J. Appl. Math. & Informatics Vol. **28***(2010), No. 5 - 6, pp. 1323-1329*
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POSITIVE PSEUDO-SYMMETRIC SOLUTIONS FOR THREE-POINT BOUNDARY VALUE PROBLEMS WITH DEPENDENCE ON THE FIRST ORDER DERIVATIVE†

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Abstract. In this paper, a new fixed point theorem in cone is applied to obtain the existence of at least one positive pseudo-symmetric solution for the second order three-point boundary value problem

$$\begin{cases} x'' + f(t, x, x') = 0, \quad t \in (0,1), \\ x(0) = 0, x(1) = x(\eta), \end{cases}$$

where f is nonnegative continuous function; $\eta \in (0,1)$ and $f(t, u, v) = f(1 + \eta - t, u, -v)$.

AMS Mathematics Subject Classification : 34B10, 34B15.
Key words and phrases : three-point boundary value problem, Fixed point theorem in cone, Pseudo-symmetric solution.

1. Introduction

The multi-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. In the past few years, there has been much attention focused on questions of three-point boundary value problems for nonlinear differential equations; see, to name a few [1-7]. Avery and Henderson had the existence of three positive pseudo-symmetric solutions for a One dimensional p-Laplacian.

Recently Guo [8] used a new fixed point theorem in cone to prove the existence of positive solution for the second order three -point boundary value problem

$$\begin{cases} x'' + f(t, x, x') = 0, \quad t \in (0,1), \\ x(0) = 0, x(1) = \alpha x(\eta), \end{cases} \tag{1}$$

Received November 20, 2009. Revised November 27, 2009. Accepted December 14, 2009.
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†The project is supported by the Natural Science Foundation of China (70671034), the Natural Science Foundation of Hebei Province (A2009000664).

where $\alpha > 0,\ 0 < \eta < 1$ and $1 - \alpha\eta > 0,\ f : [0,1] \times [0,\infty) \times R \to [0,\infty)$ is continuous.

Sun [9] applied a monotone method to prove the existence of positive pseudo-symmetric solution for a three-point boundary value problem with dependence on the first-order derivative

$$\begin{cases} (\phi_p((u'(t)))' + q(t)f(t,u,u') = 0, \quad t \in (0,1), \\ \quad u(0) = 0,\ u(1) = u(\eta). \end{cases}$$

So, motivated by all the works above, in this paper we get the the existence of at leat one positive pseudo-symmetric solution for a three-point boundary value problem with dependence on the first-order derivative by the new fixed point theorem

$$\begin{cases} x'' + f(t,x,x') = 0, \quad t \in (0,1), \\ x(0) = 0, x(1) = x(\eta), \end{cases} \tag{2}$$

where f is nonnegative continuous function, $\eta \in (0,1)$, and $f(t,u,v) = f(1 + \eta - t, u, -v)$.

2. Preliminary Definition and Lemmas

Definition 1. *Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that*
i) $au + bv \in P$ for $u, v \in P$ and all $a \geq 0, b \geq 0$,
ii) $u, -u \in P$ implies $u = 0$.

Definition 2. *Suppose K is a cone in a Banach. The map α is a nonnegative continuous concave functional on K, provided $\alpha : K \to [0,\infty)$ is continuous and*

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v) \quad \text{for} \quad u, v \in K,\ t \in [0,1].$$

Definition 3. *Let E be a real Banach space. For $\eta \in [0,1]$, a function $u \in E$ is said to be pseudo-symmetric about η on $[0,1]$, if u is symmetric over the interval $[\eta,1]$, we have $u(t) = u(1-(t-\eta))$.*

Let X be a Banach space and $K \subset X$ be a cone. Suppose $\alpha, \beta : X \to R^+$ are two continuous convex functionals satisfying

$$\alpha(\lambda x) = |\lambda|\alpha(x),\ \beta(\lambda x) = |\lambda|\beta(x), \text{ for } x \in X,\ \lambda \in R,$$

$$\|x\| \leq M \max\{\alpha(x), \beta(x)\}, \text{ for } x \in X,$$

$$\alpha(x) \leq \alpha(y), \text{ for } x, y \in K,\ x \leq y,$$

where $M > 0$ is a constant.

Lemma 1. *Let $r, L > 0$ be constants and $\Omega = \{x \in X : \alpha(x) < r, \beta(x) < L\}$, $D = \{x \in X : \alpha(x) = r\}$, $E = \{x \in X : \alpha(x) \leq r, \beta(x) = L\}$.*
Assume $T : K \to K$ is a completely continuous operator satisfying
(A_1) $\alpha(Tu) < r,\ u \in D \cap K$; (A_2) $\beta(Tu) < L,\ u \in E \cap K$.
Then $deg\{I - T, \Omega \cap K, 0\} = 1$.

Lemma 2. *In Lemma 1. suppose* (A_1) *and* (A_2) *are replaced by*

$$(A_3)\ \alpha(Tu) > r,\ u \in D \cap K;\ (A_4)\ \beta(Tu) < L,\ u \in K;$$

and there is a $p \in (\Omega \cap K) \setminus \{0\}$ *such that* $\alpha(p) \neq 0$, *and* $\alpha(x + \lambda p) \geq \alpha(x)$ *for all* $x \in K$ *and* $\lambda \geq 0$. *Then* $deg\{I - T, \Omega \cap K, 0\} = 0$.

We need a result whose proof can be found in [8, p. 291].

Theorem 1. *Let* $r_2 > r_1 > 0,\ L > 0$ *be constants and*

$$\Omega_i = \{x \in X : \alpha(x) < r_i, \beta(x) < L\},\ i = 1, 2,$$

two bounded open sets in X. *Set* $D_i = \{x \in X : \alpha(x) = r_i\}$. *Assume* $T : K \to K$ *is a completely continuous operator satisfying*
(A_5) $\alpha(Tu) < r_1,\ u \in D_1 \bigcap K;\ \alpha(Tu) > r_2,\ u \in D_2 \bigcap K;$
(A_6) $\beta(Tu) < L,\ u \in K;$
(A_7) *there is a* $p \in (\Omega_2 \bigcap K) \setminus \{0\}$ *such that* $\alpha(p) \neq 0$ *and* $\alpha(x + \lambda p) \geq \alpha(x)$ *for all* $x \in K$ *and* $\lambda \geq 0$.
Then T *has at least one fixed point in* $(\Omega_2 \setminus \overline{\Omega_1}) \bigcap K$.

3. Main results

Lemma 3. *Let* $0 < \eta < 1$, *If* $y \in C[0, 1]$ *and* $y \geq 0$, *then the unique solution* x *of the problem*

$$\begin{cases} x'' + y(t) = 0, & t \in (0, 1), \\ x(0) = 0,\ x(1) = x(\eta), \end{cases} \tag{3}$$

satisfies $\min_{t \in [\eta, 1]} x(t) \geq \eta \|x\|$.

Proof. From (3) we can know that there is a point σ and $x(t)$ is maximum at $t = \sigma$. Then $\|x\| = x(\sigma)$. And $x(1) = x(\eta)$ is minimum for $t \in [\eta, 1]$, from the concavity of x we get

$$\frac{x(\sigma) - x(0)}{\sigma - 0} < \frac{x(\eta) - x(0)}{\eta - 0},$$

$$\frac{x(\sigma)}{\sigma} < \frac{x(\eta)}{\eta},$$

$$x(\eta) > \frac{\eta}{\sigma} x(\sigma) > \eta x(\sigma).$$

That completes the proof of the Lemma 3. □

Let $X = C^1[0, 1]$ with $\|x\| = \max_{0 \leq t \leq 1} [x^2(t) + (x'(t))^2]^{1/2}$, K=$\{x \in X : x(t) \geq 0$, x is concave on $[0, 1]$ and pseudo-symmetric about η on $[0, 1]\}$.
Define functionals $\alpha(x) = \max_{0 \leq t \leq 1} |x(t)|$ and $\beta(x) = \max_{0 \leq t \leq 1} |x'(t)|$ for each $x \in X$, then

$$\|x\| \leq \sqrt{2}\max\{\alpha(x), \beta(x)\},$$
$$\alpha(\lambda x) = |\lambda|\alpha(x),\ \beta(\lambda x) = |\lambda|\beta(x), \text{ for } x \in X,\ \lambda \in R,$$
$$\alpha(x) \leq \alpha(y) \text{ for } x, y \in K,\ x \leq y.$$

In the following, we denote

$$M = \frac{8}{(1+\eta)^2},\ m = \frac{2}{\eta},\ Q = \frac{2}{1+\eta}.$$

We will suppose that there are $L > b > \eta b > c > 0$ such that $f(t, u, v)$ satisfies the following growth conditions:
(H_1) $f : [0, 1] \times [0, \infty) \times R \to [0, \infty)$ is continuous;
(H_2) $f(t, u, v) < c/M$ for $(t, u, v) \in [0, 1] \times [0, c] \times [-L, L]$;
(H_3) $f(t, u, v) \geq b/m$ for $(t, u, v) \in [0, 1] \times [\eta b, b] \times [-L, L]$;
(H_4) $f(t, u, v) < L/Q$ for $(t, u, v) \in [0, 1] \times [0, b] \times [-L, L]$;
(H_5) For any $u, v \in K, f(t, u, v) = f((1 + \eta - t), u, -v)$.
Let

$$f^*(t,u,v) = \begin{cases} f(t,u,v), & (t,u,v) \in [0,1] \times [0,b] \times (-\infty,\infty) \\ f(t,b,v), & (t,u,v) \in [0,1] \times (b,\infty) \times (-\infty,\infty) \end{cases}$$

and

$$f_1(t,u,v) = \begin{cases} f^*(t,u,v), & (t,u,v) \in [0,1] \times [0,\infty) \times [-L,L] \\ f^*(t,u,-L), & (t,u,v) \in [0,1] \times [0,\infty) \times (-\infty,-L) \\ f^*(t,u,L), & (t,u,v) \in [0,1] \times [0,\infty) \times (L,\infty) \end{cases}$$

Then $f_1 \in C([0, 1] \times [0, \infty) \times R, R^+)$. Define

$$(Tx)(t) = \begin{cases} \int_0^t (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds, & 0 \leq t \leq \frac{1+\eta}{2}, \\ \\ \int_0^\eta (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\ + \int_t^1 (\int_{\frac{1+\eta}{2}}^s f_1(r,u(r),u'(r))\,dr)ds, & \frac{1+\eta}{2} \leq t \leq 1. \end{cases} \tag{4}$$

Lemma 4. *It is easy to see that T is well defined $T : K \to K$.*

Proof. Obviously, $(Tx)(t) \geq 0$, for all $x \in K$. Since

$$(Tx)'(t) = \begin{cases} \int_t^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr, & 0 \leq t \leq \frac{1+\eta}{2}, \\ -\int_{\frac{1+\eta}{2}}^t f_1(r,u(r),u'(r))\,dr, & \frac{1+\eta}{2} \leq t \leq 1, \end{cases}$$

we can get that $(Tu)'$ is nonincreasing on $[0, 1]$. So we have Tu is concave and $Tu \in C^1[0, 1]$.

In fact, for all $t \in [\eta, \frac{1+\eta}{2}]$, we note that $1-(t-\eta) \in [\frac{1+\eta}{2}, 1]$, so we have

$$\begin{aligned}
(Tu)(1-(t-\eta)) &= \int_0^{\eta} (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\
&\quad + \int_{1-(t-\eta)}^{1} (\int_{\frac{1+\eta}{2}}^{s} f_1(r,u(r),u'(r))\,dr)ds \\
&= \int_0^{\eta} (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\
&\quad + \int_{\eta}^{t} (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\
&= \int_0^{t} (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\
&= (Tu)(t),
\end{aligned}$$

and for all $t \in [\frac{1+\eta}{2}, 1]$, we note that $1-(t-\eta) \in [\eta, \frac{1+\eta}{2}]$, we have

$$\begin{aligned}
(Tu)(1-(t-\eta)) &= \int_0^{1-(t-\eta)} (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\
&= \int_0^{\eta} (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\
&\quad + \int_{\eta}^{1-(t-\eta)} (\int_{\frac{1+\eta}{2}}^{1-(s-\eta)} f_1(r,u(r),u'(r))\,dr)ds \\
&= \int_0^{\eta} (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\
&\quad - \int_t^{1} (\int_{1-(s-\eta)}^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\
&= \int_0^{\eta} (\int_s^{\frac{1+\eta}{2}} f_1(r,u(r),u'(r))\,dr)ds \\
&\quad + \int_t^{1} (\int_{\frac{1+\eta}{2}}^{1-(s-\eta)} f_1(r,u(r),u'(r))\,dr)ds \\
&= (Tu)(t).
\end{aligned}$$

So , $T : K \to K$. That completes the proof of the Lemma 4. □

Theorem 2. *Suppose $(H_1 - H_5)$ hold, then BVP (2) has at least one positive solution $x(t)$ satisfying*

$$c < \alpha(x) < b, \ |x'(t)| < L.$$

Proof. Take

$$\Omega_1 = \{x \in X : |x(t)| < c, |x'(t)| < L\},\ \Omega_2 = \{x \in X : |x(t)| < b, |x'(t)| < L\},$$

two bounded open sets in X, and

$$D_1 = \{x \in X : \alpha(x) = c\},\ D_2 = \{x \in X : \alpha(x) = b\}.$$

Obviously, $T : K \to K$ is completely continuous, and there is a $p \in (\Omega_2 \bigcap K)\backslash\{0\}$ such that $\alpha(x+\lambda p) \geq \alpha(x)$ for all $x \in K$ and $\lambda \geq 0$. For $x \in (D_1 \bigcap K), \alpha(x) = c$. From (H_2), we get

$$\begin{aligned}\alpha(Tx) &= \max_{t\in[0,1]} |Tu| = Tu(\frac{1+\eta}{2})\\ &= \int_0^{\frac{1+\eta}{2}} \left(\int_s^{\frac{1+\eta}{2}} f_1\big(r, u(r), u'(r)\big)\, dr\right) ds\\ &< \left(\int_0^{\frac{1+\eta}{2}} \left(\int_s^{\frac{1+\eta}{2}} dr\right) ds\right) \frac{c}{M} = c.\end{aligned}$$

Whereas for $x \in (D_2 \bigcap K), \alpha(x) = b$. From Lemma 3. we have $x(t) \geq \eta\alpha(x) = \eta b$ for $t \in [\eta, 1]$. So, from (H_3), we get

$$\begin{aligned}\alpha(Tx) &= \max_{t\in[0,1]} |Tu|\\ &> |Tu(\eta)|\\ &= \int_0^{\eta} \left(\int_s^{\frac{1+\eta}{2}} f_1\big(r, u(r), u'(r)\big)\, dr\right) ds\\ &> \left(\int_0^{\eta} \left(\int_s^{\frac{1+\eta}{2}} dr\right) ds\right) \frac{b}{m} = b.\end{aligned}$$

For $x \in K$, from (H_4), we have

$$\begin{aligned}\beta(Tu) &= \max_{t\in[0,1]} |(Tu)'(t)| = \max\big\{(Tu)'(0), -(Tu)'(1)\big\}\\ &= \max\left\{\int_0^{\frac{1+\eta}{2}} f_1(r, u(r), u'(r))dr, \int_{\frac{1+\eta}{2}}^1 f_1(r, u(r), u'(r))dr\right\}\\ &< \max\left\{\int_0^{\frac{1+\eta}{2}} dr, \int_{\frac{1+\eta}{2}}^1 dr\right\} \frac{L}{Q} = L.\end{aligned}$$

Theorem 1. implies that there is a point $x \in (\Omega_2 \setminus \overline{\Omega_1}) \bigcap K$ such that $x = Tx$. So, x is a positive solution for BVP (2) satisfying

$$c < \alpha(x) < b,\ |x'(t)| < L.$$

That completes the proof of Theorem 2. □

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