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A NOTE ON EXTREMAL LENGTH AND CONFORMAL IMBEDDINGS†

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Abstract. Let D be a plane domain whose boundary consists of n components and C_1, C_2 two boundary components of D. We consider the family F_1 of conformal mappings f satisfying $f(D) \subset \{1 < |w| < \mu(f)\}$, $f(C_1) = \{|w| = 1\}$, $f(C_2) = \{|w| = \mu(f)\}$. There are conformal mappings g_0, $g_1(\in F_1)$ onto a radial and a circular slit annulus respectively. We obtain the following theorem,

$$\{\mu(f)|f \in F_1\} = \{\mu|\mu(g_1) \le \mu \le \mu(g_0)\}.$$

And we consider the family F_n of conformal mappings $\tilde{f}$ from D onto a covering surfaces of the Riemann sphere satisfying some conditions. We obtain the following theorems,

$$\{\mu|1 < \mu \le \mu(g_1)\} \subset \{\mu(\tilde{f})|\tilde{f} \in F_2\} \subset \{\mu(\tilde{f})|\tilde{f} \in F_n\}$$

and $\mu(\tilde{f}) \le \mu(g_0)^n$.

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1. Introduction

The method of extremal length is a useful tool in a wide variety of areas. Especially, it has been successfully applied to conformal mappings, analytic functions. Extremal length was introduced as a conformally invariant measure of curve families. This development appeared in Ahlfors and Beurling[7].

Let D be a plane domain whose boundary consists of non-degenerate $n(2 \le n < \infty)$ components. Let C_1, C_2 be two boundary components of D.
We consider the family $F_1 = F_1(D)$ of univalent conformal mappings f on D satisfying the following conditions (1), (2) and (3).

(1) $f(D) \subset \{1 < |w| < \mu(f)\}$

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(2) $f(C_1) = \{|w| = 1\}$
(3) $f(C_2) = \{|w| = \mu(f)\}$

Condition (2) (resp. (3)) means that $z \to C_1$ (resp. C_2) if and only if $f(z) \to \{|w| = 1\}$ (resp. $\{|w| = \mu(f)\}$).

In the family F_1, there are conformal mappings g_0 and g_1 onto a radial and a circular slit annulus respectively. That is,

$$g_0(D) \subset \{1 < |w| < \mu(g_0)\}$$

$$g_0(C_1) = \{|w| = 1\}, \qquad g_0(C_2) = \{|w| = \mu(g_0)\}$$

and

$$\{1 < |w| < \mu(g_0)\} - g_0(D)$$

consists of $(n-2)$ concentric radial slits. Similarly

$$\{1 < |w| < \mu(g_1)\} - g_1(D)$$

consists of $(n-2)$ concentric circular slits.

Since g_0 and g_1 are determined uniquely up to rotations about the origin, $\mu(g_0)$ and $\mu(g_1)$ are determined uniquely. We say that g_0(resp. g_1) is the normalized radial (resp. circular) slit mapping on D. Extremal properties of g_0 and g_1 imply

$$\{\mu(f)|f \in F_1\} \subset \{\mu|\mu(g_1) \le \mu \le \mu(g_0)\}.$$

(See [11] for the extremal properties.)

In this note, we use the method of extremal length of a curve family to the boundary behavior of conformal mappings. we will prove that

$$\{\mu(f)|f \in F_1\} = \{\mu|\mu(g_1) \le \mu \le \mu(g_0)\}.$$

And we consider conformal mappings $\tilde{f}$ from D onto a covering surfaces of the Riemann sphere such that

$$\tilde{f}(C_1) = \{|w| = 1\}, \quad \tilde{f}(C_2) = \{|w| = \mu(\tilde{f})\}$$

(see section 2 for the definition). We shall study the range of $\mu(\tilde{f})$.

2. Extremal length and extremal property

Let Γ be a family whose elements γ are curves in a domain D and $\rho(z)$ a non-negative Borel measurable function. For γ and D, we have

$$L(\gamma, \rho) = \int_\gamma \rho\,|dz|, \qquad A(D, \rho) = \iint_D \rho^2\,dx\,dy\,.$$

We introduce the minimum length

$$L(\Gamma, \rho) = \inf_{\gamma\in\Gamma} L(\gamma, \rho).$$

Definition 2.1 ([1])**.** The *extremal length* of Γ in D is defined by

$$\lambda(\Gamma) = \sup_\rho \frac{L^2(\Gamma, \rho)}{A(D, \rho)}.$$

Proposition 2.2 ([1]). (Comparison principle of extremal length) *For two curve families* Γ_1, Γ_2, *if every* $\gamma_2 \in \Gamma_2$ *contains a* $\gamma_1 \in \Gamma_1$, *then*

$$\lambda(\Gamma_1) \leq \lambda(\Gamma_2)\,.$$

Proposition 2.3 ([1]). *Let* Γ *be a family of curves in* D, *and* f *an analytic function in* D *such that* $f'(z) \neq 0$. *Then*

$$\lambda(\Gamma) \leq \lambda[f(\Gamma)]\,.$$

Proposition 2.4 ([9]). *Suppose there exist disjoint open sets* G_n *containing the curves in* Γ_n. *If* $\cup_n \Gamma_n \subset \Gamma$, *then*

$$\sum_n \frac{1}{\lambda(\Gamma_n)} \leq \frac{1}{\lambda(\Gamma)}.$$

Example 2.5 ([9]). Let Γ_s be the family of closed curves in D separating C_1 from C_2, and Γ_j the family of arcs joining C_1 and C_2 in D. We know the following,

$$\lambda(\Gamma_s) = \frac{2\pi}{\log \mu(g_1)}, \qquad \lambda(\Gamma_j) = \frac{\log \mu(g_0)}{2\pi}.$$

Then We have the following.

Theorem 2.6. $\{\mu(f) | f \in F_1\} = \{\mu | \mu(g_1) \leq \mu \leq \mu(g_0)\}$.

Proof. Let $f \in F_1$. Consider the family Γ_s^* of closed curves in

$$\{w | 1 < |w| < \mu(f)\}$$

separating $\{|w| = 1\}$ from $\{|w| = \mu(f)\}$. Since

$$f(\Gamma_s) = \{f(\gamma) | \gamma \in \Gamma_s\} \subset \Gamma_s^*,$$

we have

$$\frac{2\pi}{\log \mu(g_1)} = \lambda(\Gamma_s) = \lambda(f(\Gamma_s)) \geq \lambda(\Gamma_s^*) = \frac{2\pi}{\log \mu(f)}$$

Hence we have

$$\mu(g_1) \leq \mu(f).$$

Similarly consider the family Γ_j^* of arcs in

$$\{w | 1 < |w| < \mu(f)\}$$

joining $\{|w| = 1\}$ and $\{|w| = \mu(f)\}$ to get the inequality

$$\mu(f) \leq \mu(g_0).$$

Thus

$$\{\mu(f) | f \in F_1\} \subset \{\mu | \mu(g_1) \leq \mu \leq \mu(g_0)\}.$$

In order to prove the converse, we may assume that D is a radial slit annulus. That is,

$$C_1 = \{|w| = 1\}, C_2 = \{|w| = \mu(g_0)\}$$

and each $C_k(3 \leq k \leq n)$ is a radial slit. Let l_k be the length of C_k and ζ_k be the midpoint of the slit C_k. now, for each $0 < t \leq 1$, let D_t be the annulus

$$\{1 < |w| < \mu(g_0)\}$$

with $(n-2)$ radial slits of length $t \cdot l_k$ with center at ζ_k on C_k. Denote it by $C_{k,t}$. Then, $D_1 = D$ and D_0 is the annulus

$$\{1 < |w| < \mu(g_0)\}$$

with $(n-2)$ punctures. note that D is a subregion of D_t.

Let $g_{1,t}$ be the normalized circular slit mapping on D_t. Denote by f_t the restriction of $g_{1,t}$ onto D. Then

$$f_t \in F_1$$

and the boundary of $f_t(D)$ consists of $\{|w| = 1\}$, $\{|w| = \mu(f_t)\}$ and $(n-2)$ 'cross'-shaped slits. We assert that the outer radius $\mu(f_t)$ is a continuous function of t, $\lim_{t\to 1} \mu(f_t) = \mu(g_1)$ and $\lim_{t\to 0} \mu(f_t) = \mu(g_0)$.

Let Γ_t be the family of closed curves in D_t separating C_1 from C_2. Then

$$\lambda(\Gamma_t) = \frac{2\pi}{\log \mu(f_t)}.$$

For any $0 < t,\ t' \leq 1$, we can easily construct a $K_{t,t'}$ quasiconformal mapping $\phi_{t,t'}$ from D_t onto $D_{t'}$ such that

$$\lim_{t\to t'} K_{t,t'} = 1.$$

Since

$$\frac{\lambda(\Gamma_t)}{K_{t,t'}} \leq \lambda(\Gamma_{t'}) = \lambda(\phi_{t,t'}(\Gamma_t)) \leq K_{t,t'}\lambda(\Gamma_t),$$

we get first and second assertions.

For the proof of the last assertion, note that Γ_0 is the family of closed curves in the punctured annulus D_0 separating C_1 from C_2. Then

$$\lambda(\Gamma_0) = \frac{2\pi}{\log \mu(g_0)}.$$

We know that

$$\Gamma_t \subset \Gamma_0$$

and each

$$\gamma \in \Gamma_0 - \Gamma_t$$

crosses at least one of $C_{k,t}$. Then

$$\frac{1}{\lambda(\Gamma_t)} \leq \frac{1}{\lambda(\Gamma_0)} \leq \frac{1}{\lambda(\Gamma_0 - \Gamma_t)} + \frac{1}{\lambda(\Gamma_t)}.$$

It is easy to see that

$$\lambda(\Gamma_0 - \Gamma_t) \to \infty$$

as $t \to 0$. Hence we get

$$\lim_{t\to 0} \mu(f_t) = \mu(g_0)$$

□

3. Some boundary behavior of conformal mappings

We consider the family $F_n = F_n(D)$ of conformal mapping $\tilde{f}$ from D onto a covering surfaces of the Riemann sphere $\hat{\mathbf{C}}$ satisfying the following conditions (4), (5), (6), (7) and (8). Denote by p the projection from $\tilde{f}(D)$ into $\hat{\mathbf{C}}$. For simplicity, we assume that C_1 and C_2 are simple closed analytic curves and each $\tilde{f}$ is analytic on $C_1 \cup C_2$.

(4) $\tilde{f}(D)$ is a covering surface of $\hat{\mathbf{C}}$ of at most n sheets.
(5) $(p \circ \tilde{f})(C_1) = \{|w| = 1\}$.
(6) $(p \circ \tilde{f})(z)$ rounds $\{|w| = 1\}$ one time clockwisely as z rounds C_1 one time positively with respect to D.
(7) $(p \circ \tilde{f})(C_2) = \{|w| = \mu(\tilde{f})\}$.
(8) $(p \circ \tilde{f})(z)$ rounds $\{|w| = \mu(\tilde{f})\}$ one time anti-clockwisely as z rounds C_2 one time positively with respect to D.

Since
$$F_n \supset F_1$$
we have
$$\{\mu(\tilde{f}) | \tilde{f} \in F_n\} \supset \{\mu | \mu(g_1) \leq \mu \leq \mu(g_0)\}.$$

First, we consider a covering surface with only two boundary components. Let $\Omega_{\alpha,\nu}$ be an annulus
$$\{1 < |z| < \nu\}$$
with a circular slit
$$l_{\alpha,\nu} = \{((\nu + 1)/2)e^{i\theta} |\ |\theta| \leq \alpha\}$$
for $0 < \alpha < \pi$. Sewing $\Omega_{\alpha,\nu}$ and $\hat{\mathbf{C}} - l_{\alpha,\nu}$ along $l_{\alpha,\nu}$ crosswisely, we obtain a covering surface $\tilde{\Omega}_{\alpha,\nu}$ of $\hat{\mathbf{C}}$. Then
$$\tilde{\Omega}_{\alpha,\nu}$$
is mapped conformally onto the annulus
$$\{1 < |w| < \mu_{\alpha,\nu}\}.$$
The outer radius $\mu_{\alpha,\nu}$ is uniquely determined. We have the following properties of $\mu_{\alpha,\nu}$.

Lemma 3.1. *$\mu_{\alpha,\nu}$ is a continuous function of α. Moreover,*
$$\lim_{\alpha \to 0} \mu_{\alpha,\nu} = \nu$$
and
$$\lim_{\alpha \to \pi} \mu_{\alpha,\nu} = \infty.$$

Proof. Since each $\tilde{\Omega}_{\alpha,\nu}$ is quasiconformally equivalent, we can prove the first assertion by a similar argument as in Theorem 1.6.

Let $\tilde{\Gamma}^*_{\alpha,\nu}$ be the family of arcs in $\tilde{\Omega}_{\alpha,\nu}$ joining two boundaries of $\tilde{\Omega}_{\alpha,\nu}$. Then

$$\frac{\log \mu_{\alpha,\nu}}{2\pi} = \lambda(\tilde{\Gamma}^*_{\alpha,\nu})$$

Let $\tilde{\Gamma}'_{\alpha,\nu}$ be the subfamily of $\tilde{\Gamma}^*_{\alpha,\nu}$ consisting of arcs not crossing $p^{-1}(l_\alpha)$, where p is the projection from $\tilde{\Omega}_{\alpha,\nu}$ into $\hat{\mathbf{C}} - l_{\alpha,\nu}$. Then

$$\frac{1}{\lambda(\tilde{\Gamma}'_{\alpha,\nu})} \leq \frac{1}{\lambda(\tilde{\Gamma}^*_{\alpha,\nu})} \leq \frac{1}{\lambda(\tilde{\Gamma}'_{\alpha,\nu})} + \frac{1}{\lambda(\tilde{\Gamma}^*_{\alpha,\nu} - \tilde{\Gamma}'_{\alpha,\nu})}$$

$$\lambda(\tilde{\Gamma}^*_{\alpha,\nu} - \tilde{\Gamma}'_{\alpha,\nu}) \to \infty$$

and

$$\lambda(\tilde{\Gamma}'_{\alpha,\nu}) \to \frac{\log \nu}{2\pi}$$

as $\alpha \to 0$. Thus we get the second assertion.

In order to prove the last assertion, we note that each arc in $\tilde{\Gamma}^*_{\alpha,\nu}$ crosses

$$p^{-1}(\{((\nu+1)/2)e^{i\theta} | \alpha \leq \theta \leq 2\pi - \alpha\})$$

and that

$$\{((\nu+1)/2)e^{i\theta} | \alpha \leq \theta \leq 2\pi - \alpha\}$$

reduces to one point as $\alpha \to \pi$. □

Theorem 3.2. $\{\mu | 1 < \mu \leq \mu(g_1)\} \subset \{\mu(\tilde{f}) | \tilde{f} \in F_2\}$.

Proof. We may assume that D is the annulus

$$\{1 < |z| < \mu(g_1)\}$$

with circular slits. For any fixed $1 < \nu < \mu(g_1)$, there is a covering surface $\tilde{\Omega}_{\alpha,\nu}$ of $\hat{\mathbf{C}}$ conformally equivalent to

$$\{1 < |z| < \mu(g_1)\}$$

by Lemma 2.1. Let $\tilde{f}$ be the restriction to D of the conformal mapping from

$$\{1 < |z| < \mu(g_1)\}$$

onto

$$\tilde{\Omega}_{\alpha,\nu}$$

Then

$$\mu(\tilde{f}) = \nu$$

□

Next we give an upper bound.

Theorem 3.3. *If* $\tilde{f} \in F_n$ *then*

$$\mu(\tilde{f}) \leq \mu(g_0)^n$$

Proof. Let $\tilde{\Gamma}^*$ be the family of arcs in $\tilde{f}(D)$ joining C_1 and C_2. Define the density $\tilde{\rho}(\zeta)$ on $\tilde{f}(D)$ so that

$$\tilde{\rho}(\zeta) = \frac{1}{|p(\zeta)|}$$

if $p(\zeta) \in \{1 < |z| < \mu(\tilde{f})\}$ and $\tilde{\rho}(\zeta) = 0$ otherwise. Then for any arcs $\tilde{\gamma} \in \tilde{\Gamma}^*$,

$$\int_{\tilde{\gamma}} \tilde{\rho}(\zeta)\ |d\zeta| \geq \log \mu(\tilde{f}).$$

And

$$\iint_{\tilde{f}(D)} \tilde{\rho}^2(\zeta)\ d\xi d\eta \leq 2\pi n \log \mu(\tilde{f}).$$

Hence

$$\frac{\log \mu(g_0)}{2\pi} = \lambda(\tilde{\Gamma}^*) \geq \frac{\log \mu(\tilde{f})}{2\pi n}.$$

Here, we assume that D has at least 3 boundary components, that is, $n \geq 3$. Then, we say that a conformal mapping $\tilde{f}$ belongs to F'_n if $\tilde{f} \in F_n$ and if

(9) $p \circ \tilde{f}(D) \subset \{1 < |w| < \mu(\tilde{f})\}$.

Clearly

$$F_n \supset F'_n \supset F_1.$$

Further, if

$$\tilde{f} \in F'_n$$

then

$$\mu(\tilde{f}) \leq \mu(g_0)^n$$

□

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