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REFLECTED BSDE DRIVEN BY A LÉVY PROCESS WITH STOCHASTIC LIPSCHITZ COEFFICIENT[†]

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Abstract. In this paper, we deal with a class of one-dimensional reflected backward stochastic differential equations driven by a Brownian motion and the martingales of Teugels associated with an independent Lévy process having a stochastic Lipschitz coefficient. We derive the existence and uniqueness of solutions for these equations via Snell envelope and the fixed point theorem.

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1. Introduction

El Karoui et al. [1] introduced the notion of one barrier reflected BSDE (RBSDE in short), which is actually a backward equation but the solution is forced to stay above a given obstacle. More precisely, a solution of such an equation is a triple of processes (Y, Z, K) such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s, \quad Y_t \geq S_t,$$

where the obstacle S is a given stochastic process. The increasing process K is introduced to pushes the process Y upwards with minimal energy so that it may remain above the obstacle S, i.e. $\int_0^T (Y_t - S_t)dK_t = 0$. This type of BSDEs is motivated by pricing American options (see e.g. [2]) and studying the mixed game problems(see e.g. [3], [4]). The existence and uniqueness theorem of solution of RBSDE in [1] was proved under the Lipschitz assumption on the coefficient.

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Recently, Nualart and Schoutens [5] gave a martingale representation theorem associated to a Lévy process. Furthermore, they showed the existence and uniqueness of solutions to BSDEs driven by Teugels martingales associated with a Lévy process with moments of all orders in [6]. Following this way, Bahlali et al. [7] established the existence and uniqueness of solution for BSDEs driven by a Brownian motion and the martingales of Teugels associated with an independent Lévy process, having a Lipschitz or a locally Lipschitz coefficient. As a natural extension, Ren and Hu [8] showed the same result for the RBSDEs driven by Lévy processes with Lipschitz coefficient.

However, the Lipschitz condition is too restrictive to be assumed in many applications. Due to this limitation, many papers have devoted to relax the Lipschitz condition (see e.g. [9], [10] and the references therein). El Karoui and Huang [11] considered BSDEs driven by a general càdlàg martingale with stochastic Lipschitz coefficient, they established a general result of existence and uniqueness by strengthening the integrability conditions on the coefficient and the terminal condition. Later, under the same assumptions on the coefficient, Bender and Kohlmann [12] showed the same result for BSDEs driven only by a Brownian motion. Motivated by the above works, the purpose of the present paper is to consider a class of one-dimensional RBSDEs driven by Lévy processes with stochastic Lipschitz coefficient. We try to get the existence and uniqueness of solutions for those RBSDEs by means of the Snell envelope and the fixed point theorem.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries including some spaces. Section 3 is devoted to prove the existence and uniqueness of solutions to RBSDEs with stochastic Lipschitz coefficient.

2. Preliminaries

Let $T > 0$ be a given real number. We first introduce the following two mutually independent processes:

- $\{B_t : t \in [0,T]\}$: a standard Brownian motion in R;
- A R-valued Lévy process of the form $L_t = bt + \ell_t$ corresponding to a standard Lévy measure ν satisfying the following conditions:
  (i) $\int_R (1 \wedge y^2)\nu(\mathrm{d}y) < \infty$,
  (ii) $\int_{]-\varepsilon,\varepsilon[^c} e^{\lambda|y|}\nu(\mathrm{d}y) < \infty$, for every $\varepsilon > 0$ and for some $\lambda > 0$.

We denote by $(\Omega, \mathcal{F}, P)$ a complete probability space and $\mathcal{F}_t$ the filtration generated by the Brownian motion B and the Lévy process defined above, i.e.

$$\mathcal{F}_t = \sigma\{B_s, 0 \le s \le t\} \vee \sigma\{L_s, 0 \le s \le t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets. The Euclidean norm of a vector $y \in R^n$ will be defined by $|y|$.

Let $(a_t)_{t\geq 0}$ be a nonnegative $\mathcal{F}_t$-adapted process, define

$$A(t) = \int_0^t a^2(s)\mathrm{d}s, \quad 0 \leq t \leq T.$$

For $\beta \geq 0$, let's introduce the following spaces:

- $L^2(\beta, a)$ the space of $\mathcal{F}_T$-measurable random variables ξ such that
$$E[e^{\beta A(T)}|\xi|^2] < \infty.$$
- $S^2(\beta, a)$ the space of $\mathcal{F}_t$-progressively measurable processes $\{\psi_t : t \in [0,T]\}$ such that
$$E[e^{\beta A(T)} \sup_{0\leq t\leq T} |\psi_t|^2] < \infty.$$
- $H^2(\beta, a)$ the space of predictable processes $\{\psi_t : t \in [0,T]\}$ such that
$$E\int_0^T e^{\beta A(t)}|\psi_t|^2\mathrm{d}t < \infty.$$
- l^2: the space of real valued sequences $x = (x_n)_{n\geq 1}$ such that
$$\|x\|^2 = \sum_{i=1}^{\infty} x_i^2 < \infty.$$
- $H^2(\beta, a; l^2)$ the corresponding space of l^2-valued processes $\{\psi_t : t \in [0,T]\}$ such that
$$E\int_0^T e^{\beta A(t)}\|\psi_t\|^2\mathrm{d}t = \sum_{i=1}^{\infty} E\int_0^T e^{\beta A(t)}|\psi_t^{(i)}|^2\mathrm{d}t < \infty.$$

Let $(H^{(i)})_{i\geq 1}$ denote the Teugels martingales associated with a Lévy process $\{L_t :\ t \in [0,T]\}$. More precisely

$$H_t^{(i)} = c_{i,i}Y_t^{(i)} + c_{i,i-1}Y_t^{(i-1)} + \cdots + c_{i,1}Y_t^{(1)},$$

where $Y_t^{(i)} = L_t^{(i)} - E[L_t^{(i)}] = L_t^{(i)} - tE[L_1^{(i)}]$ for all $i \geq 1$ and $L_t^{(i)}$ are so called power-jump processes, i.e., $L_t^{(1)} = L_t$ and $L_t^{(i)} = \sum_{0\leq s\leq t}(\Delta L_t)^i$ for $i \geq 2$. It was shown in Nualart and Schoutens [6] that the coefficient $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$:

$$q_{i-1} = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}.$$

We set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \cdots + c_{i,1}x.$$

The martingale $(H^{(i)})_{i\geq 1}$ can be chosen to be pairwise strongly orthonormal martingales.

The following result is the general martingale representation theorem which due to Bahlali et al. [7].

Proposition 1. *Let* $\{M_t : t \in [0,T]\}$ *be an* $\mathcal{F}_t$*-adapted square integrable martingale. Then, there exist* $Z \in H^2(\beta, a)$ *and* $U \in H^2(\beta, a; l^2)$ *such that*

$$M_t = E[M_t] + \int_0^t Z_s \mathrm{d}B_s + \sum_{i=1}^{\infty} \int_0^t U_s^{(i)} \mathrm{d}H_s^{(i)}.$$

In this paper, we consider the following RBSDE:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_{s-}, Z_s, U_s)\mathrm{d}s + K_T - K_t \\ \qquad - \int_t^T Z_s \mathrm{d}B_s - \sum_{i=1}^{\infty} \int_t^T U_s^{(i)} \mathrm{d}H_s^{(i)},\ 0 \le t \le T, \\ Y_t \ge S_t,\ \forall\, 0 \le t \le T \text{ a.s. and } \int_0^T (Y_{t-} - S_{t-})\mathrm{d}K_t = 0, \text{a.s.} \end{cases} \tag{1}$$

where the coefficient $f : \Omega \times [0,T] \times R \times R^d \times R \to R$ is progressively measurable. For $\beta > 0$, we make the following assumptions:
(H1) $\forall t \in [0,T], (y_i, z_i, u_i) \in R \times R^d \times l^2, i = 1, 2$, there are three nonnegative $\mathcal{F}_t$-adapted processes $\mu(t)$, $\gamma(t)$ and $\eta(t)$ such that

$$|f(t, y_1, z_1, u_1) - f(t, y_2, z_2, u_2)| \le \mu(t)|y_1 - y_2| + \gamma(t)|z_1 - z_2| + \eta(t)\|u_1 - u_2\|; \tag{2}$$

(H2) $\exists\, \epsilon > 0$ such that $a^2(t) := \mu(t) + \gamma^2(t) + \eta^2(t) \ge \epsilon$;
(H3) $\forall t \in [0,T], \dfrac{f(t,0,0,0)}{a} \in H^2(\beta, a)$.

We refer to (H1) as the stochastic Lipschitz condition on the coefficient f. Furthermore, we assume:
(H4) The terminal value $\xi \in L^2(\beta, a)$;
(H5) The obstacle $\{S_t, 0 \le t \le T\}$ is a *rcll*(right continuous with left limits) progressively measurable real-valued process satisfying $S_T \le \xi$ a.s. and $E[\sup_{0 \le t \le T} e^{2\beta A(t)} (S_t^+)^2] < \infty$, where $S_t^+ = \max\{S_t, 0\}$. Moreover, we assume that its jumping times are inaccessible stopping times.

We now present the definition of the solutions for RBSDE (1).
Definition 1. Let $\beta > 0$ and a a nonnegative $\mathcal{F}_t$-adapted process. A solution for RBSDE (1) is a triple (Y, Z, U, K) satisfying (1) such that $(Y, Z, U) \in S^2(\beta, a) \times H^2(\beta, a) \times H^2(\beta, a; l^2)$ and K is continuous and increasing such that $K_0 = 0$ and $E|K_T|^2 < \infty$.

3. Main results

3.1. A priori estimate

We first give a priori estimate of the solutions of RBSDE (1).

Lemma 1. *Let* $\beta > 0$ *large enough and assume* ***(H1)****-****(H5)*** *hold, let* $(Y_t, Z_t, K_t)_{0 \le t \le T}$ *be a solution of RBSDE (1) with data* (ξ, f, S)*. Then there* *exists a constant $C_\beta > 0$ depending only on β such that*

$$E\left[\sup_{0\le t\le T} e^{\beta A(t)}|Y_t|^2 + \int_0^T e^{\beta A(s)}(a^2(s)|Y_s|^2 + |Z_s|^2 + \|U_s\|^2)\mathrm{d}s + |K_T|^2\right]$$
$$\le C_\beta E\left[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(s)}\frac{|f(s,0,0,0)|^2}{a^2(s)}\mathrm{d}s + \sup_{0\le t\le T} e^{2\beta A(t)}(S_t^+)^2\right].$$

Proof. Applying Itô's formula to $e^{\beta A(t)}|Y_t|^2$, we have

$$\begin{aligned}
&e^{\beta A(t)}|Y_t|^2 + \beta\int_t^T a^2(s)e^{\beta A(s)}|Y_s|^2\mathrm{d}s + \int_t^T e^{\beta A(s)}(|Z_s|^2 + \|U_s\|^2)\mathrm{d}s\\
&= e^{\beta A(T)}|\xi|^2 + 2\int_t^T e^{\beta A(s)}Y_s f(s, Y_{s-}, Z_s, U_s)\mathrm{d}s + 2\int_t^T e^{\beta A(s)}Y_s\mathrm{d}K_s\\
&\quad - 2\int_t^T e^{\beta A(s)}Y_s Z_s\mathrm{d}B_s - 2\sum_{i=1}^\infty\int_t^T e^{\beta A(s)}Y_s U_s^{(i)}\mathrm{d}H_s^{(i)}\\
&\quad - \sum_{i=1}^\infty\sum_{j=1}^\infty\int_t^T e^{\beta A(s)}U_s^{(i)}U_s^{(j)}\mathrm{d}\left([H^{(i)}, H^{(j)}]_s - \langle H^{(i)}, H^{(j)}\rangle_s\right)\\
&\le e^{\beta A(T)}|\xi|^2 + \frac{\beta}{2}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s + \frac{8}{\beta}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0,0)|^2}{a^2(s)}\mathrm{d}s\\
&\quad + \frac{8}{\beta}\int_t^T e^{\beta A(s)}(a^2(s)|Y_s|^2 + |Z_s|^2 + \|U_s\|^2)\mathrm{d}s + 2\int_t^T e^{\beta A(s)}Y_s\mathrm{d}K_s\\
&\quad - 2\int_t^T e^{\beta A(s)}Y_s Z_s\mathrm{d}B_s - 2\sum_{i=1}^\infty\int_t^T e^{\beta A(s)}Y_s U_s^{(i)}\mathrm{d}H_s^{(i)}\\
&\quad - \sum_{i=1}^\infty\sum_{j=1}^\infty\int_t^T e^{\beta A(s)}U_s^{(i)}U_s^{(j)}\mathrm{d}\left([H^{(i)}, H^{(j)}]_s - \langle H^{(i)}, H^{(j)}\rangle_s\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&e^{\beta A(t)}|Y_t|^2 + (\frac{\beta}{2} - \frac{8}{\beta})\int_t^T a^2(s)e^{\beta A(s)}|Y_s|^2\mathrm{d}s\\
&\qquad + (1 - \frac{8}{\beta})\int_t^T e^{\beta A(s)}(|Z_s|^2 + \|U_s\|^2)\mathrm{d}s\\
&\le e^{\beta A(T)}|\xi|^2 + \frac{8}{\beta}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0,0)|^2}{a^2(s)}\mathrm{d}s + 2\int_t^T e^{\beta A(s)}S_s\mathrm{d}K_s \qquad (3)\\
&\quad - 2\int_t^T e^{\beta A(s)}Y_s Z_s\mathrm{d}B_s - 2\sum_{i=1}^\infty\int_t^T e^{\beta A(s)}Y_s U_s^{(i)}\mathrm{d}H_s^{(i)}\\
&\quad - \sum_{i=1}^\infty\sum_{j=1}^\infty\int_t^T e^{\beta A(s)}U_s^{(i)}U_s^{(j)}\mathrm{d}\left([H^{(i)}, H^{(j)}]_s - \langle H^{(i)}, H^{(j)}\rangle_s\right),
\end{aligned}$$

where we have used the stochastic Lipschitz property of f and the facts that $\mathrm{d}K_s = I_{[Y_s=S_s]}\mathrm{d}K_s$ and $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t$.

For a sufficient large $\beta > 0$, taking expectation on both sides of inequality (3), we get

$$\begin{aligned}
&E[e^{\beta A(t)}|Y_t|^2 + \int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s + \int_t^T e^{\beta A(s)}(|Z_s|^2 + \|U_s\|^2)\mathrm{d}s] \\
&\le c_\beta E[e^{\beta A(T)}|\xi|^2 + \int_t^T e^{\beta A(s)}\frac{|f(s,0,0,0)|^2}{a^2(s)}\mathrm{d}s + 2\int_t^T e^{\beta A(s)}S_s^+\mathrm{d}K_s] \\
&\quad \le c_\beta E[e^{\beta A(T)}|\xi|^2 + \int_t^T e^{\beta A(s)}\frac{|f(s,0,0,0)|^2}{a^2(s)}\mathrm{d}s \\
&\quad + \frac{1}{\theta}\sup_{0\le t\le T} e^{2\beta A(t)}(S_t^+)^2 + \theta(K_T - K_t)^2],
\end{aligned} \tag{4}$$

where $c_\beta > 0$ is a constant depending only on β and $\theta > 0$ is a constant.

On the other hand, from the equation

$$K_T - K_t = Y_t - \xi - \int_t^T f(s, Y_{s-}, Z_s, U_s)\mathrm{d}s + \int_t^T Z_s\mathrm{d}B_s + \sum_{i=1}^\infty \int_t^T U_s^{(i)}\mathrm{d}H_s^{(i)}$$

and the stochastic Lipschitz property of f, we have

$$\begin{aligned}
&E[|K_T - K_t|^2] \\
&\le 5\,E[|Y_t|^2 + |\xi|^2 + \int_t^T (|Z_s|^2 + \|U_s\|^2)\mathrm{d}s \\
&\quad + |\int_t^T f(s, Y_{s-}, Z_s, U_s)\mathrm{d}s|^2] \\
&\le 5\,E[|Y_t|^2 + |\xi|^2 + \int_t^T (|Z_s|^2 + \|U_s\|^2)\mathrm{d}s \\
&\quad + \int_t^T e^{-\beta A(s)}a^2(s)\mathrm{d}s \int_t^T e^{\beta A(s)}\frac{|f(s, Y_{s-}, Z_s, U_s)|^2}{a^2(s)}\mathrm{d}s] \\
&\le 5\,E[|Y_t|^2 + |\xi|^2 + (1 + \frac{6}{\beta})\int_t^T e^{\beta A(s)}(|Z_s|^2 + \|U_s\|^2)\mathrm{d}s \\
&\quad + \frac{2}{\beta}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0,0)|^2}{a^2(s)}\mathrm{d}s + \frac{6}{\beta}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2\mathrm{d}s].
\end{aligned} \tag{5}$$

Combining this with (4), choosing $\theta > 0$ small enough, we derive that there exists a constant $k_\beta > 0$ depending only on β such that

$$E[|K_T - K_t|^2] \le k_\beta E[e^{\beta A(T)}|\xi|^2 + \int_t^T e^{\beta A(s)}\frac{|f(s,0,0,0)|^2}{a^2(s)}\mathrm{d}s + \sup_{0\le t\le T} e^{2\beta A(t)}(S_t^+)^2].$$

We then get the desired result by combining Itô's formula and Burkhölder-Davis-Gundy's inequality. The proof is complete. □

3.2. Existence and uniqueness of solution

We first consider the special case that is the coefficient does not depend on (Y, Z), i.e. $f(\omega, t, y, z) \equiv g(\omega, t)$. We have the following result.

Theorem 1. *Let $\beta > 0$ large enough and $(a(t))_{t\geq 0}$ a nonnegative $\mathcal{F}_t$-adapted process. Assume that $\frac{g}{a} \in H^2(\beta, a)$ and (H4)-(H5) hold. Then RBSDE (1) with data (ξ, g, S) has a solution.*

Proof. For $0 \leq t \leq T$, we define

$$\widetilde{Y}_t = \operatorname*{ess\,sup}_{\nu\geq t} E[\int_0^\nu g(s)ds + S_\nu I_{\{\nu<T\}} + \xi I_{\{\nu=T\}}|\mathcal{F}_t],$$

where ν is an $\mathcal{F}_t$-stopping time. The process $\widetilde{Y}_t$ is called the Snell envelope of the process which is inside ess sup.

By assumptions of the theorem, it is easy to see that $\xi \in L^2(0, a)$, $S_t^+ \in S^2(0, a)$ and $(\int_0^t |g(s)|\mathrm{d}s)_{0\leq t\leq T} \in L^2(0, a)$. Consequently, by Doob-Meyer decomposition theorem in Dellacherie and Meyer [13], there exists a continuous increasing process $(K_t)_{0\leq t\leq T}$ which satisfies $E[|K_T|^2] < \infty$ $(K_0 = 0)$ and a martingale $M_t \in S^2(0, a)$ such that

$$\forall t \in [0, T], \quad \widetilde{Y}_t = M_t - K_t.$$

By Proposition 1, there exists $Z_t \in H^2(0, a)$ and $U_t \in H^2(0, a; l^2)$ such that

$$M_t = M_0 + \int_0^t Z_s \mathrm{d}B_s + \sum_{i=1}^\infty \int_0^t U_s^{(i)} \mathrm{d}H_s^{(i)}, \quad \forall t \in [0, T].$$

Let

$$Y_t = \widetilde{Y}_t - \int_0^t f(s)\mathrm{d}s, \quad \forall t \in [0, T].$$

According to Theorem 3.1 of Ren and Hu [8], we derive that (Y, Z, U, K) verifies

$$Y_t = \xi + \int_t^T g(s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s - \sum_{i=1}^\infty \int_t^T U_s^{(i)} \mathrm{d}H_s^{(i)}$$

and

$$\forall t \in [0, T], Y_t \geq S_t, \int_0^T (Y_{t-} - S_{t-})\mathrm{d}K_t = 0.$$

By Lemma 1, $(Y_t, Z_t, U_t, K_t)_{0\leq t\leq T}$ is a solution of RBSDE (1). □

Furthermore, we have the following uniqueness result.

Proposition 2. *With the same assumptions of Theorem 1, the RBSDE (1) with data (ξ, g, S) has at most one solution.*

Proof. Let (Y,Z,U,K) and (Y',Z',U',K') be two solutions of the RBSDE (1). Let

$$\Delta Y = Y - Y',\ \Delta Z = Z - Z',\ \Delta K = K - K',\ \Delta U = U - U'.$$

For $0 \le t \le T$, we have

$$\Delta Y_t = \Delta K_T - \Delta K_t - \int_t^T \Delta Z_s \mathrm{d}B_s - \sum_{i=1}^{\infty} \int_t^T \Delta U_s^{(i)} \mathrm{d}H_s^{(i)}.$$

Applying Itô's formula to $e^{\beta A(t)}|\Delta Y_t|^2$, we obtain

$$\begin{aligned} -E[e^{\beta A(t)}|\Delta Y_t|^2] = -2\,E[\int_t^T e^{\beta A(s)} \Delta Y_s \mathrm{d}(\Delta K_s)] + E[\int_t^T e^{\beta A(s)} |\Delta Z_s|^2 \mathrm{d}s] \\ + E[\int_t^T e^{\beta A(s)} \|\Delta U_s\|^2 \mathrm{d}s] \end{aligned}$$

Noting that $\int_t^T e^{\beta A(s)} \Delta Y_s \mathrm{d}(\Delta K_s) \le 0$, it follows that $\Delta Y_t = \Delta Z_t = \Delta U_t = 0$ and thus $\Delta K_t = 0$, $0 \le t \le T$ a.s. □

We can now state and prove our main result.

Theorem 2. *Assume (H1)-(H5) hold for a sufficient large β. Then RBSDE (1) with data (ξ, f, S) has a unique solution.*

Proof. Let $\mathcal{H}(\beta,a) = S^2(\beta,a) \times H^2(\beta,a) \times L^2(\beta,a)$. Given $(y,z,u) \in \mathcal{H}(\beta,a)$, consider the following RBSDE:

$$Y_t = \xi + \int_t^T f(s, y_s, z_s, u_s)\mathrm{d}s + K_T - K_t - \int_t^T Z_s \mathrm{d}B_s - \sum_{i=1}^{\infty} \int_t^T U_s^{(i)} \mathrm{d}H_s^{(i)}. \tag{6}$$

By the stochastic Lipschitz assumption on f, we have

$$\frac{|f(t,y_t,z_t,u_t)|^2}{a^2(t)} \le 6[a^2(t)|y_t|^2 + |z_t|^2 + \|u_t\|^2] + 2\frac{|f(t,0,0,0)|^2}{a^2(t)},$$

it follows from (H3) and Theorem 1 that the RBSDE (6) has a unique solution.

Define a mapping Φ from $\mathcal{H}(\beta,a)$ to itself. Let (y',z',u') be another element in $\mathcal{H}(\beta,a)$, set

$$(Y,Z,U) = \Phi(y,z,u),\ (Y',Z',U') = \Phi(y',z',u'),$$

where (Y,Z,U,K) (resp. (Y',Z',U',K')) is the unique solution of the RBSDE (6) associated with data $(\xi, f(t,y_t,z_t,u_t), S)$ (resp. $(\xi, f(t,y'_t,z'_t,u'_t), S)$).

Let

$$\begin{aligned} &\Delta Y = Y - Y', \quad \Delta Z = Z - Z', \quad \Delta U = U - U', \quad \Delta K = K - K' \\ &\Delta y = y - y', \qquad \Delta z = z - z', \qquad \Delta u = u - u' \end{aligned}$$

and

$$\Delta f_s = f(s, y_s, z_s, u_s) - f(s, y'_s, z'_s, u'_s).$$

For $0 \leq t \leq T$, we have

$$\Delta Y_t = \int_t^T \Delta f_s \mathrm{d}s + \Delta K_T - \Delta K_t - \int_t^T \Delta Z_s \mathrm{d}B_s - \sum_{i=1}^{\infty} \int_t^T \Delta U_s^{(i)} \mathrm{d}H_s^{(i)}.$$

Applying Itô's formula to $e^{\beta A(t)}|\Delta Y_t|^2$, using (H1) and the facts $\mathrm{d}K_s = I_{[Y_s=S_s]}\mathrm{d}K_s$ and $\mathrm{d}K'_s = I_{[Y'_s=S_s]}\mathrm{d}K'_s$, we get

$$\begin{aligned}
& e^{\beta A(t)}|\Delta Y_t|^2 + \beta \int_t^T a(s)^2 e^{\beta A(s)} |\Delta Y_s|^2 \mathrm{d}s + \int_t^T e^{\beta A(s)}(|\Delta Z_s|^2 + \|\Delta U_s\|^2)\mathrm{d}s \\
\leq {} & 2\int_t^T e^{\beta A(s)} \Delta Y_s \Delta f_s \mathrm{d}s + 2\int_t^T e^{\beta A(s)} \Delta Y_s \mathrm{d}(\Delta K_s) - 2\int_t^T e^{\beta A(s)} \Delta Y_s \Delta Z_s \mathrm{d}B_s \\
& - 2\sum_{i=1}^{\infty} \int_t^T e^{\beta A(s)} \Delta Y_s \Delta U_s^{(i)} \mathrm{d}H_s^{(i)} - (N_T - N_t) \\
\leq {} & 2\int_t^T e^{\beta A(s)} \Delta Y_s \Delta f_s \mathrm{d}s - 2\int_t^T e^{\beta A(s)} \Delta Y_s \Delta Z_s \mathrm{d}B_s \\
& - 2\sum_{i=1}^{\infty} \int_t^T e^{\beta A(s)} \Delta Y_s \Delta U_s^{(i)} \mathrm{d}H_s^{(i)} - (N_T - N_t) \\
\leq {} & \frac{\beta}{2}\int_t^T a(s)^2 e^{\beta A(s)} |\Delta Y_s|^2 \mathrm{d}s + \frac{6}{\beta}\int_t^T e^{\beta A(s)} |(a(s)^2|\Delta y_s|^2 + |\Delta z_s|^2 + \|\Delta u\|^2)\mathrm{d}s \\
& - 2\int_t^T e^{\beta A(s)} \Delta Y_s \Delta Z_s \mathrm{d}B_s - 2\sum_{i=1}^{\infty} \int_t^T e^{\beta A(s)} \Delta Y_s \Delta U_s^{(i)} \mathrm{d}H_s^{(i)} - (N_T - N_t),
\end{aligned}$$

where $\{N_t,\ 0 \leq t \leq T\}$ is a martingale given by

$$N_t = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_0^t e^{\beta A(s)} \Delta U_s^{(i)} \Delta U_s^{(j)} \mathrm{d}([H^{(i)}H^{(j)}]_s - \langle H^{(i)}H^{(j)}\rangle_s).$$

It follows that

$$\begin{aligned}
& E\left[\int_t^T e^{\beta A(s)}(a(s)^2|\Delta Y_s|^2 + |\Delta Z_s|^2 + \|\Delta U_s\|^2)\mathrm{d}s\right] \\
& \leq (\frac{12}{\beta^2} + \frac{6}{\beta})E\left[\int_t^T e^{\beta A(s)}(a(s)^2|\Delta y_s|^2 + |\Delta z_s|^2 + \|\Delta u_s\|^2)\mathrm{d}s\right].
\end{aligned}$$

For $\beta > 0$ large enough, one can easily to check that Φ is a contraction mapping with the norm

$$\|(Y, Z, U)\|_\beta^2 = E\left[\int_0^T e^{\beta A(s)}(a(s)^2|Y_s|^2 + |Z_s|^2 + \|U_s\|^2)ds\right].$$

Thus, ϕ has a unique fixed point which is the unique solution of RBSDE (1). The theorem is proved. □

Remark 1. When the coefficient f satisfy the standard Lipschitz condition, one can easily to check that assumptions (H3)-(H5) are equivalent to:
(H3') For all $(y, z) \in R \times R^d$, the process $f(\cdot, \cdot, y, z)$ is progressively measurable and such that $\forall t \in [0, T], f(t, 0, 0) \in H^2(0, a)$;
(H4') The terminal condition $\xi \in L^2(0, a)$;
(H5') The obstacle S satisfying $E[\sup_{0 \leq t \leq T}(S_t^+)^2] < \infty$ and $S_T \geq \xi$ a.s.

As a result, Theorem 2 covers the result in Ren and Hu [8] in the case of standard setting.

References

1. N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng and M. Quenez, *Reflected backward SDE's, and related obstacle problems for PDE's*, Ann. Probab. **25** (1997), 702-737.
2. N. El Karoui, E.Pardoux and M. Quenez, *Reflected backward SDEs and American options, in: Robers, L., Talay, D. (eds.), Numerical Methods in Finance*, Cambridge University Press, Cambridge, 1997.
3. J. Cvitanic and I. Karatzas, *Backward stochastic differential equations with reflection and Dynkin games*, Ann. Probab. **24** (1996), 2024-2056.
4. S. Hamadène and J. Lepeltier, *Reflected BSDEs and mixed game problem*, Stoch. Proc. Appl. **85** (2000), 177-188.
5. D. Nualart and W. Schoutens, *Backward stochastic differential equations and Feynman-Kac formula for Lévy processes, with applications in finance*, Bernoulli. **7** (2001), 761-776.
6. D. Nualart and W. Schoutens, *Chaotic and predictable representations for Lévy processes*, Stochastic Process. Appl. **90** (2000), 109-122.
7. K. Bahlali, M. Eddahbi and E. Essaky, *BSDE associated with Lévy processes and application to PDIE*, J. Appl. Math. Stoch. Anal. **16** (2003), 1-17.
8. Y. Ren and L. Hu, *Reflected backward stochastic differential equation driven by Lévy processes*, Statist. Probab. Lett. **77** (2007), 1559-1566.
9. A. Matoussi, *Reflected solutions of backward stochastic differential equations with continuous coefficient.* Statist. Probab. Lett. **34** (1997), 347-354.
10. J. Lepeltier, A. Matoussi and M. Xu, *Reflected backward stochastic differential equations under monotonicity and general increasing growth conditions*, Adv. in Appl. Probab. **37** (2005), 134-159.
11. N. El Karoui and S. Huang, *A general result of existence and uniqueness of backward stochastic differential equations, in: El Karoui, N., Mazliak, L. (Eds.), Backward Stochastic Differential Equations.* Pitman Research Notes Mathematical Series, **364**, Longman, Harlow, 1997.
12. Bender, C., Kohlmann, M. *BSDEs with stochastic Lipschitz condition*, http://cofe.uni-konstanz.de/Papers/dp00_08.pdf (2000).
13. C. Dellacherie and P. Meyer, *Probabilités et Potentiel V-VIII*, Hermann, Paris, 1980.

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