REFLECTION SYMMETRIES OF THE q-GENOCCHI POLYNOMIALS

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ABSTRACT. One purpose of this paper is to consider the reflection symmetries of the q-Genocchi polynomials $G_{n,q}^*(x)$. We also observe the structure of the roots of q-Genocchi polynomials, $G_{n,q}^*(x)$, using numerical investigation. By numerical experiments, we demonstrate a remarkably regular structure of the real roots of $G_{n,q}^*(x)$.

AMS Mathematics Subject Classification: 11B68, 11S40, 11S80
Key words and phrases : Euler numbers, Euler polynomials, q-Genocchi polynomials, Roots of q-Genocchi polynomials, Reflection symmetries of the q-Genocchi polynomials

1. **Introduction**

Many mathematicians have studied Genocchi polynomials and Genocchi numbers. Genocchi polynomials and Genocchi numbers posses many interesting properties and arising in many areas of mathematics and physics. In [1], T. Kim constructed the q- Genocchi $G_{n,q}$ and polynomials $G_{n,q}(x)$ using generating functions. In order to study the q-Genocchi polynomials $G_{n,q}(x)$, we must understand the structure of the q- Genocchi polynomials $G_{n,q}(x)$. Therefore, using computer, a realistic study for the q-Genocchi polynomials $G_{n,q}(x)$ is very interesting. It is the aim of this paper to consider the reflection symmetries of the q-Genocchi polynomials $G_{n,q}(x)$. We also observe the structure of the roots of the q-Genocchi polynomials $G_{n,q}(x)$, using numerical investigation. By numerical experiments, we demonstrate a remarkably regular structure of the real roots of $G_{n,q}(x)$. Finally, we give a table for the solutions of our q- Genocchi polynomials. The outline of this paper is as follows. We introduce the q-Genocchi polynomials $G_{n,q}(x)$. In Section 2, we consider reflection symmetries

of the q-Genocchi polynomials $G_{n,q}(x)$ using a numerical investigation. Finally, we investigate the roots of the q-Genocchi polynomials $G_{n,q}^*(x)$. We introduce the ordinary Euler numbers and Euler polynomials. The usual Euler numbers E_n are defined by

$$F(t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

where the symbol E_k is interpreted to mean that E^k must be replaced by E_k when we expand the one on the left. For any complex number x, it is well known that the familiar Euler polynomials $E_n(x)$ are defined by means of the following generating function:

$$F(x,t) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

In [1], Kim constructed q-analogue of Euler numbers and polynomials. We now introduce the q-extension of Euler numbers $E_{q,n}$ and polynomials $E_{n,q}(x)$ (see [1]). Let us consider a complex number $q \in \mathbb{C}$ with $|q| < 1$ as an indeterminate. The q-analogue of n is denoted by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + q^3 + \cdots + q^{n-1}.$$

Let us consider the q-Euler polynomials as follows:

$$F_q(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}.$$

Note that $\lim_{q \to 1} F_q(t) = F(t)$ and $\lim_{q \to 1} E_{n,q} = E_n$. The q-Euler polynomials $E_{n,q}(x)$ are defined by

$$F_q(x,t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.$$

We easily see that $\lim_{q \to 1} F_q(x,t) = F(x,t)$ and $\lim_{q \to 1} E_{n,q}(x) = E_n(x)$. The Genocchi numbers G_n are defined by the generating function:

$$G(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, (|t| < \pi), \text{ cf. [1]} \tag{1}$$

where we use the technique method notation by replacing G^n by $G_n (n \geq 0)$ symbolically. For $x \in \mathbb{R}$ (= the field of real numbers), we consider the Genocchi polynomials $G_n(x)$ as follows:

$$G(x,t) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{2}$$

Note that $G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}$. In the special case $x = 0$, we define $G_n(0) = G_n$. Since

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2t}{(e^t+1)t} e^{xt} = \sum_{i=0}^{\infty} G_i \frac{t^{i-1}}{i!} \sum_{j=0}^{\infty} x^j \frac{t^j}{j!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{G_{k+1}}{k+1} \frac{t^k}{k!} x^{n-k} \frac{t^{n-k}}{(n-k)!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{G_{k+1}}{k+1} x^{n-k} \right) \frac{t^n}{n!},$$

we have

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{G_{k+1}}{k+1} x^{n-k}.$$

In [1], Kim constructed q-analogue of Genocchi numbers and polynomials. We now introduce the q-extension of Genocchi numbers $G_{n,q}$ and polynomials $G_{n,q}(x)$ (see [4]). We consider the following generating functions:

$$G_q(t) = [2]_q t e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{(-1)^n}{1+q^{n+1}} \left(\frac{1}{1-q} \right)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \qquad (3)$$

and

$$G_q(x,t) = [2]_q q^x t e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{(-1)^n}{1+q^{n+1}} q^{nx} \left(\frac{1}{1-q} \right)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}, \qquad (4)$$

By simple calculation in (4), we obtain

$$G_{n,q}(x) = [2]_q n \left(\frac{1}{1-q} \right)^{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(-1)^i}{1+q^{i+1}} q^{(i+1)x}.$$

We obtain the first value of the q- Genocchi polynomials $G_{n,q}(x)$:

$$G_{1,q}(x) = q^x, \quad G_{2,q}(x) = -\frac{2q^x(1+q^2-q^x-q^{1+x})}{(-1+q)(1+q^2)},$$

$$G_{3,q}(x) = \frac{3q^x(1-q+2q^2-q^3+q^4-2q^x+q^{2x}-2q^{3+x}+q^{2+2x})}{(-1+q)^2(1+q^2)(1-q+q^2)}, \cdots,$$

When $x = 0$, we write $G_{n,q} = G_{n,q}(0)$, which are called the q-Genocchi numbers. $G_{n,q}(x)$ is a polynomial of degree $= n$ in q^x. We obtain the first value of the q-Genocchi numbers $G_{n,q}$:

$$G_{1,q} = 1, \quad G_{2,q} = -\frac{2q}{1+q^2},$$

$$G_{3,q} = \frac{3(-1+q)q}{(1+q^2)(1-q+q^2)}, \quad G_{4,q} = -\frac{4q(1-q-q^2-q^3+q^4)}{(1+q^2)(1-q+q^2)(1+q^4)}, \cdots,$$

2. **Reflection symmetries on the q-Genocchi polynomials**

In this section we consider the reflection symmetries of the q-Genocchi polynomials $G_{n,q}(x)$. Since

$$\sum_{n=0}^{\infty} G_n(1-x)\frac{(-t)^n}{n!} = \frac{-2t}{e^{-t}+1}e^{(1-x)(-t)} = \frac{-2t}{e^t+1}e^{xt} = -\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!},$$

we obtain that

$$G_n(x) = (-1)^{n+1}G_n(1-x). \tag{5}$$

We prove that $G_n(x), x \in \mathbb{C}$, has $Re(x) = \frac{1}{2}$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions. The question is: what happens with the reflection symmetry (5), when one considers the q-Genocchi polynomials? We are going now to reflection at $\frac{1}{2}$ of x on the q-Genocchi polynomials. Since

$$G_{n,q}(x) = [2]_q n \left(\frac{1}{1-q}\right)^{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(-1)^i}{1+q^{i+1}} q^{(i+1)x},$$

by simple calculation, we have

$$G_{n,q^{-1}}(1-x)$$

$$= n(1+q^{-1}) \left(\frac{1}{1-q^{-1}}\right)^{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (q^{-1})^{(i+1)(1-x)} \frac{1}{1+(q^{-1})^{i+1}}$$

$$= n(1+q^{-1}) \left(\frac{q}{q-1}\right)^{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (q^{-1})^{(i+1)(1-x)} \frac{1}{1+(q^{-1})^{i+1}}$$

$$= n(1+q^{-1}) \left(\frac{-q}{1-q}\right)^{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i q^{-(i+1)} q^{(i+1)x} \frac{1}{q^{-(i+1)}(1+q^{i+1})}$$

$$= (-1)^{n-1}q^{n-2}[2]_q n \left(\frac{1}{1-q}\right)^{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(-1)^i}{1+q^{i+1}} q^{(i+1)x}.$$

Hence we obtain the following theorem.

Theorem 1. For $n \geq 0$, we have

$$G_{n,q}^*(x) \equiv G_{n,q^{-1}}(1-x) = (-1)^{n-1}q^{n-2}G_{n,q}(x). \tag{6}$$

(6) is the q-analog of the classical reflection formula (5). Prove or disprove: $G_{n,q}^*(x)$ has $Im(x) = 0$ reflection symmetry analytic complex functions (Figures 6, 8). $G_{n,q}^*(x)$ has not $Re(x) = 1/2$ reflection symmetry (Figures 6, 8). The open question is: what happens with the reflection symmetry (6), when one considers the q-extension of Genocchi polynomials ? We have the first value of

the q- Genocchi polynomials We have the first value of the q- Genocchi numbers $G^*_{n,q}$:

$$G^*_{1,q} = \frac{1}{q}, \quad G^*_{2,q} = \frac{2q}{1+q^2},$$

$$G^*_{3,q} = \frac{3(-1+q)q^2}{(1+q^2)(1-q+q^2)}, \quad G^*_{4,q} = -\frac{4q^3(1-q-q^2-q^3+q^4)}{(1+q^2)(1-q+q^2)(1+q^4)}, \cdots,$$

For $n = 1, \cdots, 10$, we can draw a plot of the q- Genocchi numbers $E^*_{n,q}$, respectively. This shows the ten plots combined into one. We display the shape of $G_{n,q}, G^*_{n,q}, -9/10 \leq q \leq 9/10$ (Figures 1, 2). We obtain the first value of the q-Genocchi polynomials $G^*_{n,q}(x)$:

$$G^*_{1,q}(x) = q^{-1+x},$$

$$G^*_{2,q}(x) = \frac{2q^x(1+q^2-q^x-q^{1+x})}{(-1+q)(1+q^2)},$$

$$G^*_{3,q}(x) = \frac{3q^{1+x}(1-q+2q^2-q^3+q^4-2q^x+q^{2x}-2q^{3+x}+q^{2+2x})}{(-1+q)^2(1+q^2)(1-q+q^2)}, \cdots$$

First, we display the shapes of the q-Genocchi polynomials $G^*_{n,q}(x)$ and we investigate the zeros of the q-Genocchi polynomials $G^*_{n,q}(x)$. For $n = 1, \cdots, 10$, we can draw a plot of the q-Genocchi polynomials $G^*_{n,q}(x)$, respectively. This shows the ten plots combined into one. We display the shape of $G_{n,q}(x), G^*_{n,q}(x), n = 1, \cdots, 10, q = 1/2. -1 \leq x \leq 1$.

Table 1. Numbers of real and complex zeros of $G^*_{n,q}(x)$

degree n	$q = \frac{1}{3}$ real zeros	complex zeros	$q = \frac{1}{2}$ real zeros	complex zeros
2	1	0	1	0
3	2	0	2	0
4	1	2	3	0
5	2	2	2	2
6	3	2	3	2
7	2	4	2	4
8	3	4	3	4
9	2	6	4	4
10	3	6	3	6
11	2	8	4	6

We observe a remarkably regular structure of the complex roots of the q-Genocchi polynomials $G^*_{n,q}(x)$. We hope to verify a remarkably regular structure of the complex roots of the q-Genocchi polynomials $G^*_{n,q}(x)$. (Table 1). Next, we calculate an approximate solution satisfying $G^*_{n,q}(x), q = 1/2, x \in \mathbb{R}$. The results are given in Table 2.
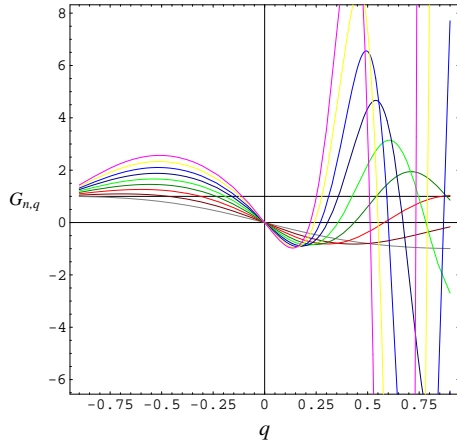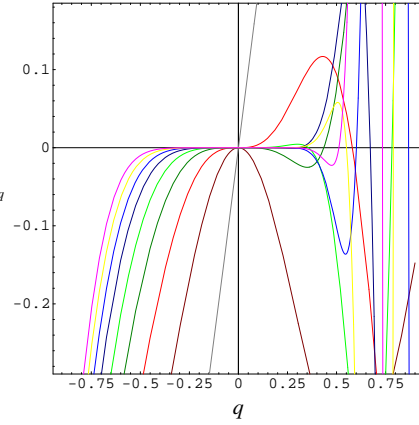
FIGURE 1.   Curvers of $G_{n,q}$

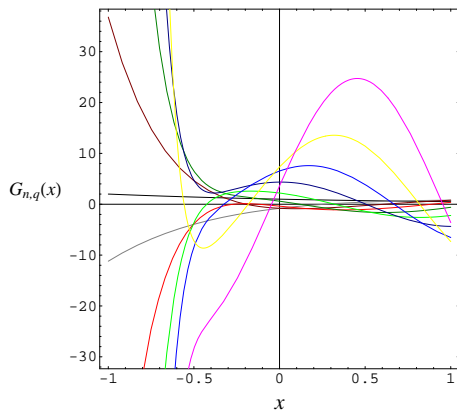FIGURE 2.   Curvers of $G^*_{n,q}$

FIGURE 3.   Curvers of $G_{n,q}(x)$
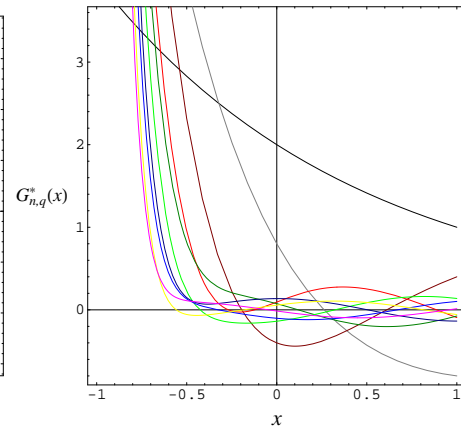
FIGURE 4.   Curvers of $G^*_{n,q}(x)$

Table 2. Approximate solutions of $G^*_{n,1/2}(x) = 0, x \in \mathbb{R}$

degree n	x
2	0.263034
3	$-0.195283,\quad 0.610321$
4	$-0.268477,\quad -0.122484,\quad 0.888461$
5	$0.119298,\quad 1.11848$
6	$-0.426196,\quad 0.322836,\quad 1.31475$
7	$0.505176,\quad 1.48613$
8	$-0.325429,\quad 0.66923,\quad 1.63838$
9	$-0.557938,\quad -0.182186,\quad 0.817824,\quad 1.77545$
10	$-0.0466248,\quad 0.953375,\quad 1.90015$
11	$-0.605574,\quad 0.0778538,\quad 1.07785,\quad 2.01459$

We investigate the beautiful zeros of the $G_{n,q}^*(x)$ by using a computer. We plot the zeros of q-Genocchi polynomials $G_{n,q}(x), G_{n,q}^*(x), q = 1/2, n = 20, n = 40, x \in \mathbb{C}$.
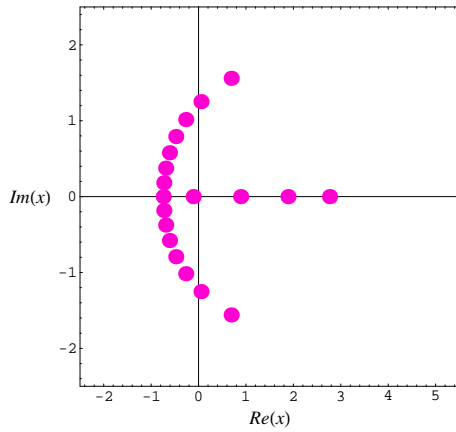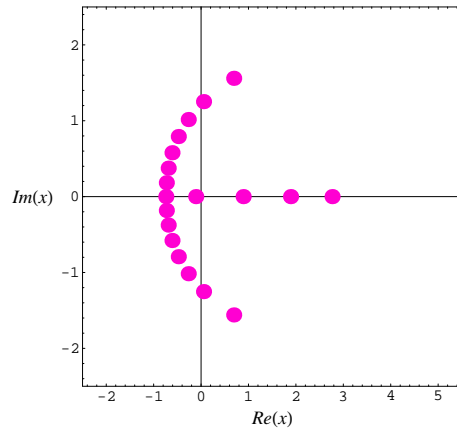

FIGURE 5. Zeros of $G_{20,q}(x)$
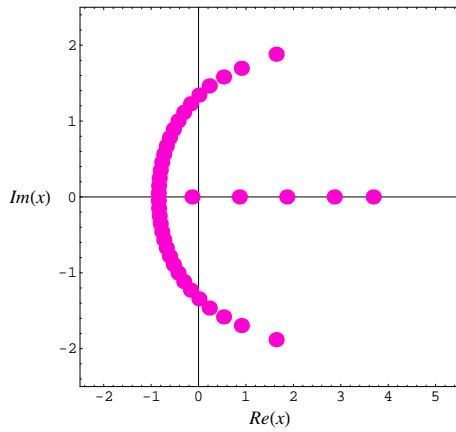

FIGURE 6. Zeros of $G_{20,q}^*(x)$
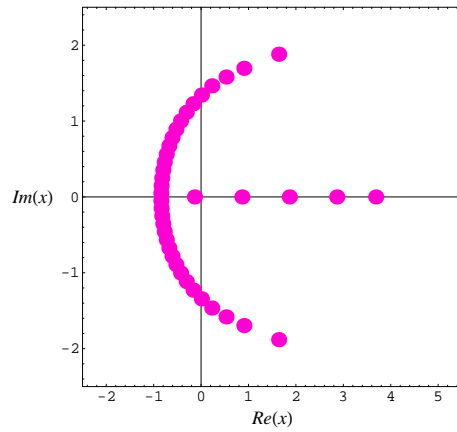

FIGURE 7. Zeros of $G_{40,q}(x)$

FIGURE 8. Zeros of $G_{40,q}^*(x)$

3. Directions for Further Research

In Figure 5, we choose $q = 1/2$ and $n = 20$. In Figure 6, we choose $q = 1/2$ and $n = 20$. Obviously, both figures reveal the same zero behaviors. In order to

understand zero behavior better, we propose Figure 7 and Figure 8. In Figure 7, we choose $q = 1/2$ and $n = 40$. In Figure 8, we choose $q = 1/2$ and $n = 40$. More interesting patterns of zero are visualized. Obviously, the zero behavior is bounded. The theoretical prediction on the boundedness and behavior of zero is await for further study.

Finally, we shall consider the more general problems. In general, how many roots does $G_{n,q}^*(x)$ have ? This is open problem. Prove or disprove: $G_{n,q}^*(x) = 0$ has $n - 1$ distinct solutions. Find the numbers of complex zeros $C_{G_{n,q}^*(x)}$ of $G_{n,q}^*(x), Im(x) \neq 0$. Since $n - 1$ is the degree of the polynomial $G_{n,q}^*(x)$, the number of real zeros $R_{G_{n,q}^*(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{G_{n,q}^*(x)} = n - 1 - C_{G_{n,q}^*(x)}$, where $C_{E_{n,q}^*(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{G_{n,q}^*(x)}$ and $C_{G_{n,q}^*(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the q-Genocchi polynomials $G_{n,q}^*(x)$ to appear in mathematics and physics. The reader may refer to [2-7] for the details

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