

OSCILLATION OF SECOND-ORDER FUNCTIONAL DYNAMIC EQUATIONS OF EMDEN-FOWLER-TYPE ON TIME SCALES

S. H. SAKER

ABSTRACT. The purpose of this paper is to establish some sufficient conditions for oscillation of solutions of the second-order functional dynamic equation of Emden-Fowler type

$$\left[a(t)x^\Delta(t) \right]^\Delta + p(t) |x^\gamma(\tau(t))| |x^\Delta(t)|^{1-\gamma} \operatorname{sgn} x(\tau(t)) = 0, \quad t \geq t_0,$$

on a time scale \mathbb{T} , where $\gamma \in (0, 1]$, a , p and τ are positive rd-continuous functions defined on \mathbb{T} , and $\lim_{t \rightarrow \infty} \tau(t) = \infty$. Our results include some previously obtained results for differential equations when $\mathbb{T} = \mathbb{R}$. When $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$, the results are essentially new for difference and q -difference equations and can be applied on different types of time scales. Some examples are worked out to demonstrate the main results.

AMS Mathematics Subject Classification : 34K11, 39A10, 39A99.

Key word and phrases : Oscillation, dynamic equations, time scales.

1. Introduction

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [10] is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice - once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale \mathbb{T} , which may be

Received November 30, 2009. Revised December 9, 2009. Accepted December 23, 2009.

© 2010 Korean SIGCAM and KSCAM.

an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [13]), i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [5]). There are applications of dynamic equations on time scales to *quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics*. A recent cover story article in New Scientist [26] discusses several possible applications. Since then several authors have expounded on various aspects of this new theory [6]. The book on the subject of time scale, i.e., measure chain, by Bohner and Peterson [5] summarizes and organizes much of time scale calculus.

In this paper, we are concerned with oscillation of the second-order functional dynamic equation of Emden–Fowler type

$$(ax^\Delta)^\Delta(t) + p(t) |x^\gamma(\tau(t))| |x^\Delta(t)|^{1-\gamma} \operatorname{sgn} x(\tau(t)) = 0, \quad (1.1)$$

on a time scale \mathbb{T} . Throughout this paper, we will assume the following hypotheses:

- (h_1). $\gamma \in (0, 1]$, a and p are positive, rd -continuous functions,
- (h_2). $\tau : \mathbb{T} \rightarrow \mathbb{T}$, and $\lim_{t \rightarrow \infty} \tau(t) = \infty$.

Equation (1.1) is called a delay dynamic equation if $\tau(t) < t$ and is called an advanced dynamic equation if $\tau(t) > t$. Since, we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1), we mean a function $x^{[0]}(t) \in C_r^1([t_0, \infty), \mathbf{R})$ such that $x^{[1]}(t) \in C_r^1([t_0, \infty), \mathbf{R})$ satisfying (1.1) for all $t \geq t_0$, where C_r is the space of rd -continuous functions and

$$x^{[0]} = x, \quad x^{[1]} = ax^\Delta \quad \text{and} \quad x^{[2]} = \left(x^{[1]}\right)^\Delta$$

are called the Δ -quasi derivatives of the solution $x(t)$. Any solution of (1.1) is said to be proper if it is defined on the interval $[t_0, \infty)_{\mathbb{T}}$ and is nontrivial in any neighborhood of infinity. So the solutions vanishing in some neighborhood of infinity will be excluded from our consideration and we are interesting only in the asymptotic behavior of the proper solutions. A proper solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory in case every solution of is oscillatory and is nonoscillatory if all its solutions are nonoscillatory.

In the last few years, there has been increasing interest in obtaining sufficient conditions for the qualitative properties of solutions of different classes of dynamic equations on time scales, for contribution we refer the readers to the papers [1, 2, 3, 4, 7, 8, 9, 12, 17, 19, 20, 21, 22, 23, 24, 25] and the references cited therein. For completeness in the following, we recall some of the related results that has been established for the second-order dynamic equations on time scales that serve and motivate the contents of this paper. In [4] Akin-Bohner and Hoffacker considered the second order dynamic equation

$$x^{\Delta\Delta}(t) + p(t)(x^\sigma)^\gamma = 0 \tag{1.2}$$

of Emden-Fowler type and established some necessary and sufficient conditions for oscillation of all solutions when $\gamma > 1$ and $0 < \gamma < 1$, where $\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}$ is the forward jump operator defined on the time scale. Their results cannot be applied in the case when $\gamma = 1$ and applied only on discrete time scales.

In [1] Agarwal, Bohner and Saker considered the delay dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0 \tag{1.3}$$

on a time scale \mathbb{T} , where the function p is rd-continuous such that $p(t) > 0$ for all $t \in \mathbb{T}$, $\tau : \mathbb{T} \rightarrow \mathbb{T}$ and proved that if there exists a differentiable function δ such that

$$\lim_{t \rightarrow \infty} \int_{t_0}^t \left\{ \delta^2(\sigma(s))p(s) \frac{\tau(s)}{\sigma(s)} - (\delta^\Delta(s))^2 \right\} \Delta s = \infty, \tag{1.4}$$

then every solution of (1.3) is oscillatory.

In [3] Akin-Bohner, Bohner and Saker considered the Emden-Fowler dynamic equation

$$(ax^\Delta)^\Delta(t) + p(t)x^\gamma(\sigma(t)) = 0 \tag{1.5}$$

and proved that if

$$\int_{t_0}^\infty \frac{\Delta t}{a(t)} = \infty \tag{1.6}$$

and there exists a differentiable function δ such that

$$\limsup_{t \rightarrow \infty} \int_a^t \left[p(s) (\delta^\sigma(s))^2 - K^{\gamma-1} a(s) (\delta^\Delta(s))^2 \right] \Delta s = \infty, \text{ for } \gamma > 1, \tag{1.7}$$

$$\limsup_{t \rightarrow \infty} \int_a^t \left[p(s) (\delta^\sigma(s))^2 - K^{\gamma-1} (\sigma(s))^{\gamma-1} a(s) (\delta^\Delta(s))^2 \right] \Delta s = \infty, \text{ } 0 < \gamma \leq 1 \tag{1.8}$$

for all constants $K > 0$ then every solution of (1.5) is oscillatory.

In [12] Han, Sun and Shi considered the delay equation

$$x^{\Delta\Delta}(t) + p(t)x^\gamma(\tau(t)) = 0, \tag{1.9}$$

where γ is a quotient of odd positive integers, p is a positive, real-valued rd-continuous function, $\tau(t) : \mathbb{T} \rightarrow \mathbb{T}$, is a positive, real-valued rd-continuous function such that $\tau(t) \leq t$ and $\lim_{t \rightarrow \infty} \tau(t) = \infty$. The authors extended the conditions (1.7) and (1.8) and proved that if (1.6) hold and there exists a differentiable

function δ such that

$$\limsup_{t \rightarrow \infty} \int_a^t \left[p(s) \left(\frac{\tau(s)}{\sigma(s)} \right)^\gamma (\delta^\sigma(s))^2 - K^{\gamma-1} (\delta^\Delta(s))^2 \right] \Delta s = \infty, \text{ for } \gamma > 1, \quad (1.10)$$

$$\limsup_{t \rightarrow \infty} \int_a^t \left[p(s) \left(\frac{\tau(s)}{\sigma(s)} \right)^\gamma (\delta^\sigma(s))^2 - \frac{(\sigma(s))^{\gamma-1} (\delta^\Delta(s))^2}{K^{1-\gamma}} \right] \Delta s = \infty, \text{ for } \gamma < 1 \quad (1.11)$$

for all constants $K > 0$, then every solution of (1.9) is oscillatory. We Note that all above results are given for the ordinary or the delay equations and nothing is known regarding the oscillation of advanced equations. So one of our aims in this paper is to consider this case and establish some sufficient conditions for oscillation of Emden-Fowler advanced dynamic equations.

Dynamic equation (1.1) in its general form, includes second-order differential and difference equations depends on the time scale \mathbb{T} . When $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\mu(t) = 0$, $x^\Delta(t) = x'(t)$ and (1.1) becomes the functional differential equation

$$(a(t)x'(t))' + p(t)|x^\gamma(\tau(t))| \left| x'(t) \right|^{1-\gamma} \operatorname{sgn} x(\tau(t)), \quad [t_0, \infty) \quad (1.12)$$

When $\tau(t) = t$ and $a(t) = 1$, the equation (1.12) becomes

$$x''(t) + p(t)|x^\gamma(t)| \left| x'(t) \right|^{1-\gamma} \operatorname{sgn} x(t) = 0, \quad t \in [t_0, \infty). \quad (1.13)$$

Oscillation and asymptotic properties of (1.13), has been investigated in the literature by some authors, we refer the reader to the papers [15, 16]. When $\gamma = 1$, the equation (1.13) becomes

$$x''(t) + p(t)x(t) = 0, \quad t \in [t_0, \infty). \quad (1.14)$$

This equation has been investigated in the literature by many authors. Here we present some of these results that serve and motivate the contents of this paper. Hille [11] proved that every solution of (1.14) oscillates if

$$\liminf_{t \rightarrow \infty} t \int_t^\infty p(s) ds > \frac{1}{4}. \quad (1.15)$$

Nehari [18] proved that if

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t s^2 p(s) ds > \frac{1}{4}, \quad (1.16)$$

then every solution of (1.14) oscillates. Lomtatidze [16], extended the condition (1.15) and proved that if

$$\liminf_{t \rightarrow \infty} t^\gamma \int_t^\infty p(s) ds > \frac{\gamma^{\gamma-1}}{(\gamma+1)^{\gamma+1}}, \quad (1.17)$$

then every solution of (1.13) oscillates. Lomtatidze [16] extended the condition (1.16) and proved that if

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t s^{\gamma+1} p(s) ds > \frac{\gamma^\gamma}{(\gamma + 1)^{\gamma+1}}, \tag{1.18}$$

then every solution of (1.13) oscillates. Also it was proved that

$$\gamma \limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_0}^t s^{\gamma+1} p(s) ds > 1, \tag{1.19}$$

then every solution of (1.13) oscillates.

When $\mathbb{T} = \mathbb{N}$, $\sigma(n) = n + 1$, $\mu(n) = 1$, $x^\Delta(n) = \Delta x(n) = x(n + 1) - x(n)$. In this case, (1.1) becomes the difference equation

$$\Delta(a(n)\Delta x(n)) + p(n) |x^\gamma(\tau(n))| |\Delta x(n)|^{1-\gamma} \operatorname{sgn} x(\tau(n)) = 0, \quad n \in [n_0, \infty)_{\mathbb{N}} \tag{1.20}$$

If $\mathbb{T} = h\mathbb{N}_0$, $h > 0$, $\sigma(t) = t + h$, $\mu(t) = h$, $x^\Delta(t) = \Delta_h x(t) := \frac{x(t+h) - x(t)}{h}$. In this case, (1.1) becomes the difference equation with step size h

$$\Delta_h(a(t)\Delta_h x(t)) + p(t) |x^\gamma(\tau(t))| |\Delta_h x(t)|^{1-\gamma} \operatorname{sgn} x(\tau(t)) = 0, \quad t \in [0, \infty)_{h\mathbb{N}_0} \tag{1.21}$$

If $\mathbb{T} = q^{\mathbb{N}_0} = \{t : t = q^k, k \in \mathbb{N}_0, q > 1\}$, $\sigma(t) = qt$, $\mu(t) = (q - 1)t$, $x^\Delta(t) = D_q x(t) := \frac{x(qt) - x(t)}{(q-1)t}$ (D_q is the so-called quantum derivative which has important applications in quantum mechanics [13]). In this case (1.1) becomes

$$D_q(a(t)D_q x(t)) + p(t) |x^\gamma(\tau(t))| |D_q x(t)|^{1-\gamma} \operatorname{sgn} x(\tau(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}} \tag{1.22}$$

Also, the results can be applied to many other time scales. For example, if $\mathbb{T} = \mathbb{N}_0^2 = \{t = n^2 : n \in \mathbb{N}_0\}$, we have $\sigma(t) = (\sqrt{t} + 1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$, $x^\Delta(t) = \Delta_0 x(t) = \frac{x((\sqrt{t}+1)^2) - x(t)}{1+2\sqrt{t}}$, and (1.1) becomes

$$\Delta_0(a(t)\Delta_0 x(t)) + p(t) |x^\gamma(\tau(t))| |\Delta_0 x(t)|^{1-\gamma} \operatorname{sgn} x(\tau(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}} \tag{1.23}$$

When $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}\}$ where t_n are the so-called harmonic numbers defined by

$$t_0 = 0, \quad t_n = \sum_{k=1}^n \frac{1}{k}, \quad n \in \mathbb{N}_0,$$

we have $\sigma(t_n) = t_{n+1}$, $\mu(t_n) = \frac{1}{n+1}$, $y^\Delta(t) = \Delta_{t_n} y(t_n) = (n+1)\Delta y(t_n)$ and (1.1) becomes

$$\Delta_{t_n}(a(t)\Delta_{t_n} x(t)) + p(t) |x^\gamma(\tau(t))| |\Delta_{t_n} x(t)|^{1-\gamma} \operatorname{sgn} x(\tau(t)) = 0. \tag{1.24}$$

When $\mathbb{T} = \mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt{t^2 + 1}$ and $\mu(t) = \sqrt{t^2 + 1} - t$, $x^\Delta(t) = \Delta_2 x(t) = (x(\sqrt{t^2 + 1}) - x(t))/\sqrt{t^2 + 1} - t$, and (1.1) becomes the second-order difference equation

$$\Delta_2(a(t)\Delta_2 x(t)) + p(t) |x^\gamma(\tau(t))| |\Delta_2 x(t)|^{1-\gamma} \operatorname{sgn} x(\tau(t)) = 0. \tag{1.25}$$

When $\mathbb{T} = \mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt[3]{t^3 + 1}$ and $\mu(t) = \sqrt[3]{t^3 + 1} - t$, $x^\Delta(t) = \Delta_3 x(t) = (x(\sqrt[3]{t^3 + 1}) - x(t)) / \sqrt[3]{t^3 + 1} - t$, and (1.1) becomes the second-order difference equation

$$\Delta_3(a(t)\Delta_3 x(t)) + p(t)|x^\gamma(\tau(t))||\Delta_3 x(t)|^{1-\gamma} \operatorname{sgn} x(\tau(t)) = 0. \tag{1.26}$$

The natural question now is: If the oscillation conditions (1.17), (1.18) and (1.19) due Lomtatadiz for second order differential equation (1.13) can be extended to the functional dynamic equation (1.1) on time scales?. The purpose of this paper is to give an affirmative answer to this question and establish some sufficient conditions for oscillation of the equation (1.1). Our results are new for the second order dynamic equations and include the previously obtained results for differential equations (1.13) and (1.14). For the equations (1.12), (1.20)-(1.26) our results are essentially new. The paper is organized as follows: In section 2, we prove the main results. In Section 3, we give some applications of the results and establish some sufficient conditions for oscillation of different types of equations on different time scales. In Section4, we give some examples to demonstrate the main results.

2. Main Results

In this section, we establish some sufficient conditions for oscillation of (1.1). We note that if $x(t)$ is a solution of (1.1) then $z = -x$ is also solution of (1.1). Thus, concerning nonoscillatory solutions of (1.1) we can restrict our attention only to the positive ones.

2.1. Case when $\tau(t) > t$. In this subsection, we establish some sufficient conditions for oscillation for advanced equation. We introduce the following notations:

$$\begin{aligned} p_* &:= \liminf_{t \rightarrow \infty} \frac{t^\gamma}{a^\gamma(t)} \int_{\sigma(t)}^\infty Q(s) \Delta s & q_* &:= \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{a^\gamma(t)} Q(s) \Delta s, \\ r_* &:= \liminf_{t \rightarrow \infty} \frac{t^\gamma w^\sigma(t)}{a^\gamma(t)}, & R &:= \limsup_{t \rightarrow \infty} \frac{t^\gamma w^\sigma(t)}{a^\gamma(t)}, \\ Q(t) &:= \gamma p(t) \left(\frac{a(t)A(t,T)}{a(t)A(t,T) + \sigma(t) - t} \right)^\gamma, & A(t,T) &:= \int_T^t \left(\frac{1}{a(\tau)} \right) \Delta \tau, \end{aligned}$$

and assume that $l := \liminf_{t \rightarrow \infty} \frac{t}{\sigma(t)}$. Note that $0 \leq l \leq 1$.

Theorem 2.1. *Assume that $(h_1) - (h_2)$, (1.6) hold and $\tau(t) > t$. Let $x(t)$ be a solution of (1.1) and make the Riccati substitution*

$$w := \left(\frac{x^{[1]}}{x} \right)^\gamma.$$

Then $w(t) > 0$ and

$$w^\Delta(t) + Q(t) + \frac{\gamma}{a(t)} (w^\sigma)^{1+\frac{1}{\gamma}}(t) \leq 0, \quad \text{for } t \in [T, \infty)_{\mathbb{T}}. \tag{2.1}$$

Proof. Let x be as in the statement of this theorem and without loss of generality we assume that there is $t_1 > t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$. Then from (1.1), we see that $x^{[2]}(t) = -p(t) |x^\gamma(\tau(t))| |x^\Delta(t)|^{1-\gamma} < 0$ for $t \geq t_1$ and there exists $T > t_1$ such that $x^{[1]}(t)$ is decreasing and either positive or negative for $t \geq T$. We claim that $x^{[1]}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not, then there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $x^{[1]}(t_2) =: c < 0$. Then $x^{[1]}(t) \leq x^{[2]}(t_2) = c$, for $t \in [t_2, \infty)_{\mathbb{T}}$, and therefore $x^\Delta(t) \leq \frac{c}{a(t)}$, for $t \in [t_2, \infty)_{\mathbb{T}}$. Integrating, we find by (1.6) that

$$x(t) = x(t_2) + \int_{t_2}^t x^\Delta(s) \Delta s \leq x(t_2) + c \int_{t_2}^t \frac{\Delta s}{a(s)} \rightarrow -\infty \text{ as } t \rightarrow \infty$$

which implies that $x(t)$ is eventually negative and this is a contradiction. Then there exists $T > t_1$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) < 0$ for $t \geq T$. By the quotient rule ([5, Theorem 1.20]), we have

$$\begin{aligned} w^\Delta(t) &= \left(\frac{(x^{[1]})^\gamma}{x^\gamma} \right)^\Delta = \frac{x^\gamma \left((x^{[1]})^\gamma \right)^\Delta - (x^\gamma)^\Delta (x^{[1]})^\gamma}{x^\gamma (x^\sigma)^\gamma} \\ &= \frac{\left((x^{[1]})^\gamma \right)^\Delta}{(x^\sigma)^\gamma} - \frac{(x^\gamma)^\Delta (x^{[1]})^\gamma}{x^\gamma (x^\sigma)^\gamma} \end{aligned} \tag{2.2}$$

By the Pötzsche chain rule ([5, Theorem 1.90]), if $f^\Delta(t) < 0$ and $0 < \gamma \leq 1$, we obtain

$$\begin{aligned} (f^\gamma(t))^\Delta &= \gamma f^\Delta(t) \int_0^1 [f(t) + h\mu(t)f^\Delta(t)]^{\gamma-1} dh \\ &= \gamma f^\Delta(t) \int_0^1 [(1-h)f(t) + hf^\sigma(t)]^{\gamma-1} dh \\ &\leq \gamma f^\Delta(t) \int_0^1 (f(t))^{\gamma-1} dh = \gamma (f(t))^{\gamma-1} f^\Delta(t) \end{aligned} \tag{2.3}$$

By putting $f(t) = x^{[1]}(t)$, since x is increasing and $x^{[1]}$ is decreasing, we have

$$\left((x^{[1]})^\gamma \right)^\Delta \leq \gamma (x^{[1]})^{\gamma-1} (x^{[1]})^\Delta = \gamma (x^{[1]})^{\gamma-1} (x^{[2]}).$$

This, (1.1) and (2.2) implies that

$$w^\Delta(t) \leq -\gamma p(t) \left(\frac{x(\tau)}{x^\sigma} \right)^\gamma - \frac{(x^\gamma)^\Delta (x^{[1]})^\gamma}{x^\gamma (x^\sigma)^\gamma}. \tag{2.4}$$

By the Pötzsche chain rule ([5, Theorem 1.90]), if $f^\Delta(t) > 0$ and $0 < \gamma \leq 1$, we obtain

$$\begin{aligned} (f^\gamma(t))^\Delta &= \gamma f^\Delta(t) \int_0^1 [f(t) + h\mu(t)f^\Delta(t)]^{\gamma-1} dh \\ &= \gamma f^\Delta(t) \int_0^1 [(1-h)f(t) + hf^\sigma(t)]^{\gamma-1} dh \\ &\geq \gamma f^\Delta(t) \int_0^1 (f^\sigma(t))^{\gamma-1} dh = \gamma (f^\sigma(t))^{\gamma-1} f^\Delta(t) \end{aligned} \quad (2.5)$$

So that from (2.5), by putting $f(t) = x(t)$, since x is increasing and $x^{[1]}$ is decreasing, we have for $0 < \gamma \leq 1$

$$\begin{aligned} \frac{(x^\gamma)^\Delta (x^{[1]})^\gamma}{x^\gamma (x^\sigma)^\gamma} &\geq \frac{\gamma x^{[1]} (x^\sigma)^{\gamma-1} (x^{[1]})^\gamma}{ax^\gamma (x^\sigma)^\gamma} = \frac{\gamma x^{[1]} (x^{[1]})^\gamma}{ax^\gamma (x^\sigma)} \\ &\geq \frac{\gamma (x^{[1]})^\sigma ((x^{[1]})^\sigma)^\gamma}{a (x^\sigma)^\gamma x^\sigma} = \frac{\gamma}{a} (w^\sigma)^{1+\frac{1}{\gamma}}. \end{aligned}$$

Substituting in (2.4), we have

$$w^\Delta(t) \leq -\gamma p(t) \left(\frac{x(\tau)}{x^\sigma} \right)^\gamma - \frac{\gamma}{a} (w^\sigma)^{1+\frac{1}{\gamma}}. \quad (2.6)$$

Next consider the coefficient of $p(t)$ in (2.6). Since $x^\sigma = x(t) + \mu(t)x^\Delta$, we have

$$\frac{x^\sigma}{x(t)} = 1 + \mu(t) \frac{x^\Delta}{x(t)} = 1 + \frac{\mu(t)}{a(t)} \frac{x^{[1]}(t)}{x(t)}.$$

Also since $x^{[1]}(t)$ is decreasing, we have

$$x(t) = x(T) + \int_T^t \frac{x^{[1]}(\tau)}{a(\tau)} \Delta\tau \geq x(T) + x^{[1]}(t) \int_T^t \frac{1}{a(\tau)} \Delta\tau > x^{[1]}(t) \int_T^t \left(\frac{1}{a(\tau)} \right) \Delta\tau.$$

It follows that

$$x(t)/x^{[1]}(t) \geq \int_T^t \left(\frac{1}{a(\tau)} \right) \Delta\tau = A(t, T). \quad (2.7)$$

Hence

$$\frac{x^\sigma}{x(t)} = 1 + \mu(t) \frac{x^\Delta}{x(t)} = 1 + \frac{\mu(t)}{a(t)} \frac{x^{[1]}(t)}{x(t)} \leq \frac{a(t)A(t, T) + \mu(t)}{a(t)A(t, T)}.$$

Hence, we have

$$\frac{x(t)}{x^\sigma} \geq \frac{a(t)A(t, T)}{a(t)A(t, T) + \mu(t)} = \frac{a(t)A(t, T)}{a(t)A(t, T) + \sigma(t) - t}.$$

So that

$$\frac{x(\tau)}{x^\sigma} = \frac{x(\tau)}{x} \frac{x}{x^\sigma} \geq \frac{x(\tau)}{x(t)} \frac{a(t)A(t, T)}{a(t)A(t, T) + \sigma(t) - t}. \quad (2.8)$$

Now, since $\tau(t) > t$ and $x(t)$ is increasing, we have $x(\tau)/x(t) > 1$. This and (2.8) show that

$$x(\tau)/(x)^\sigma \geq a(t)A(t, T)/(a(t)A(t, T) + \sigma(t) - t). \tag{2.9}$$

Substituting from (2.9) into (2.6), we have the inequality (2.1) and this completes the proof. \square

In order for the definition of p_* to make sense, we assume that

$$\int_{t_0}^\infty Q(s)\Delta s < \infty. \tag{2.10}$$

Theorem 2.2. Assume that $(h_1) - (h_2)$, (1.6) hold and $a^\Delta(t) \geq 0$. Furthermore, assume that

$$p_* > \frac{\gamma^\gamma}{l^{\gamma^2(\gamma+1)^{\gamma+1}}}, \tag{2.11}$$

or

$$p_* + q_* > \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.12}$$

Then every solution of (1.1) is oscillatory.

Proof. Suppose the contrary and assume that (1.1) has a nooscillatory solution $x(t)$. Without loss of generality we may assume that $x(t) > 0$, $x(\tau(t)) > 0$ for $t \geq T$ where T is chosen so large. Define the function $w(t)$ by the Riccati substitution as in Theorem 2.1. Then, we get from (2.1) that

$$-w^\Delta(t) > Q(t) + \gamma a^{-\frac{1}{\gamma}}(t)(w^\sigma(t))^{\frac{\gamma+1}{\gamma}}, \quad \text{for } t \in [T, \infty)_{\mathbb{T}}. \tag{2.13}$$

Since $x^{[1]}(t) = a(t)x^\Delta(t)$, integrating in (T, t) , we obtain

$$x(t) = x(T) + \int_T^t \frac{x^{[1]}}{a(s)}(s)\Delta s.$$

Taking into account that $x^{[1]}(t)$ is positive and decreasing, we get,

$$x(t) \geq x(T) + x^{[1]}(t) \int_T^t \frac{1}{a(s)}\Delta s > x^{[1]}(t) \int_T^t \frac{1}{a(s)}\Delta s.$$

It follows that

$$w(t) = \left(x^{[1]}/x\right)^\gamma < \left(\int_{t_0}^t \frac{1}{a(s)}\Delta s\right)^{-\gamma}, \quad \text{for } t \in [T, \infty)_{\mathbb{T}},$$

which implies using (1.6) that $\lim_{t \rightarrow \infty} w(t) = 0$. First, we assume (2.11) holds. Integrating (2.13) from $\sigma(t)$ to ∞ and using $\lim_{t \rightarrow \infty} w(t) = 0$, we have

$$w^\sigma(t) \geq \int_{\sigma(t)}^\infty Q(s)\Delta s + \gamma \int_{\sigma(t)}^\infty r^{-\frac{1}{\gamma}}(s)(w^\sigma(s))^{\frac{1}{\gamma}}w^\sigma(s)\Delta s. \tag{2.14}$$

where $r(t) = a^\gamma(t)$. It follows from (2.14) that

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \geq \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty Q(s)\Delta s + \gamma \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty r^{-\frac{1}{\gamma}}(s)(w^\sigma(s))^{\frac{1}{\gamma}}w^\sigma(s)\Delta s. \tag{2.15}$$

Let $\epsilon > 0$, then by the definition of p_* and r_* we can pick $t_1 \in [T, \infty)_{\mathbb{T}}$, sufficiently large, so that

$$\frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty Q(s)\Delta s \geq p_* - \epsilon, \quad \text{and} \quad \frac{t^\gamma w^\sigma(t)}{r(t)} \geq r_* - \epsilon, \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \quad (2.16)$$

From (2.15) and (2.16) and using the fact $r^\Delta(t) \geq 0$, we get that

$$\begin{aligned} \frac{t^\gamma w^\sigma(t)}{r(t)} &\geq (p_* - \epsilon) + \gamma \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty r^{-\frac{1}{\gamma}}(s) \frac{s(w^\sigma(s))^{\frac{1}{\gamma}} s^\gamma w^\sigma(s)}{s^{\gamma+1}} \Delta s \\ &\geq (p_* - \epsilon) + (r_* - \epsilon)^{1+\frac{1}{\gamma}} \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty \frac{\gamma r(s)}{s^{\gamma+1}} \Delta s \\ &\geq (p_* - \epsilon) + (r_* - \epsilon)^{1+\frac{1}{\gamma}} t^\gamma \int_{\sigma(t)}^\infty \frac{\gamma}{s^{\gamma+1}} \Delta s \end{aligned} \quad (2.17)$$

Using the Pötzsche chain rule ([5, Theorem 1.90]), we get

$$\left(\frac{-1}{s^\gamma}\right)^\Delta = \gamma \int_0^1 \frac{1}{[s + h\mu(s)]^{\gamma+1}} dh \leq \int_0^1 \left(\frac{\gamma}{s^{\gamma+1}}\right) dh = \frac{\gamma}{s^{\gamma+1}}. \quad (2.18)$$

Then from (2.17) and (2.18), we have

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \geq (p_* - \epsilon) + (r_* - \epsilon)^{1+\frac{1}{\gamma}} \left(\frac{t}{\sigma(t)}\right)^\gamma.$$

Taking the \liminf of both sides as $t \rightarrow \infty$ we get that $r_* \geq a_* - \epsilon + (r_* - \epsilon)^{1+\frac{1}{\gamma}} l^\gamma$. Since $\epsilon > 0$ is arbitrary, we get

$$p_* \leq r_* - r_*^{1+\frac{1}{\gamma}} l^\gamma. \quad (2.19)$$

Using the inequality $Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}$ with $B = 1$ and $A = l^\gamma$, we get that

$$p_* \leq \frac{\gamma^\gamma}{l^{\gamma^2(\gamma+1)\gamma+1}},$$

which contradicts (2.11). Next, we assume (2.12) holds. Multiplying both sides (2.13) by $\frac{t^{\gamma+1}}{r(t)}$, and integrating from T to t ($t \geq T$), we get

$$\int_T^t \frac{s^{\gamma+1}}{r(s)} w^\Delta(s) \Delta s \leq - \int_T^t \frac{s^{\gamma+1}}{r(s)} Q(s) \Delta s - \gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s.$$

Using integration by parts, we obtain

$$\begin{aligned} \frac{t^{\gamma+1} w(t)}{r(t)} &\leq \frac{T^{\gamma+1} w(T)}{r(T)} + \int_T^t \left(\frac{s^{\gamma+1}}{r(s)}\right)^\Delta w^\sigma(s) \Delta s - \int_T^t \frac{s^{\gamma+1}}{r(s)} Q(s) \Delta s \\ &\quad - \gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s. \end{aligned}$$

By the quotient rule and applying the Pötzsche chain rule,

$$\left(\frac{s^{\gamma+1}}{r(s)}\right)^\Delta = \frac{(s^{\gamma+1})^\Delta}{r^\sigma(s)} - \frac{s^{\gamma+1}r^\Delta(s)}{r(s)r^\sigma(s)} \leq \frac{(\gamma+1)\sigma^\gamma(s)}{r^\sigma(s)} \leq \frac{(\gamma+1)\sigma^\gamma(s)}{r(s)}. \tag{2.20}$$

Hence

$$\begin{aligned} \frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s + \int_T^t (\gamma+1) \left(\frac{\sigma^\gamma(s)w^\sigma(s)}{r(s)}\right) \Delta s \\ &\quad - \gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s. \end{aligned}$$

Let $\epsilon > 0$ be given, then using the definition of l , we can assume, without loss of generality, that T is sufficiently large so that $\frac{s}{\sigma(s)} > l - \epsilon$, $s \geq T$. It follows that

$$\sigma(s) \leq Ks, \quad s \geq T \quad \text{where} \quad K := \frac{1}{l - \epsilon}.$$

We then get that

$$\begin{aligned} \frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s \\ &\quad + \int_T^t \left\{ (\gamma+1)K^\gamma \frac{s^\gamma w^\sigma(s)}{r(s)} - \gamma \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \right\} \Delta s. \end{aligned}$$

Let $u(s) := \frac{s^\gamma w^\sigma(s)}{r(s)}$, then $u^\lambda(s) = \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^\lambda$, where $\lambda = \frac{\gamma+1}{\gamma}$. It follows that

$$\frac{t^{\gamma+1}w(t)}{r(t)} \leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s + \int_T^t \{(\gamma+1)K^\gamma u(s) - \gamma u^\lambda(s)\} \Delta s.$$

Again, using the inequality $Bu - Au^\lambda \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}$, where A, B are constants, we get

$$\begin{aligned} \frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s + \int_T^t \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{[(\gamma+1)K^\gamma]^{\gamma+1}}{\gamma^\gamma} \Delta s \\ &\leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s + K^{\gamma(\gamma+1)}(t - T). \end{aligned}$$

It follows from this that

$$\frac{t^\gamma w(t)}{r(t)} \leq \frac{T^{\gamma+1}w(T)}{tr(T)} - \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{r(s)}p(s)\Delta s + K^{\gamma(\gamma+1)}\left(1 - \frac{T}{t}\right).$$

Since $w^\sigma(t) \leq w(t)$, we get

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \leq \frac{T^{\gamma+1}w(T)}{tr(T)} - \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{r(s)}Q(s)\Delta s + K^{\gamma(\gamma+1)}\left(1 - \frac{T}{t}\right).$$

Taking the lim sup of both sides as $t \rightarrow \infty$ we obtain $R \leq -q_* + K^{\gamma(\gamma+1)} = -q_* + \frac{1}{(l-\epsilon)^{\gamma(\gamma+1)}}$. Since $\epsilon > 0$ is arbitrary, we get that $R \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}}$. Using this and the inequality (2.19), we get

$$p_* \leq r_* - l^\gamma r_*^{1+\frac{1}{\gamma}} \leq r_* \leq R \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Therefore

$$p_* + q_* \leq \frac{1}{l^{\gamma(\gamma+1)}},$$

which contradicts (2.12). The proof is complete. \square

From Theorem 2.1, we have the following results immediately.

Corollary 2.1. *Assume that $(h_1) - (h_2)$, (1.6) hold, and $a^\Delta(t) \geq 0$. Furthermore, assume that*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{a^\gamma(s)} Q(s) \Delta s > \frac{\gamma^\gamma}{l^{\gamma^2(\gamma+1)\gamma+1}}. \quad (2.21)$$

Then every solution of (1.1) is oscillatory.

Corollary 2.2. *Assume that $(h_1) - (h_2)$, (1.6) hold, and $a^\Delta(t) \geq 0$. Furthermore, assume that*

$$\liminf_{t \rightarrow \infty} \frac{t^\gamma}{a^\gamma(t)} \int_{\sigma(t)}^\infty Q(s) \Delta s > \frac{1}{l^{\gamma(\gamma+1)}}. \quad (2.22)$$

Then every solution of (1.1) is oscillatory.

Corollary 2.3. *Assume that $(h_1) - (h_2)$, (1.6) hold and $a^\Delta(t) \geq 0$. Furthermore, assume that*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{a^\gamma(s)} Q(s) \Delta s > \frac{1}{l^{\gamma(\gamma+1)}}. \quad (2.23)$$

Then every solution of (1.1) is oscillatory.

2.2. Case when $\tau(t) \leq t$. In this section, we consider the case when $\tau(t) \leq t$ and establish some sufficient conditions for oscillation. First, we will prove the following lemma which will be useful in the proof of the main results.

Lemma 2.1. *Assume that (1.6) holds, $a^\Delta(t) \geq 0$, and*

$$\int_{t_0}^\infty \tau^\gamma(t) p(t) \Delta t = \infty. \quad (2.24)$$

Suppose that (1.1) has a positive solution x on $[t_0, \infty)_{\mathbb{T}}$. Then there exists a $T \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, so that

- (i). $x^{\Delta\Delta}(t) < 0$, $x(t) > tx^\Delta(t)$ for $t \in [T, \infty)_{\mathbb{T}}$;
- (ii). $x(t)/t$ is strictly decreasing on $[T, \infty)_{\mathbb{T}}$.

Proof. Assume x is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. (Note that in the case when $x(t)$ is negative the proof is similar, since the transformation $y(t) = -x(t)$ transforms the (1.1) into the same form). Since x is a positive solution of (1.1), we see from Theorem 2.1 that $x^{[1]}(t) > 0$ and strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. We show that $x^{\Delta\Delta}(t) < 0$. Since $(x^{[1]}(t))^{\Delta} < 0$ on $[t_1, \infty)_{\mathbb{T}}$, we have after differentiation that

$$a^{\Delta}(t)x^{\Delta}(t) + a^{\sigma}x^{\Delta\Delta}(t) < 0. \tag{2.25}$$

since $a^{\Delta}(t) \geq 0$, we have $x^{\Delta\Delta}(t) < 0$. Next, we show that $x(t)/t$ is strictly decreasing. To do this, let $U(t) := x(t) - tx^{\Delta}(t)$, so that $U^{\Delta}(t) = -\sigma(t)x^{\Delta\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. This implies that $U(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. We claim there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Assume not, then $U(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{2.26}$$

which implies that $x(t)/t$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Pick $t_3 \in [t_1, \infty)_{\mathbb{T}}$ so that $\tau(t) \geq \tau(t_1)$, for $t \geq t_3$. Then $x(\tau(t))/\tau(t) \geq x(\tau(t_1))/\tau(t_1) =: d > 0$, $x(t)/t \geq x(t_1)/t_1 = d_1 > 0$, so that $x(\tau(t)) \geq d\tau(t)$ and $x(t) > d_1t$ for $t \geq t_3$. Now by integrating both sides of (1.1) from t_3 to t , we have

$$a(t)x^{\Delta}(t) - a(t_3)x^{\Delta}(t_3) + \int_{t_3}^t p(s)x^{\gamma}(\tau(s))(x^{\Delta}(s))^{1-\gamma}\Delta s = 0.$$

Since by assumption $x(t) < tx^{\Delta}(t)$, this implies that

$$\begin{aligned} a(t_3)x^{\Delta}(t_3) &\geq \int_{t_3}^t p(s)x^{\gamma}(\tau(s))(x^{\Delta}(s))^{1-\gamma}\Delta s \\ &\geq d^{\gamma} \int_{t_3}^t p(s)\tau^{\gamma}(s)(x^{\Delta}(s))^{1-\gamma}\Delta s \\ &\geq d^{\gamma} \int_{t_3}^t p(s)\tau^{\gamma}(s)\left(\frac{x(s)}{s}\right)^{1-\gamma}\Delta s \\ &\geq d^{\gamma}d_1^{1-\gamma} \int_{t_3}^t p(s)\tau^{\gamma}(s)\Delta s, \end{aligned}$$

which contradicts (2.24). Hence there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \quad t \in [t_2, \infty)_{\mathbb{T}}$$

and we have that $\frac{x(t)}{t}$ is strictly decreasing on $[t_2, \infty)_{\mathbb{T}}$. The proof is complete. □

For the delay case we introduce the following notations:

$$A_* := \liminf_{t \rightarrow \infty} \frac{t^\gamma}{a^\gamma(t)} \int_{\sigma(t)}^{\infty} A(s) \Delta s, \quad B_* := \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{a^\gamma(s)} A(s) \Delta s,$$

$$A(t) := \gamma p(t) \left(\frac{\tau(t)}{\sigma(t)} \right)^\gamma.$$

Theorem 2.3. Assume that $(h_1) - (h_2)$, (1.6), (2.24) hold and $a^\Delta(t) \geq 0$. Let $x(t)$ be a solution of (1.1) and make the Riccati substitution $w(t)$ be as in Theorem 2.1. Then

$$w^\Delta(t) + A(t) + \frac{\gamma}{a(t)} (w^\sigma)^{1+\frac{1}{\gamma}}(t) \leq 0, \quad \text{for } t \in [T, \infty)_{\mathbb{T}}. \quad (2.27)$$

Proof. Let x be as in the statement of this theorem and without loss of generality we assume that there is $t_1 > t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$. Now, since $a^\Delta(t) \geq 0$ then there exists $T > t_1$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) < 0$ for $t \geq T$. From the definition of $w(t)$, by quotient rule [?, Theorem 1.20] and continue as in the proof of Theorem 2.1, we get

$$w^\Delta(t) \leq -\alpha p(t) \left(\frac{x(\tau)}{x} \right)^\gamma - \gamma a^{-1} (w^\sigma)^{1+\frac{1}{\gamma}}. \quad (2.28)$$

Now we consider the coefficient of $p(t)$ in (2.28). From Lemma 2.1, since $x(t)/t$ is decreasing and $\tau(t) \leq t \leq \sigma(t)$, we have

$$\frac{x(\tau)}{x^\sigma} \geq \frac{\tau(t)}{\sigma(t)}, \quad (2.29)$$

Substituting from (2.29) into (2.28), we have the inequality (2.27) and this completes the proof. \square

In order for the definition of A_* to make sense, we assume that

$$\int_{t_0}^{\infty} A(s) \Delta s < \infty.$$

Theorem 2.4. Assume that $(h_1) - (h_2)$, (1.6), (2.24) hold and $a^\Delta(t) \geq 0$. Furthermore, assume that

$$A_* > \frac{\gamma^\gamma}{l^{\gamma^2} (\gamma + 1)^{\gamma+1}}, \quad (2.30)$$

or

$$A_* + B_* > \frac{1}{l^{\gamma(\gamma+1)}}. \quad (2.31)$$

Then every solution of (1.1) is oscillatory..

Proof. The proof is similar to the proof of Theorem 2.2, by replacing $p(t)$ by $A(t)$ and hence is omitted. \square

Corollary 2.4. Assume that $(h_1) - (h_2)$, (1.6), (2.24) hold and $a^\Delta(t) \geq 0$. Furthermore, assume that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{a^\gamma(s)} A(s) \Delta s > \frac{\gamma^\gamma}{l^{\gamma^2(\gamma+1)^{\gamma+1}}}. \tag{2.32}$$

Then every solution of (1.1) is oscillatory.

Corollary 2.5. Assume that $(h_1) - (h_2)$, (1.6), (2.24) hold and $a^\Delta(t) \geq 0$. Furthermore, assume that

$$\liminf_{t \rightarrow \infty} \frac{t^\gamma}{a^\gamma(t)} \int_{\sigma(t)}^\infty A(s) \Delta s > \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.33}$$

Then every solution of (1.1) is oscillatory.

Corollary 2.6. Assume that $(h_1) - (h_2)$, (1.6), (2.24) hold and $a^\Delta(t) \geq 0$. Furthermore, assume that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{a^\gamma(s)} A(s) \Delta s > \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.34}$$

Then every solution of (1.1) is oscillatory.

3. Applications

In this section, we use Theorems 2.2 and 2.4 to establish some sufficient conditions for oscillation of (1.12), (1.20), (1.21) and (1.22). First, we consider the case when $\mathbb{T} = \mathbb{R}$. We restate the assumptions for this case:

- (H₁). $\gamma \in (0, 1]$, a and p are positive continuous functions,
- (H₂). $\tau : \mathbb{T} \rightarrow \mathbb{T}$, and $\lim_{t \rightarrow \infty} \tau(t) = \infty$,

$$\int_{t_0}^\infty \frac{1}{a(t)} dt = \infty. \tag{3.1}$$

The following theorem gives some sufficient conditions for oscillation of (1.12) when $\tau(t) > t$. Note that when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$ and then $Q(t) = \gamma p(t)$.

Theorem 3.1. Assume that $(H_1) - (H_2)$, (3.1) hold, and $a'(t) \geq 0$, $\tau(t) > t$. Furthermore, assume that

$$\gamma \liminf_{t \rightarrow \infty} \frac{t^\gamma}{a^\gamma(t)} \int_t^\infty p(s) ds > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}, \tag{3.2}$$

or

$$\gamma \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{a^\gamma(s)} p(s) ds > 1. \tag{3.3}$$

Then every solution of (1.12) is oscillatory.

Corollary 3.1. Assume that $(H_1) - (H_2)$, $\tau(t) > t$. Furthermore, assume that

$$\gamma \liminf_{t \rightarrow \infty} t^\gamma \int_t^\infty p(s) ds > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}, \tag{3.4}$$

or

$$\gamma \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t s^{\gamma+1} p(s) ds > 1. \tag{3.5}$$

Then every solution of (1.13) is oscillatory.

The following theorem gives some sufficient conditions for oscillation of (1.12) when $\tau(t) \leq t$.

Theorem 3.2. Assume that $(H_1) - (H_2)$, (3.1) hold, $a'(t) \geq 0$, $\tau(t) \leq t$ and

$$\int_{t_0}^{\infty} \tau^\gamma(t) p(t) dt = \infty. \tag{3.6}$$

Furthermore, assume that

$$\gamma \liminf_{t \rightarrow \infty} \frac{t^\gamma}{a^\gamma(t)} \int_t^{\infty} p(s) \left(\frac{\tau(s)}{s} \right)^\gamma ds > \frac{\gamma^\gamma}{(\gamma + 1)^{\gamma+1}}, \tag{3.7}$$

or

$$\gamma \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{a^\gamma(s)} p(s) \left(\frac{\tau(s)}{s} \right)^\gamma ds > 1, \tag{3.8}$$

then (1.12) is oscillatory.

Corollary 3.2. Assume that $(H_1) - (H_2)$, (3.1), (3.6) hold and $a'(t) \geq 0$, $\tau(t) \leq t$. Furthermore, assume that

$$\gamma \liminf_{t \rightarrow \infty} t^\gamma \int_t^{\infty} p(s) \left(\frac{\tau(s)}{s} \right)^\gamma ds > \frac{\gamma^\gamma}{(\gamma + 1)^{\gamma+1}}, \tag{3.9}$$

or

$$\gamma \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t s^{\gamma+1} p(s) \left(\frac{\tau(s)}{s} \right)^\gamma ds > 1. \tag{3.10}$$

Then (1.13) is oscillatory.

Remark. Note that when $\tau(t) = t$, the results in Corollary 3.2 become the results that has been established in [15, 16].

Next, we consider the case when $\mathbb{T} = \mathbb{Z}$ and assume that:

(D₁). $\gamma \in (0, 1]$, a and p are positive sequences,

(D₂). $\tau : \mathbb{T} \rightarrow \mathbb{T}$, and $\lim_{t \rightarrow \infty} \tau(t) = \infty$,

$$\sum_{t=t_0}^{\infty} \frac{1}{a(t)} = \infty. \tag{3.11}$$

The following theorem gives some sufficient conditions for oscillation of (1.20) when $\tau(t) > t$

Theorem 3.3. Assume that $(D_1) - (D_2)$ (3.11) hold, $\Delta a(t) \geq 0$ and $\tau(t) > t$. Furthermore, assume that

$$\gamma \liminf_{t \rightarrow \infty} \frac{t^\gamma}{a(t)} \sum_{s=t+1}^{\infty} D(s) > \frac{\gamma^\gamma}{(\gamma + 1)^{\gamma+1}}, \tag{3.12}$$

or

$$\gamma \liminf_{t \rightarrow \infty} \frac{t^\gamma}{a(t)} \sum_{s=t+1}^{\infty} D(s) + \gamma \liminf_{t \rightarrow \infty} \frac{t^\gamma}{a(t)} \sum_{s=t+1}^{\infty} D(s) > 1, \tag{3.13}$$

where

$$D(t) := \gamma p(t) \left(\frac{a(t)P(t, T)}{a(t)P(t, T) + 1} \right)^\gamma, P(t, T) = \sum_T^{t-1} \frac{1}{a(\tau)}.$$

Then every solution of equation (1.20) is oscillatory.

The following theorem gives some sufficient conditions for oscillation of (1.20) when $\tau(t) \leq t$.

Theorem 3.4. Assume that $(D_1) - (D_2)$, (3.11) hold, $\Delta a(t) \geq 0$ and $\tau(t) \leq t$ and

$$\sum_{t=t_0}^{\infty} \tau^\gamma(t)p(t) = \infty. \tag{3.14}$$

Furthermore, assume that

$$\gamma \liminf_{t \rightarrow \infty} \frac{t^\gamma}{r(t)} \sum_{s=t+1}^{\infty} \left(\frac{\tau(s)}{s+1} \right)^\gamma p(s) > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \tag{3.15}$$

or

$$\gamma \liminf_{t \rightarrow \infty} \frac{t^\gamma}{r(t)} \sum_{s=t+1}^{\infty} \left(\frac{\tau(s)}{s+1} \right)^\gamma p(s) + \gamma \liminf_{t \rightarrow \infty} \frac{t^\gamma}{r(t)} \sum_{s=t+1}^{\infty} \left(\frac{\tau(s)}{s+1} \right)^\gamma p(s) > 1 \tag{3.16}$$

Then every solution of equation (1.20) is oscillatory.

Now, we consider the case when $\mathbb{T} = h\mathbb{Z}$, $h > 0$. The following theorem gives some sufficient conditions for oscillation of (1.21) when $\tau(t) > t$ and the case when $\tau(t) \leq t$ will be left to the interested reader due to the limited space. Also one can derive some sufficient conditions for oscillation of (1.22) from Theorem 2.2 and Theorem 2.4.

Theorem 3.5. Assume that $(D_1) - (D_2)$ hold, $\Delta_h a(t) \geq 0$, $\tau(t) \leq t$, and

$$\sum_{k=0}^{\infty} \tau^\gamma(t_0 + kh)p(t_0 + kh) = \infty, \sum_{k=0}^{\infty} \frac{h}{a(t_0 + kh)} = \infty, \tag{3.17}$$

$$\sum_{k=0}^{\infty} \left(\frac{\tau(t_0 + kh)}{t_0 + kh + h} \right)^\gamma p(t_0 + kh) < \infty.$$

Furthermore, assume that

$$\liminf_{t \rightarrow \infty} \frac{t^\gamma}{r(t)} \sum_{k=0}^{\infty} \left(\frac{\tau(t + kh + h)}{t + kh + 2h} \right)^\gamma p(t + kh + h)h > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}, \tag{3.18}$$

or

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{k=0}^{\frac{t-t_0-h}{h}} \frac{(N+kh)^{\gamma+1}}{r(N+kh)} \left(\frac{\tau(N+kh)}{N+kh+h} \right)^\gamma p(N+kh)h > 1, \tag{3.19}$$

where N is sufficiently large. Then every solution of equation (1.21) is oscillatory.

4. Examples

In this section we give some examples to illustrate the main results. The first example is for the advanced equation and the second example is for the delay equations.

Example 1. Consider the dynamic equation

$$x^{\Delta\Delta}(t) + \frac{\alpha\sqrt{\sigma(t)-1}}{t^{3/2}\sqrt{t-1}} |x^{1/2}(2t)| |x^\Delta(t)|^{1/2} \operatorname{sgn}x(2t) = 0, \text{ for } t \in [1, \infty)_{\mathbb{T}}. \tag{4.1}$$

Here $\gamma = 1/2$, $a(t) = 1$, $p(t) = \frac{\alpha\sqrt{\sigma(t)-1}}{t^{3/2}\sqrt{t-1}}$ and $\tau(t) = 2t > t$. So that

$$P(t, T) = \int_T^t \left(\frac{1}{a(\tau)} \right)^{\frac{1}{\gamma}} \Delta\tau = (t-1),$$

$$\begin{aligned} Q(t) &= \gamma p(t) \left(\frac{a(t)P(t, 1)}{a(t)P(t, 1) + \sigma(t) - t} \right)^\gamma \\ &= \frac{\alpha\sqrt{\sigma(t)-1}}{2t^{3/2}\sqrt{t-1}} \left(\frac{t-1}{(t-1) + \sigma(t) - t} \right)^{\frac{1}{2}} = \frac{\alpha}{2t^{3/2}}. \end{aligned}$$

Now, we apply Theorem 2.2. In this case it is clear that the conditions $(h_1) - (h_2)$ hold. It remains to satisfy the condition (2.11). In this case the condition reads

$$\begin{aligned} p_* &:= \liminf_{t \rightarrow \infty} \frac{t^{\frac{1}{2}}}{a(t)} \int_{\sigma(t)}^\infty Q(s) \Delta s = \frac{\alpha}{2} \liminf_{t \rightarrow \infty} t^{\frac{1}{2}} \int_{\sigma(t)}^\infty \frac{1}{s^{3/2}} \Delta s \\ &\geq \frac{\alpha}{2} \liminf_{t \rightarrow \infty} t^{1/2} \int_{\sigma(t)}^\infty \frac{1}{\sigma(s)s^{1/2}} \Delta s \\ &\geq \frac{\alpha}{2} \liminf_{t \rightarrow \infty} t^{1/2} \int_{\sigma(t)}^\infty \left(\frac{-1}{s^{1/2}} \right)^\Delta \Delta s = \frac{\alpha}{2} t^{1/2}. \end{aligned}$$

Then by Theorem 2.2, if

$$\alpha > \frac{4}{9} \frac{\sqrt{6}}{\sqrt{l^4 l}},$$

then every solution of the solution $x(t)$ of (4.1) oscillates.

Example 2. Consider the third-order dynamic equation

$$x^{\Delta\Delta}(t) + \frac{\alpha\sigma^{\frac{1}{2}}}{t^{3/2}\tau^{\frac{1}{2}}(t)} \left| x^{1/2}(\tau(t)) \right| |x^\Delta(t)|^{1/2} \operatorname{sgn}x(\tau(t)) = 0, \quad \tau(t) \leq t, \quad (4.2)$$

for $t \in [1, \infty)_{\mathbb{T}}$ $\lim_{t \rightarrow \infty} \tau(t) = \infty$. Here $a(t) = 1$, and $p(t) = \frac{\alpha\sigma(t)^{\frac{1}{2}}}{t^{3/2}\tau^{\frac{1}{2}}(t)}$. It is clear that $(h_1) - (h_2)$ holds. To apply Theorem 2.4, it remains to prove that (2.30) holds. For equation (4.2), we have

$$\begin{aligned} & \liminf_{t \rightarrow \infty} t^{1/2} \int_{\sigma(t)}^{\infty} \gamma p(s) \left(\frac{\tau(s)}{\sigma(s)} \right)^\gamma \Delta s \\ &= \frac{\alpha}{2} \liminf_{t \rightarrow \infty} t^{1/2} \int_{\sigma(t)}^{\infty} \frac{\sigma(t)^{\frac{1}{2}}}{t^{3/2}\tau^{\frac{1}{2}}(t)} \left(\frac{\tau(s)}{\sigma(s)} \right)^\gamma \Delta s \\ &= \frac{\alpha}{2} \liminf_{t \rightarrow \infty} t^{1/2} \int_{\sigma(t)}^{\infty} \frac{1}{s^{3/2}} \Delta s \geq \frac{\alpha}{2} \liminf_{t \rightarrow \infty} t^{1/2} \int_{\sigma(t)}^{\infty} \frac{1}{\sigma(s)s^{1/2}} \Delta s \\ &\geq \frac{\alpha}{2} \liminf_{t \rightarrow \infty} t^{1/2} \int_{\sigma(t)}^{\infty} \left(\frac{-1}{s^{1/2}} \right)^\Delta \Delta s = \frac{\alpha}{2} l^{1/2}. \end{aligned}$$

Then by Theorem 2.4, the solutions of (4.2) are oscillatory if $\alpha > \frac{4}{9} \frac{\sqrt{6}}{\sqrt{l}^{3/4}}$.

REFERENCES

1. R. P. Agarwal, M. Bohner, and S. H. Saker, *Oscillation criteria for second order delay dynamic equation*, *Canad. Appl. Math. Quart.* 13 (2005), 1-19.
2. R. P. Agarwal, D. O'Regan and S. H. Saker, *Oscillation criteria for second-order nonlinear neutral delay dynamic equations*, *J. Math. Anal. Appl.* 300 (2004) 203-217.
3. E. Akin-Bohner, M. Bohner, and S. H. Saker, *Oscillation for a certain of class of second order Emden-Fowler dynamic equations*, *Elect. Trans. Numer. Anal.* 27 (2007), 1-12.
4. E. Akin-Bohner and J. Hoffacker, *Oscillation properties of an Emden-Fowler type equation on discrete time scales*, *J. Diff. Eqns. Appl.* 9 (2003), 603-612.
5. M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
6. M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
7. M. Bohner and S. H. Saker, *Oscillation of second order nonlinear dynamic equations on time scales*, *Rocky Mountain J. Math.*, 34(4) (2004) 1239-1254.
8. M. Bohner and S. H. Saker, *Oscillation criteria for perturbed nonlinear dynamic equations*, *Mathl. Comp. Modeling* 40 (2004) 249-260.
9. L. Erbe, A. Peterson, and S. H. Saker, *Oscillation criteria for second-order nonlinear dynamic equations on time scales*, *J. London Math. Soc.* 76 (2003) 701-714.
10. S. Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, *Results Math.*, 18 (1990) 18-56.
11. E. Hille, *Non-oscillation theorems*, *Trans. Amer. Math. Soc.*, 64 (1948) 234-252.
12. Z. Han, S. Sun and B. Shi, *Oscillation criteria for a class of second-order Emden-Fowler delay dynamic equations on time scales*, *J. Math. Anal. Appl.* 334 (2007), 847-858.
13. V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2001.
14. W. Kelley and A. Peterson, *Difference Equations: An Introduction With Applications*, Second edition, Harcourt/Academic Press, San Diego, 2001.

15. N. Kandelaki, A. Lomtadze and D. Ugulava, *On oscillation of and nonoscillation of a second order half-linear equation*, Georgian Math. J. 7 (2000), 329-346.
16. A. Lomtadze, *Oscillation of and nonoscillation of Emden-Fowler type equation of second order*, Archivum Math. 32 (1996), 181-193.
17. M. Morelli and A Peterson, *A third-order differential equation on a time scale*, Math. Comp. Modelling 32 (2000), 565-570.
18. Z. Nehari, *Oscillation criteria for second-order linear differential equations*, Trans. Amer. Math. Soc., 85 (1957) 428-445.
19. S. H. Saker, *Oscillation of nonlinear dynamic equations on time scales*, Appl. Math. Comp. 148 (2004), 81-91.
20. S. H. Saker, *Oscillation criteria of second-order half-linear dynamic equations on time scales*, J. Comp. Appl. Math. 177 (2005), 375-387.
21. S. H. Saker, *Oscillation of second-order forced nonlinear dynamic equations on time scales*, E. Journal of Qualitative Theory Diff. Eqns. 23 (2005), 1-17.
22. S. H. Saker, *Boundedness of solutions of second-order forced nonlinear dynamic equations*, Rocky Mountain. J. Math. 36 (2006), 2027-2039.
23. S. H. Saker, *New oscillation criteria for second-order nonlinear dynamic equations on time scales*, Nonlinear Fun. Anal. Appl. 11 (3) (2006), 351-370.
24. S. H. Saker, *Oscillation of second-order neutral delay dynamic equations of Emden-Fowler type*, Dyn. Sys. Appl. 15 (2006), 629-644.
25. S. H. Saker, *Oscillation of second-order delay and neutral delay dynamic equations on time scales*, Dyn. Sys. Appl. 16 (2007), 345-360.
26. V. Spedding, *Taming Nature's Numbers*, New Scientist, July 19, 2003, 28-31.

S. H. Saker received his B. SC. (1993) and M. SC. (1997) from Mansoura university, Egypt and got his Ph. D. from Adam Mickiewicz University, Poland (2002). His research interests focus on the oscillation theory of differential and difference equations, asymptotic behavior of continuous and discrete population dynamics and oscillation theory of dynamic equations on time scales which unify the oscillation of differential and difference equations. He got Abdul Hamid Shoman Award for Young Arab Scientists, Mathematical Science, (Jordan 2003) and got the National State Prize on Basic Science (Mathematical Science, Egypt 2005). He published more than 140 papers in high level journals.

Permanent address: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

e-mail: ssaker@ksu.edu.sa, shsaker@mans.edu.eg