}{=} M_i. \tag{32}$$

From (27) and (32), we have

$$\begin{aligned} x_i(t) &\ge x_i(\eta_i^{+}) - \int_0^{\omega}|x_i'(t)|dt \\ &\ge M_i - 2\bar{R}_i\omega \overset{def}{=} M_i^{-}. \end{aligned} \tag{33}$$

From (32) and (33), we obtain

$$\sup_{t\in[0,\omega]} |x_i(t)| < \max\{|M_i^+|, |M_i^-|\} \stackrel{def}{=} H_i. \tag{34}$$

Clearly, H_i are independent of λ. Since

$$\bar{R}_i > \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\bar{a}_{ij}}{\bar{a}_{jj}} \bar{R}_j e^{2\bar{R}_i\omega} > \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\bar{a}_{ij}}{\bar{a}_{jj}} \bar{R}_j;$$

by using the Lemma 4.1.1 of [10] and the assumptions of Theorem 8, it is not difficult to show that the system of algebraic equations

$$\sum_{j=1}^{3} \int_0^{\omega} \bar{a}_{ij} e^{x_j} dt = \bar{R}_i\omega \quad i = 1, 2, 3 \tag{35}$$

has a unique solution $(x_1^*, x_2^*, x_3^*)^\top \in R^3$. Set $H = \|(H_1, H_2, H_3)^\top\| + C$, where C is taken sufficiently large such that the unique solution of (35) satisfies $\|(x_1^*, x_2^*, x_3^*)^\top\| < C$, then $\|x\| < H$. Let

$$\emptyset = \{x(t) = (x_1, x_2, x_3)^\top \in X : \|x(t)\| < H\}.$$

It is clear that $\emptyset$ satisfies the requirement (a) of Lemma 5. If $x \in \mathrm{Ker}L \cap \partial\emptyset$, then x is a constant vector in R^n with $\|x\| = H$; therefore, we have

$$QNx = \left(\begin{pmatrix} \bar{r}_1 - \sum\limits_{j=1}^{3} \bar{a}_{1j} e^{x_j} + 0 \\ \bar{r}_2 - \sum\limits_{j=1}^{3} \bar{a}_{2j} e^{x_j} + \frac{1}{\omega} \sum\limits_{k=1}^{q} \ln(1 + b_{2k}) \\ \bar{r}_3 - \sum\limits_{j=1}^{3} \bar{a}_{3j} e^{x_j} + \frac{1}{\omega} \sum\limits_{k=1}^{q} \ln(1 + b_{3k}) \end{pmatrix}, 0, \cdots, 0 \right) \neq 0.$$

Let $J : \mathrm{Im}Q \to \mathrm{Ker}L, (r, 0, \cdots, 0) \to r$; then, we get

$$JQNx = \begin{pmatrix} \bar{r}_1 - \sum\limits_{j=1}^{3} \bar{a}_{1j} e^{x_j} + 0 \\ \bar{r}_2 - \sum\limits_{j=1}^{3} \bar{a}_{2j} e^{x_j} + \frac{1}{\omega} \sum\limits_{k=1}^{q} \ln(1 + b_{2k}) \\ \bar{r}_3 - \sum\limits_{j=1}^{3} \bar{a}_{3j} e^{x_j} + \frac{1}{\omega} \sum\limits_{k=1}^{q} \ln(1 + b_{3k}) \end{pmatrix}.$$

Furthermore, in view of the assumptions of Theorem 8, it is easy to prove that

$$\deg\{JQNx, \emptyset \cap \mathrm{Ker}L, 0\} = \mathrm{sgn}\{(-1)^3[\det(\bar{a}_{ij})_{3\times 3}] e^{\sum_{j=1}^{3} e^{x_j^*}}\} \neq 0.$$

Therefore, we have proved that ω satisfies all the requirements in Lemma 5. Hence, system (9) has at least one ω-periodic solution in $\bar{\Omega}$. □

8. Numerical examples

In this section we give some numerical examples in support to our analytical findings; for instance, the following example is given in order to show the stable trivial solution of system (9) as claimed in subsection 4.1.

Example 1.

$$\begin{cases} y_1'(t) = y_1(t)(-0.1 - (0.11 + 0.1sin(2\pi t))y(1) - (0.012 + 0.01sin(2\pi t))y(2) \\ \qquad\quad -(0.013 + 0.01cos(2\pi t))y(3)) \\ y_2'(t) = x(2)(0.11 - (0.011 + 0.01sin(2\pi t))y(1) - (0.12 + 0.1sin(2\pi t))y(2) \\ \qquad\quad -(0.013 + 0.01cos(2\pi t))y(3)); \\ y_3'(t) = y(3)(0.3 - (0.011 + 0.01sin(2\pi t))y(1) - (0.012 + 0.01sin(2\pi t))y(2) \\ \qquad\quad -(0.13 + 0.1cos(2\pi t))y(3)), t \neq \tau_k \\ y_2(\tau_k^+) = (1 - 0.2)y_2(\tau_k^-);\ \ y_3(\tau_k^+) = (1 - 0.4)y_3(\tau_k^-), t = \tau_k \end{cases}$$

with the initial value $(0.1, 0.2, 0.3)$.

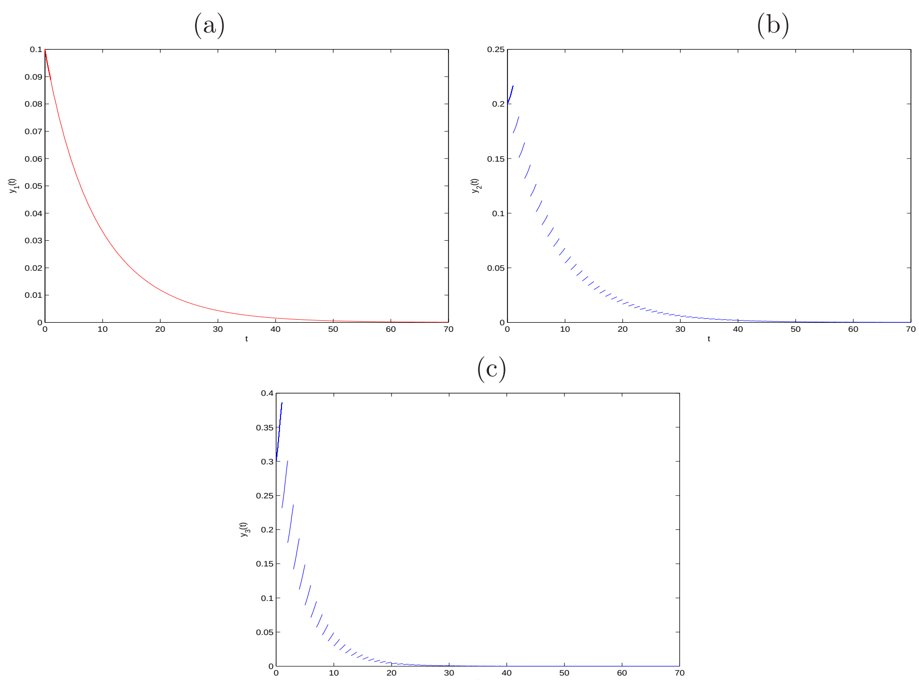

Fig 1. (a) Time-series of the species y_1 with t over $[0, 70]$. (b) Time-series of the species y_2 with t over $[0, 70]$. (c) Time-series of the species y_3 with t over $[0, 70]$.

The following example supports the claim of theorem 8.

Example 2.

$$\begin{cases} y_1'(t) = y_1(t)(1.5 - (0.21 + 0.1sin(2\pi t))y(1) - (0.012 + 0.01sin(2\pi t))y(2) \\ \qquad -(0.013 + 0.01cos(2\pi t))y(3)) \\ y_2'(t) = x(2)(1.2 - (0.011 + 0.01sin(2\pi t))y(1) - (0.22 + 0.1sin(2\pi t))y(2) \\ \qquad -(0.013 + 0.01cos(2\pi t))y(3)); \\ y_3'(t) = y(3)(1.3 - (0.011 + 0.01sin(2\pi t))y(1) - (0.012 + 0.01sin(2\pi t))y(2) \\ \qquad -(0.33 + 0.1cos(2\pi t))y(3)), t \neq \tau_k \\ y_2(\tau_k^+) = (1 - 0.01)y_2(\tau_k^-);\ y_3(\tau_k^+) = (1 - 0.02)y_3(\tau_k^-), t = \tau_k \end{cases}$$

with the initial value $(0.1, 0.2, 0.3)$.

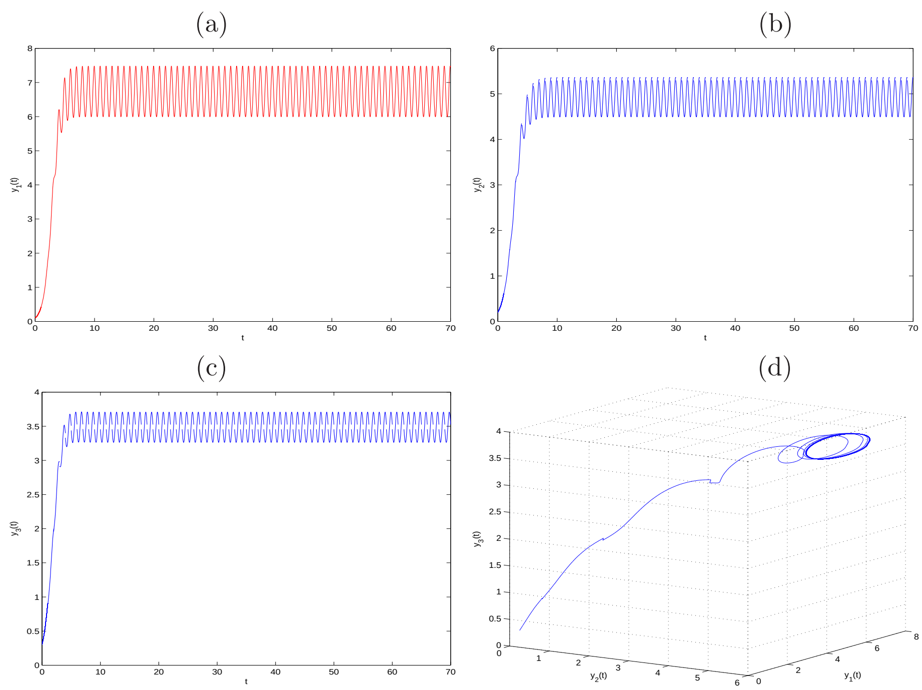

Fig 2. (a) Time-series of the species y_1 with t over $[0, 70]$. (b) Time-series of the species y_2 with t over $[0, 70]$. (c) Time-series of the species y_3 with t over $[0, 70]$. (d) Corresponding phase portrait.

9. Discussion

The disease in the predator/either species plays a crucial in predator-prey dynamics, [1]-[3], [16], [18]-[22]. However no attempt has been made to consider the effect of impulse on this issue. As we explained in the introduction that how important it is to consider these eco-epidemic models under the influence of impulse force to make them more realistic from biological point of view. In this article, we propose and analyze an eco-epidemic predator-prey system with disease in the predator/either species with impulsive effect, which in fact, covers variety of practical ecological situations.

We investigated the conditions (Theorem 1) for which the trivial equilibrium $(0,0,0)$ is stable and unstable. The undesirable case when it is stable; however the extinction of the both species is frequently observed in nature, [31]; and can be thus explained by our model (see numerical Example 1 also); and cannot be explained by the impulse free model, since in this case, trivial equilibrium $(0,0,0)$ in unstable always (see p.15 in [22]). At the same time, in Theorem 2, we have also obtained the persistence condition, which states the coexistence of all the species.

In section 5, the existence of infection free 'semi-trivial' solutions $(y_1(t),0)$ and $(0,y_2(t))$ have been given. We noticed there that infection free solution $(0,0)$ is always unstable; however with out impulsive force the resultant system has also unstable trivial solutions around $(0,0)$, see section 2.1 of [22]. Therefore, the impulsive force does not make any change in the stability of $(0,0)$ in the infection free system; however with infection it changes the unstable equilibrium $(0,0,0)$ to a stable one. Thus, infection may cause the extinction of all the species in impulsive eco-epidemic system, a natural result reproduces by our model.

The non trivial periodic solution of prey free situation has been shown in section 6. This gives conditions for which prey will extinct. In nature there are many situations where extinction of the prey species is desirable and introduction of non-fatal disease into the predator is a potential method of biological control to ensure extinction of the prey; Theorem 6 gives us the required conditions. In addition, we have also given the conditions for which this sub system has subcritical Hopf bifurcation which gives a oscillatory situation of infected predator and healthy predator.

Our main results lie in section 7. There we obtain a set of sufficient conditions for the existence of at least one strictly positive periodic solution of our proposed model by using the method of coincidence degree. This case reflects the fact where the three species prey, healthy predator and the infected predator exist in the eco-epidemic system. Our analytical claim is supported by the numerical Example 2.

Before ending this article, we remark that since this is one of the earlier attempts to show the effect of impulse on eco-epidemiological model, there is a room of improvement. Here we mention some future directions of research based on this article: (i) One may consider the situation where the disease crosses the species barrier. (ii) The time delay or latent period between contact between susceptible and infected species could also be modeled; and (iii) the effect of impulsive force for the both species is also an important factor to be considered. These are possible directions for future research.

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Zhijun Liu received his BS from Hubei University for Nationalities in 1998. MS from Hubei University in 2001 and Ph.D from Dalian University of Technology in 2007. He finished his postdoctoral training in 2009. Since 2010 he has been at Hubei University for Nationalities. His research interests focus on the applications of differential, difference and impulsive equations.

Department of Mathematics, Hubei University for Nationalities, Enshi Hubei, 445-000, P. R. China.
e-mail: zhijun_liu47@hotmail.com

Sahabuddin Sarwardi received his B.Sc Honours in Mathematics from University of Kalyani and M.Sc in Applied Mathematics from Jadavpur University, India in 2002 and 2004 respectively. Presently he is a lecturer in Mathematics of Aliah University, Kolkata, 700-091, India. His research interests focus on Non-linear Dynamics and Chaos.

Department of Mathematics, Aliah University, Kolkata, 700-016. Office: DN-41, Salt Lake City, Sector-V, Kolkata, 700-091, India.
e-mail: sarwardi.ioer@gmail.com