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FIXED POINTS SOLUTIONS OF GENERALIZED EQUILIBRIUM PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we introduce a new iterative scheme for finding a common element of the set of common fixed points of infinite family of nonexpansive mappings and the set of solutions to a generalized equilibrium problem and the set of solutions to a variational inequality problem in a real Hilbert space. Then strong convergence of the scheme to a common element of the three sets is proved. As applications, three new strong convergence theorems are obtained. Our theorems extend important recent results.

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1. Introduction

Let K be a nonempty closed convex subset of a real Hilbert space H. A mapping $A : K \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall x, y \in K. \quad (1)$$

The variational inequality problem is to find an $x^* \in K$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0,\ \forall y \in K. \quad (2)$$

(See, for example, [2-4]). We shall denote the set of solutions of the variational inequality problem (2) by $VI(K, A)$.
A mapping $A : K \to H$ is called *inverse-strongly monotone* (see, for example, [3, 10] if there exists a positive real number α such that $\langle Ax-Ay, x-y\rangle \geq \alpha||Ax-Ay||^2,\ \forall x, y \in K$. For such a case, A is called α-inverse-strongly monotone.

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A mapping $T : K \to K$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \ \forall x, y \in K.$$

A mapping $T : K \to K$ is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2,$$

for all $x, y \in K$. If $k = 0$, then the mapping T is nonexpansive. Observe that if T is a k-strictly pseudocontractive and we put $A := I - T$, where I is the identity operator defined on K, then we have that

$$||(I - A)x - (I - A)y||^2 \leq ||x - y||^2 + k||Ax - Ay||^2$$

for all $x, y \in K$ and since H is a real Hilbert space, we have that

$$||(I - A)x - (I - A)y||^2 = ||x - y||^2 + ||Ax - Ay||^2 - 2\langle x - y, Ax - Ay\rangle.$$

So,

$$\langle x - y, Ax - Ay\rangle \geq \frac{1 - k}{2}||Ax - Ay||^2.$$

Thus, if T is a k-strictly pseudocontractive mapping, then $A = I - T$ is an α-inverse strongly monotone operator with $\alpha = \frac{1-k}{2}$.

A point $x \in K$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in K : Tx = x\}$.

Let F be a bifunction of $K \times K$ into $\mathbb{R}$, the set of reals and $A : K \to H$ be a nonlinear mapping. The generalized equilibrium problem is to find $x \in K$ such that

$$F(x, y) + \langle Ax, y - x\rangle \geq 0, \tag{3}$$

for all $y \in K$. The set of solutions of this generalized equilibrium problem is denoted by EP. Thus

$$EP := \{x^* \in K : F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0, \ \forall y \in K\}.$$

In the case of $A \equiv 0$, EP is denoted by $EP(F)$ and in the case of $F \equiv 0$, EP is denoted by $VI(K, A)$. The problem (3) includes as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games, etc (see, for example, [1, 12]). Very recently, the problem of approximating fixed points of nonexpansive mappings which are also solutions to generalized equilibrium problem has become an interesting area of research for many authors in fixed point theory and many iterative schemes have been developed. Furthermore, these iterative schemes are for either single nonexpansive mapping (see, for example, [8, 10, 11, 17], [22-24] and the references contained therein) or finite family of nonexpansive mappings (see, for example, [15], [18] and the references contained therein) or infinite family of nonexpansive mappings (see, for example, [16, 21, 26] and the references contained therein). We remark here that many of the algorithms constructed for approximation of common fixed points of family of nonexpansive mappings which are also solutions to generalized equilibrium problems involve the so-called W_n-mapping generated by $T_n,\ T_{n-1},...,T_1$ mappings (see, for example, [5-7], [25] and the references contained therein). Also, the problem of approximating the common fixed points of finite family of asymptotically nonexpansive mappings which are also solutions to variational inequality problems have also been considered (see, for example, [13, 20]).

In this paper, we propose *a new iterative scheme for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings and the set of solutions to a generalized equilibrium problem and the set of solutions to a variational inequality problem in a real Hilbert space.* We show that the iterative scheme proposed converges strongly to a common element of the three sets. Then, three new strong convergence theorems are deduced. Our proposed algorithm does not involve the W_n-mappings for the family of operators considered. Furthermore, the condition: "Let $\sum_{n=1}^{\infty} \sup\{||T_{n+1}x - T_nx|| : x \in B\} < \infty$ for any bounded subset B of K and T be a mapping of K into itself defined by $Tx := \lim_{n\to\infty} T_nx$ for all $x \in K$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$" used in [14] and [19] is dispensed with in our iterative algorithm.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle .,.\rangle$ and norm $||.||$ and let K be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written $x_n \to x$ as $n \to \infty$.

For any point $u \in H$, there exists a unique point $P_Ku \in K$ such that

$$||u - P_Ku|| \le ||u - y||,\ \forall y \in K. \tag{4}$$

P_K is called the *metric projection* of H onto K. We know that P_K is a nonexpansive mapping of H onto K. It is also known that P_K satisfies

$$\langle x - y, P_Kx - P_Ky\rangle \ge ||P_Kx - P_Ky||^2,\ \forall x, y \in H. \tag{5}$$

Furthermore, P_Kx is characterized by the properties $P_Kx \in K$ and

$$\langle x - P_Kx, P_Kx - y\rangle \ge 0,\ \forall y \in K, \tag{6}$$

$$||x - P_Kx||^2 \le ||x - y||^2 - ||y - P_Kx||^2,\ \forall x \in H,\ y \in K. \tag{7}$$

In the context of the variational inequality problem, (6) implies that

$$x^* \in VI(K, A) \Leftrightarrow x^* = P_K(x^* - \lambda Ax^*),\ \forall \lambda > 0.$$

If A is α-inverse-strongly monotone mapping of K into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in K$ and $r > 0$,

$$\begin{aligned} ||(I - rA)x - (I - rA)y||^2 &= ||x - y - r(Ax - Ay)||^2 \\ &= ||x - y||^2 - 2r\langle Ax - Ay, x - y\rangle + r^2||Ax - Ay||^2 \\ &\le ||x - y||^2 + r(r - 2\alpha)||Ax - Ay||^2 \end{aligned} \tag{8}$$

So, if $r \leq 2\alpha$, then $I - rA$ is a nonexpansive mapping of K into H.
For solving the equilibrium problem for a bifunction $F : K \times K \to \mathbb{R}$, let us assume that F satisfies the following conditions:
(A1) $F(x,x) = 0$ for all $x \in K$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y, \in K$;
(A3) for each $x, y \in K$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) for each $x \in K$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 1 (Blum and Oettli, [1]). *Let K be a nonempty closed convex subset of H and let F be a bifunction of $K \times K$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad for\ all\ y \in K.$$

Lemma 2 (Combettes and Hirstoaga, [9]). *Assume that $F : K \times K \to \mathbb{R}$ satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to K$ as follows:*

$$T_r(x) = \{z \in K : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in K\}$$

for all $z \in H$. Then, the following hold:
1. T_r is single-valued;
2. T_r is firmly nonexpansive, i.e., $||T_r x - T_r y||^2 \leq \langle T_r x - T_r y, x - y\rangle$ $\forall x, y \in H$;
3. $F(T_r) = EP(F)$;
4. $EP(F)$ is closed and convex.

3. Main Results

Theorem 1. *Let K be a closed convex nonempty subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1)-(A4)$, A be an α-inverse-strongly monotone mapping of K into H, B be an β-inverse-strongly monotone mapping of K into H and for each $i = 1, 2, ...,$ let $T_i : K \to K$ be a nonexpansive mapping. Suppose $F := \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP \bigcap VI(K,B) \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, ...)$ and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in K$,*

$$\begin{cases} C_{1,i} = K,\ C_1 = \bigcap_{i=1}^{\infty} C_{1,i} \\ x_1 = P_{C_1} x_0 \\ F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0\ \forall y \in K \\ u_n = P_K(z_n - \lambda_n B z_n) \\ y_{n,i} = \alpha_{n,i} x_n + (1 - \alpha_{n,i}) T_i u_n,\ n \geq 1 \\ C_{n+1,i} = \{z \in C_{n,i} : ||y_{n,i} - z|| \leq ||x_n - z||\},\ n \geq 1 \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i} \\ x_{n+1} = P_{C_{n+1}} x_0,\ n \geq 1. \end{cases} \tag{9}$$

Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset [0,1)$ $(i = 1,2,...)$, $\{r_n\}_{n=1}^{\infty} \subset [0,2\alpha]$ *and* $\{\lambda_n\}_{n=1}^{\infty} \subset [0,2\beta]$ *satisfy*

$$0 < a \le r_n \le b < 2\alpha,\ 0 < c \le \lambda_n \le f < 2\beta,\ 0 \le \alpha_{n,i} \le d_i < 1.$$

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_F x_0$.

Proof. Put $z_n := T_{r_n}(x_n - r_n A x_n),\ n \ge 1$. Let $x^* \in F$ and $\{T_{r_n}\}_{n=1}^{\infty}$ be a sequence of mappings defined as in Lemma 2. Since both $I - r_n A$ and $I - \lambda_n B$ are nonexpansive for each $n \ge 1$ and $x^* = T_{r_n}(x^* - \lambda_n A x^*)$, we have $||u_n - x^*|| \le ||z_n - x^*||$ and from (8), we have

$$\begin{aligned}
||z_n - x^*||^2 =& ||T_{r_n}(x_n - r_n A x_n) - x^*||^2 \\
=& ||T_{r_n}(x_n - r_n A x_n) - T_{r_n}(x^* - r_n A x^*)||^2 \\
\le& ||(I - r_n A)x_n - (I - r_n A)x^*||^2 \\
\le& ||x_n - x^*||^2 + r_n(r_n - 2\alpha)||Ax_n - Ax^*||^2 \\
\le& ||x_n - x^*||^2 \quad (\text{since } r_n < 2\alpha,\ \forall n \ge 1).
\end{aligned}$$

Let $n = 1$, then $C_{1,i} = K$ is closed convex for each $i = 1,2,....$ Now assume that $C_{n,i}$ is closed convex for some $n > 1$. Then, from definition of $C_{n+1,i}$, we know that $C_{n+1,i}$ is closed convex for the same $n > 1$. Hence, $C_{n,i}$ is closed convex for $n \ge 1$ and for each $i = 1,2,....$ This implies that C_n is closed convex for $n \ge 1$ and for each $i = 1,2,....$ Furthermore, for $n = 1$, $F \subset K = C_{1,i}$. For $n \ge 2$, let $x^* \in F$. Then,

$$\begin{aligned}
||y_{n,i} - x^*|| \le& \alpha_{n,i}||x_n - x^*|| + (1 - \alpha_{n,i})||T_i u_n - x^*|| \\
\le& \alpha_{n,i}||x_n - x^*|| + (1 - \alpha_{n,i})||u_n - x^*|| \\
\le& \alpha_{n,i}||x_n - x^*|| + (1 - \alpha_{n,i})||z_n - x^*|| \\
\le& ||x_n - x^*||,
\end{aligned}$$

which shows that $x^* \in C_{n,i},\ \forall n \ge 2,\ \forall i = 1,2,...$ Thus, $F \subset C_{n,i},\ \forall n \ge 1,\ \forall i = 1,2,...$ Hence, it follows that $F \subset C_n,\ \forall n \ge 1$. Since $x_n = P_{C_n} x_0,\ \forall n \ge 1$ and $x_{n+1} \in C_{n+1} \subset C_n,\ \forall n \ge 1$, we have

$$||x_n - x_0|| \le ||x_{n+1} - x_0||,\ \forall n \ge 0. \tag{10}$$

Also, as $F \subset C_n$ by (4), it follows that

$$||x_n - x_0|| \le ||z - x_0||,\ z \in F,\ \forall n \ge 0. \tag{11}$$

From (10) and (11), we have that $\lim_{n\to\infty} ||x_n - x_0||$ exists. Hence, $\{x_n\}_{n=0}^{\infty}$ is bounded and so are $\{z_n\}_{n=1}^{\infty}$, $\{Ax_n\}_{n=1}^{\infty}$, $\{T_i u_n\}_{n=1}^{\infty}$, $\{Bz_n\}_{n=1}^{\infty}$ and $\{y_{n,i}\}_{n=1}^{\infty}$, $i = 1,2,....$ For $m > n \ge 1$, we have that $x_m = P_{C_m} x_0 \in C_m \subset C_n$. By (7), we obtain

$$||x_m - x_n||^2 \le ||x_n - x_0||^2 - ||x_m - x_0||^2. \tag{12}$$

Letting $m, n \to \infty$ and taking the limit in (12), we have $x_m - x_n \to 0,\ m, n \to \infty$, which shows that $\{x_n\}_{n=0}^{\infty}$ is Cauchy. In particular, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. Since,

$\{x_n\}_{n=0}^{\infty}$ is Cauchy, we assume that $x_n \to z \in K$. Since $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1}$, then $||y_{n,i} - x_{n+1}|| \leq ||x_n - x_{n+1}||$ and it follows that

$$||y_{n,i} - x_n|| \leq ||y_{n,i} - x_{n+1}|| + ||x_n - x_{n+1}|| \leq 2||x_n - x_{n+1}||.$$

Thus,

$$\lim_{n\to\infty} ||y_{n,i} - x_n|| = 0, \ i = 1, 2, \ldots$$

Furthermore,

$$\begin{aligned}
&||y_{n,i} - x^*||^2 \leq \alpha_{n,i}||x_n - x^*||^2 + (1-\alpha_{n,i})||T_i u_n - x^*||^2 \\
&\leq \alpha_{n,i}||x_n - x^*||^2 + (1-\alpha_{n,i})||u_n - x^*||^2 \\
&\leq \alpha_{n,i}||x_n - x^*||^2 + (1-\alpha_{n,i})||z_n - x^*||^2 \\
&\leq \alpha_{n,i}||x_n - x^*||^2 + (1-\alpha_{n,i})||T_{r_n}(x_n - r_nAx_n) - T_{r_n}(x^* - r_nAx^*)||^2 \\
&\leq \alpha_{n,i}||x_n - x^*||^2 + (1-\alpha_{n,i})||(x_n - r_nAx_n) - (x^* - r_nAx^*)||^2 \\
&\leq \alpha_{n,i}||x_n - x^*||^2 + (1-\alpha_{n,i})\Big[||x_n - x^*||^2 + r_n(r_n - 2\alpha)||Ax_n - Ax^*||^2\Big] \\
&= ||x_n - x^*||^2 + (1-\alpha_{n,i})r_n(r_n - 2\alpha)||Ax_n - Ax^*||^2.
\end{aligned}$$

Since $0 < a \leq r_n \leq b < 2\alpha$ and $0 \leq \alpha_{n,i} \leq d_i < 1$, we have

$$\begin{aligned}
(1-d_i)a(2\alpha - b)||Ax_n - Ax^*||^2 &\leq ||x_n - x^*||^2 - ||y_{n,i} - x^*||^2 \\
&\leq ||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||).
\end{aligned}$$

Hence, $\lim_{n\to\infty} ||Ax_n - Ax^*|| = 0$. From (9), we have

$$\begin{aligned}
||y_{n,i} - x^*||^2 &\leq \alpha_{n,i}||x_n - x^*||^2 + (1-\alpha_{n,i})||T_i u_n - x^*||^2 \\
&\leq \alpha_{n,i}||x_n - x^*||^2 + (1-\alpha_{n,i})||u_n - x^*||^2 \qquad (13) \\
&\leq \alpha_{n,i}||x_n - x^*||^2 + (1-\alpha_{n,i})||z_n - x^*||^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&||z_n - x^*||^2 \leq ||T_{r_n}(x_n - r_nAx_n) - T_{r_n}(x^* - r_nAx^*)||^2 \\
&\leq \langle (x_n - r_nAx_n) - (x^* - r_nAx^*), z_n - x^* \rangle \\
&= \frac{1}{2}\Big[||(x_n - r_nAx_n) - (x^* - r_nAx^*)||^2 + ||z_n - x^*||^2 \\
&\quad - ||(x_n - r_nAx_n) - (x^* - r_nAx^*) - (z_n - x^*)||^2\Big] \\
&\leq \frac{1}{2}\Big[||x_n - x^*||^2 - ||(x_n - r_nAx_n) - (x^* - r_nAx^*) - (z_n - x^*)||^2 \\
&\quad + ||z_n - x^*||^2\Big] \\
&= \frac{1}{2}\Big[||x_n - x^*||^2 + ||z_n - x^*||^2 - ||z_n - x_n||^2 + 2r_n\langle x_n - z_n, Ax_n - Ax^* \rangle \\
&\quad - r_n^2||Ax_n - Ax^*||^2\Big]
\end{aligned}$$

and hence

$$\begin{aligned}||z_n - x^*||^2 \leq& ||x_n - x^*||^2 - ||z_n - x_n||^2 + 2r_n\langle x_n - z_n, Ax_n - Ax^*\rangle \\ &- r_n^2||Ax_n - Ax^*||^2 \\ \leq& ||x_n - x^*||^2 - ||z_n - x_n||^2 + 2r_n||x_n - z_n||||Ax_n - Ax^*||.\end{aligned} \tag{14}$$

Putting (14) into (13), we have

$$||y_{n,i} - x^*||^2 \leq ||x_n - x^*||^2 - (1 - \alpha_{n,i})||z_n - x_n||^2 + 2r_n||x_n - z_n||||Ax_n - Ax^*||.$$

It follows that

$$\begin{aligned}(1 - d_i)||x_n - z_n||^2 \leq& ||x_n - x^*||^2 - ||y_{n,i} - x^*||^2 + 2r_n||x_n - z_n||||Ax_n - Ax^*|| \\ \leq& ||y_{n,i} - x_n||(||x_n - x^*|| + ||y_{n,i} - x^*||) + 2r_n||x_n - z_n||||Ax_n - Ax^*||.\end{aligned}$$

Therefore, $\lim_{n\to\infty} ||x_n - z_n|| = 0$. This implies that

$$||x_{n+1} - z_n|| \leq ||x_{n+1} - x_n|| + ||x_n - z_n|| \to 0, \ n \to \infty.$$

Since $x_{n+1} \in C_{n+1}$, then

$$||y_{n,i} - x_{n+1}|| \quad \leq \quad ||x_n - x_{n+1}||. \tag{15}$$

But $y_{n,i} = \alpha_{n,i}x_n + (1 - \alpha_{n,i})T_iu_n$ implies that

$$\begin{aligned}||y_{n,i} - x_{n+1}||^2 =& \alpha_{n,i}||x_n - x_{n+1}||^2 + (1 - \alpha_{n,i})||T_iu_n - x_{n+1}||^2 \\ &- \alpha_{n,i}(1 - \alpha_{n,i})||x_n - T_iu_n||^2.\end{aligned} \tag{16}$$

Putting (16) into (15), we have

$$(1-\alpha_{n,i})||T_iu_n - x_{n+1}||^2 \leq \alpha_{n,i}(1-\alpha_{n,i})||x_n - T_iu_n||^2 + (1-\alpha_{n,i})||x_n - x_{n+1}||^2.$$

Thus,

$$||T_iu_n - x_{n+1}||^2 \leq \alpha_{n,i}||x_n - T_iu_n||^2 + ||x_n - x_{n+1}||^2. \tag{17}$$

But

$$\begin{aligned}||T_iu_n - x_{n+1}||^2 =& ||x_{n+1} - x_n||^2 + 2\langle x_{n+1} - x_n, x_n - T_iu_n\rangle \\ &+ ||x_n - T_iu_n||^2.\end{aligned} \tag{18}$$

Putting (18) into (17), we have

$$\begin{aligned}(1 - d_i)||x_n - T_iu_n||^2 \leq& -2\langle x_{n+1} - x_n, x_n - T_iu_n\rangle \\ \leq& 2||x_{n+1} - x_n||||x_n - T_iu_n|| \to 0, \ n \to \infty.\end{aligned}$$

Hence, $\lim_{n\to\infty} ||x_n - T_iu_n|| = 0, \ i = 1, 2, ...$ Furthermore,

$$\begin{aligned}&||y_{n,i} - x^*||^2 \leq \alpha_{n,i}||x_n - x^*||^2 + (1 - \alpha_{n,i})||T_iu_n - x^*||^2 \\ &\leq \alpha_{n,i}||x_n - x^*||^2 + (1 - \alpha_{n,i})||u_n - x^*||^2 \\ &\leq \alpha_{n,i}||x_n - x^*||^2 + (1 - \alpha_{n,i})||P_K(z_n - \lambda_nBz_n) - P_K(x^* - \lambda_nBx^*)||^2 \\ &\leq \alpha_{n,i}||x_n - x^*||^2 + (1 - \alpha_{n,i})||(z_n - \lambda_nBz_n) - (x^* - \lambda_nBx^*)||^2 \\ &\leq \alpha_{n,i}||x_n - x^*||^2 + (1 - \alpha_{n,i})[||z_n - x^*||^2 + \lambda_n(\lambda_n - 2\beta)||Bz_n - Bx^*||^2] \\ &\leq ||x_n - x^*||^2 + (1 - \alpha_{n,i})\lambda_n(\lambda_n - 2\beta)||Bz_n - Bx^*||^2.\end{aligned}$$

Thus,

$$(1-\alpha_{n,i})\lambda_n(2\beta-\lambda_n)||Bz_n-Bx^*||^2 \leq ||x_n-x^*||^2-||y_{n,i}-x^*||^2$$
$$\leq ||y_{n,i}-x_n||(||x_n-x^*||+||y_{n,i}-x^*||).$$

Since $0<c\leq\lambda_n\leq f<2\beta,\ 0\leq\alpha_{n,i}\leq d_i<1$, we have that $\lim_{n\to\infty}||Bz_n-Bx^*||=0$. Now, from (5), we obtain

$$||u_n-x^*||^2 \leq ||P_K(z_n-\lambda_nBz_n)-P_K(x^*-\lambda_nBx^*)||^2$$
$$\leq \langle (z_n-\lambda_nBz_n)-(x^*-\lambda_nBx^*), u_n-x^*\rangle$$
$$=\frac{1}{2}\Big[||(z_n-\lambda_nBz_n)-(x^*-\lambda_nBx^*)||^2+||u_n-x^*||^2$$
$$-||(z_n-\lambda_nBz_n)-(x^*-\lambda_nBx^*)-(u_n-x^*)||^2\Big]$$
$$\leq \frac{1}{2}\Big[||z_n-x^*||^2+||u_n-x^*||^2-||(z_n-\lambda_nBz_n)-(x^*-\lambda_nBx^*)-(u_n-x^*)||^2\Big]$$
$$=\frac{1}{2}\Big[||x_n-x^*||^2+||u_n-x^*||^2-||u_n-z_n||^2+2\lambda_n\langle z_n-u_n, Bz_n-Bx^*\rangle$$
$$-\lambda_n^2||Bz_n-Bx^*||^2\Big].$$

Thus,

$$||u_n-x^*||^2 \leq ||x_n-x^*||^2-||u_n-z_n||^2+2\lambda_n||z_n-u_n||||Bz_n-Bx^*||.$$

Using this last inequality, we obtain from (9) that

$$||y_{n,i}-x^*||^2 \leq \alpha_{n,i}||x_n-x^*||^2+(1-\alpha_{n,i})||T_iu_n-x^*||^2$$
$$\leq \alpha_{n,i}||x_n-x^*||^2+(1-\alpha_{n,i})||u_n-x^*||^2$$
$$\leq ||x_n-x^*||^2-(1-\alpha_{n,i})||u_n-z_n||^2$$
$$+2\lambda_n(1-\alpha_{n,i})||u_n-z_n||||Bz_n-Bx^*||.$$

This implies that

$$(1-\alpha_{n,i})||u_n-z_n||^2 \leq ||x_n-x^*||^2-||y_{n,i}-x^*||^2$$
$$+2\lambda_n(1-\alpha_{n,i})||u_n-z_n||||Bz_n-Bx^*||$$
$$\leq ||y_{n,i}-x_n||(||x_n-x^*||+||y_{n,i}-x^*||)$$
$$+2\lambda_n(1-\alpha_{n,i})||u_n-z_n||||Bz_n-Bx^*||$$

Since $0\leq\alpha_{n,i}\leq d_i<1$, we have $\lim_{n\to\infty}||u_n-z_n||=0$. Consequently,

$$||x_n-T_ix_n|| \leq ||x_n-T_iu_n||+||T_iu_n-T_ix_n||$$
$$\leq ||x_n-T_iu_n||+||u_n-x_n||$$
$$\leq ||x_n-T_iu_n||+||u_n-z_n||+||z_n-x_n||.$$

Since $\lim_{n\to\infty}||x_n - z_n|| = 0$, $\lim_{n\to\infty}||u_n - z_n|| = 0$ and $\lim_{n\to\infty}||x_n - T_i u_n|| = 0$, we have that $\lim_{n\to\infty}||x_n - T_i x_n|| = 0$, $i = 1, 2, ...$ Now, by $\lim_{n\to\infty}||x_n - z|| = 0$ and $\lim_{n\to\infty}||x_n - T_i x_n|| = 0$, $i = 1, 2, ...,$ we have that $z \in \bigcap_{i=1}^{\infty} F(T_i)$.
We next show that $z \in EP$. Since $z_n := T_{r_n}(x_n - r_n A x_n)$, $n \geq 1$, we have for any $y \in K$ that

$$F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0.$$

Furthermore, replacing n by n_j in the last inequality and using $(A2)$, we obtain

$$\langle Ax_{n_j}, y - z_{n_j}\rangle + \frac{1}{r_{n_j}}\langle y - z_{n_j}, z_{n_j} - x_{n_j}\rangle \geq F(y, z_{n_j}). \tag{19}$$

Let $z_t := ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in K$. This implies that $z_t \in K$. Then, by (19), we have

$$\begin{aligned}\langle z_t - z_{n_j}, Az_t\rangle &\geq \langle z_t - z_{n_j}, Az_t\rangle - \langle z_t - z_{n_j}, Ax_{n_j}\rangle - \Big\langle z_t - z_{n_j}, \frac{z_{n_j} - x_{n_j}}{r_{n_j}}\Big\rangle + F(z_t, z_{n_j})\\ &= \langle z_t - z_{n_j}, Az_t - Az_{n_j}\rangle + \langle z_t - z_{n_j}, Az_{n_j} - Ax_{n_j}\rangle\\ &\quad - \Big\langle z_t - z_{n_j}, \frac{z_{n_j} - x_{n_j}}{r_{n_j}}\Big\rangle + F(z_t, z_{n_j}).\end{aligned}$$

Since $||x_{n_j} - z_{n_j}|| \to 0$, $j \to \infty$, we obtain $||Ax_{n_j} - Az_{n_j}|| \to 0$, $j \to \infty$. Furthermore, by the monotonicity of A, we obtain $\langle z_t - z_{n_j}, Az_t - Az_{n_j}\rangle \geq 0$. Then, by $(A4)$ we obtain

$$\langle z_t - z, Az_t\rangle \geq F(z_t, z),\ j \to \infty. \tag{20}$$

Using $(A1)$, $(A4)$ and (20) we also obtain

$$\begin{aligned}0 = F(z_t, z_t) &\leq tF(z_t, y) + (1-t)F(z_t, z)\\ &\leq tF(z_t, y) + (1-t)\langle z_t - z, Az_t\rangle\\ &= tF(z_t, y) + (1-t)t\langle y - z, Az_t\rangle\end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle y - z, Az_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq F(z, y) + \langle y - z, Az\rangle. \tag{21}$$

This implies that $z \in EP$.

Following the line of arguments of Theorem 3.1, page 346-347 of [10], we can show that $z \in VI(K, B)$. Therefore, $z \in \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP \bigcap VI(K, B)$.

Noting that $x_n = P_{C_n} x_0$, we have by (6) that

$$\langle x_0 - x_n, y - x_n\rangle \leq 0,$$

for all $y \in C_n$. Since $F \subset C_n$, we obtain from the above inequality that

$$\langle x_0 - z, y - z \rangle \leq 0,$$

for all $y \in F$. By (6), we conclude that $z = P_F x_0$. This completes the proof. □

Corollary 2. *Let K be a closed convex nonempty subset of a real Hilbert space H. Let F be a bi-function from $K \times K$ satisfying $(A1) - (A4)$ and for each $i = 1, 2, ...,$ let $T_i : K \to K$ be a nonexpansive mapping. Let S be a k-strictly pseudocontractive map of K into H. Suppose $F := \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP(F) \bigcap F(S) \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, ...)$ and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in K$,*

$$\begin{cases} C_{1,i} = K, \ C_1 = \bigcap_{i=1}^{\infty} C_{1,i} \\ x_1 = P_{C_1} x_0 \\ F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0 \ \forall y \in K, \ n \geq 1 \\ u_n = P_K((1 - \lambda_n) z_n + \lambda_n S z_n) \\ y_{n,i} = \alpha_{n,i} x_n + (1 - \alpha_{n,i}) T_i u_n, \ n \geq 1 \\ C_{n+1,i} = \{z \in C_{n,i} : ||y_{n,i} - z|| \leq ||x_n - z||\}, \ n \geq 1 \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i} \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \geq 1. \end{cases}$$

Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset [0, 1)$ $(i = 1, 2, ...)$, $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 1 - k]$ satisfy

$$\liminf_{n \to \infty} r_n > 0, \ 0 < c \leq \lambda_n \leq f < 1 - k, \ 0 \leq \alpha_{n,i} \leq d_i < 1.$$

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_F x_0$.

Proof. Let $B := I - S$, where S is k-strictly pseudocontractive map. Then, B is $\frac{1-k}{2}$ inverse-strongly monotone. Furthermore, putting $A \equiv 0$ in Theorem 1, we obtain the desired result. □

Corollary 3. *Let K be a closed convex nonempty subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of K into H, B be an β-inverse-strongly monotone mapping of K into H and for each $i = 1, 2, ...,$ let $T_i : K \to K$ be a nonexpansive mapping. Suppose $F := \bigcap_{i=1}^{\infty} F(T_i) \bigcap VI(K, A) \bigcap VI(K, B) \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \dots)$*

and $\{x_n\}_{n=0}^{\infty}$ *be generated by* $x_0 \in K$,

$$\begin{cases} C_{1,i} = K,\ C_1 = \bigcap_{i=1}^{\infty} C_{1,i} \\ x_1 = P_{C_1} x_0 \\ z_n = P_K(x_n - r_n A x_n),\ n \geq 1 \\ u_n = P_K(z_n - \lambda_n B z_n),\ n \geq 1 \\ y_{n,i} = \alpha_{n,i} x_n + (1-\alpha_{n,i}) T_i u_n,\ n \geq 1 \\ C_{n+1,i} = \{z \in C_{n,i} : ||y_{n,i} - z|| \leq ||x_n - z||\},\ n \geq 1 \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i} \\ x_{n+1} = P_{C_{n+1}} x_0,\ n \geq 1. \end{cases}$$

Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset [0,1)$ $(i = 1, 2, ...)$, $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$ *and* $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 2\beta]$ *satisfy*

$$0 < a \leq r_n \leq b < 2\alpha,\ 0 < c \leq \lambda_n \leq f < 2\beta,\ 0 \leq \alpha_{n,i} \leq d_i < 1.$$

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_F x_0$.

Proof. Taking $F(x, y) = 0,\ \forall x, y \in K$ in Theorem 1, we have

$$\langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0\ \forall y \in K,\ \forall n \geq 1.$$

Thus

$$\langle y - z_n, x_n - r_n A x_n - z_n\rangle \leq 0\ \forall y \in K,\ \forall n \geq 1.$$

This implies

$$P_K(x_n - r_n A x_n) = z_n,\ \forall n \geq 1.$$

Hence, the desired conclusion follows from Theorem 1. □

Corollary 4. *Let* K *be a closed convex nonempty subset of a real Hilbert space* H. *Let* B *be an* β*-inverse-strongly monotone mapping of* K *into* H. *For each* $i = 1, 2, ...$, *let* $T_i : K \to K$ *be a nonexpansive mapping. Suppose* $\bigcap_{i=1}^{\infty} F(T_i) \bigcap VI(K, B) \neq \emptyset$. *Let* $\{z_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, ...)$ *and* $\{x_n\}_{n=0}^{\infty}$ *be generated by* $x_0 \in K$,

$$\begin{cases} C_{1,i} = K,\ C_1 = \bigcap_{i=1}^{\infty} C_{1,i} \\ x_1 = P_{C_1} x_0 \\ u_n = P_K(x_n - \lambda_n B x_n),\ n \geq 1 \\ y_{n,i} = \alpha_{n,i} x_n + (1-\alpha_{n,i}) T_i u_n,\ n \geq 1 \\ C_{n+1,i} = \{z \in C_{n,i} : ||y_{n,i} - z|| \leq ||x_n - z||\},\ n \geq 1 \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i} \\ x_{n+1} = P_{C_{n+1}} x_0,\ n \geq 1. \end{cases}$$

Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset [0,1)$ $(i = 1, 2, ...)$ *and* $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 2\beta]$ *satisfy*

$$0 < c \leq \lambda_n \leq f < 2\beta,\ 0 \leq \alpha_{n,i} \leq d_i < 1.$$

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_F x_0$*.*

Proof. Taking $F(x,y) = 0,\ \forall x, y \in K,\ A \equiv 0$ and $r_n = 1$ in Theorem 1, we have the desired conclusion from Theorem 1. □

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