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THE CONVERGENCE OF HOMOTOPY METHODS FOR NONLINEAR KLEIN-GORDON EQUATION

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Abstract. In this paper, a Klein-Gordon equation is solved by using the homotopy analysis method (HAM), homotopy perturbation method (HPM) and modified homotopy perturbation method (MHPM). The approximation solution of this equation is calculated in the form of series which its components are computed easily. The uniqueness of the solution and the convergence of the proposed methods are proved. The accuracy of these methods are compared by solving an example.

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Key words and phrases : Klein-Gordon equation, Homotopy analysis method (HAM), Homotopy perturbation method (HPM), Modified homotopy perturbation method (MHPM).

1. Introduction

Klein-Gordon equation plays an important role in mathematical physics. The equation has attracted much attention in studying solitons and condensed matter physics [4], in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [6]. In recent years some works have been done in order to find the numerical solution of this equation. For example, spline difference method for solving Klein-Gordon equations [9], invariant-conserving finite difference algorithms for the nonlinear Klein-Gordon equation [13], a Legender spectral method [2], application of homotopy perturbation method to Klein-Gordon equation [5]. In this work, we compare the HAM, HPM and MHPM to solve the Klein-Gordon equation as follows:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -F(u), \tag{1}$$

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with the initial conditions given by :

$$u(x,0) = f(x), \quad \frac{\partial u(x,t)}{\partial t} \mid_{t=0} = g(x).$$

where, $F(u)$ is a linear or nonlinear function and $u(x,t)$ is unknown.

The paper is organized as follows. In section 2, the mentioned iterative methods are introduced for solving Eq.(1). Also, the uniqueness of the solution and convergence of the proposed in section 3. Finally, the numerical example is presented method are proved in section 4 to illustrate the accuracy of these methods.

To obtain the approximate solution of Eq.(1), by integrating 2 times from Eq.(1) with respect to t and using the initial conditions we obtain,

$$u(x,t) = G(x,t) + \int_0^t \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau\, d\tau - \int_0^t \int_0^t F(u(x,\tau))\, d\tau\, d\tau, \tag{2}$$

where

$$G(x,t) = f(x) + tg(x).$$

The double integrals in (2) can be written as [15]:

$$\int_0^t \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau\, d\tau = \int_0^t (t-\tau)\, \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau,$$
$$\int_0^t \int_0^t F(u(x,\tau))\, d\tau\, d\tau = \int_0^t (t-\tau)\, F(u(x,\tau))\, d\tau.$$

So, we can write Eq.(2) as follows:

$$u(x,t) = G(x,t) + \int_0^t (t-\tau)\, \frac{\partial^2 u(x,\tau)}{\partial x^2}\, d\tau - \int_0^t (t-\tau)\, F(u(x,\tau))\, d\tau. \tag{3}$$

Now we decompose the unknown function $u(x,t)$ by the following decomposition series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

2. Iterative methods

2.1 Description of the HAM Consider

$$N[u] = 0,$$

where N is a nonlinear operator, $u(x,t)$ is unknown function and x is an independent variable. Let $u_0(x,t)$ denote an initial guess of the exact solution $u(x,t)$, $h \neq 0$ an auxiliary parameter, $H(x,t) \neq 0$ an auxiliary function, and L an auxiliary nonlinear operator with the property $L[r(x,t)] = 0$ when $r(x,t) = 0$.

Then using $q \in [0,1]$ as an embedding parameter, we construct a homotopy as follows:

$$\begin{aligned}&(1-q)L[\phi(x,t;q)-u_0(x,t)]-qhH(x,t)N[\phi(x,t;q)]\\=&\hat{H}[\phi(x,t;q);u_0(x,t),H(x,t),h,q].\end{aligned} \tag{4}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x,t)$, the auxiliary nonlinear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x,t)$ [11,12].

Enforcing the homotopy (4) to be zero, i.e.,

$$\hat{H}[\phi(x,t;q);u_0(x,t),H(x,t),h,q]=0, \tag{5}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q)-u_0(x,t)]=qhH(x,t)N[\phi(x,t;q)]. \tag{6}$$

When $q=0$, the zero-order deformation Eq.(6) becomes

$$\phi(x,t;0)=u_0(x,t) \tag{7}$$

and when $q=1$, since $h \neq 0$ and $H(x,t) \neq 0$, the zero-order deformation Eq.(6) is equivalent to

$$\phi(x,t;1)=u(x,t). \tag{8}$$

Thus, according to (7) and (8), as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$. Such a kind of continuous variation is called deformation in homotopy.

Due to Taylor's theorem, $\phi(x,t;q)$ can be expanded in a power series of q as follows

$$\phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{9}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;q)}{\partial q^m}\mid_{q=0}.$$

Let the initial guess $u_0(x,t)$, the auxiliary nonlinear parameter L, the nonzero auxiliary parameter h and the auxiliary function $H(x,t)$ be properly chosen so that the power series (9) of $\phi(x,t;q)$ converges at $q=1$, then, we have under these assumptions the solution series

$$u(x,t)=\phi(x,t;1)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t). \tag{10}$$

From Eq.(9), we can write Eq.(4) as follows

$$(1-q)L[\phi(x,t,q)-u_0(x,t)]=(1-q)L[\sum_{m=1}^{\infty}u_m(x,t)\ q^m]=q\ h\ H(x,t)N[\phi(x,t,q)]$$

$$\Rightarrow L[\sum_{m=1}^{\infty}u_m(x,t)\ q^m]-q\ L[\sum_{m=1}^{\infty}u_m(x,t)q^m]=q\ h\ H(x,t)N[\phi(x,t,q)] \tag{11}$$

By differentiating (11) m times with respect to q, we obtain

$$\{L[\sum_{m=1}^{\infty}u_m(x,t)\ q^m]-q\ L[\sum_{m=1}^{\infty}u_m(x,t)q^m]\}^{(m)}=\{q\ h\ H(x,t)N[\phi(x,t,q)]\}^{(m)}$$

$$=m!\ L[u_m(x,t)-u_{m-1}(x,t)]=h\ H(x,t)\ m\ \frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}\ .$$

Therefore,

$$\begin{aligned}&L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hH(x,t)\Re_m(u_{m-1}(x,t)),\\&u_m(0)=0,\end{aligned} \tag{12}$$

where

$$\Re_m(u_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}, \tag{13}$$

and

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$$

Note that the high-order deformation Eq.(12) is governing the nonlinear operator L, and the term $\Re_m(u_{m-1}(x,t))$ can be expressed simply by (13) for any nonlinear operator N. To obtain the approximation solution of Eq.(1), according to HAM, let

$$N[u]=u(x,t)-G(x,t)-\int_0^t(t-\tau)D^2(u(x,\tau))d\tau+\int_0^t(t-\tau)F(u(x,\tau))d\tau$$

so,

$$\begin{aligned}\Re_m(u_{m-1}(x,t))=u_{m-1}(x,t)&-\int_0^t(t-\tau)D^2(u(x,\tau))d\tau\\&+\int_0^t(t-\tau)F(u(x,\tau))d\tau-(1-\chi_m)G(x,t)\end{aligned} \tag{14}$$

Substituting (14) into (12)

$$\begin{aligned}L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hH(x,t)\Big[u_{m-1}(x,t)&-\int_0^t(t-\tau)D^2(u(x,\tau))d\tau\\&+\int_0^t(t-\tau)F(u(x,\tau))d\tau-(1-\chi_m)G(x,t)\Big].\end{aligned} \tag{15}$$

We take an initial guess $u_0(x,t) = G(x,t)$, an auxiliary nonlinear operator $Lu = u$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H(x,t) = 1$. This is substituted into (15) to give the following recurrence relation

$$\begin{aligned} &u_0(x,t) = G(x,t), \\ &u_n(x,t) = \int_0^t (t-\tau)D^2(u_{n-1}(x,\tau))d\tau - \int_0^t (t-\tau)F(u_{n-1}(x,\tau))d\tau, \quad n \geq 1 \end{aligned} \tag{16}$$

Therefore, the solution $u(x,t)$ becomes

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\ &= G(x,t) + \sum_{n=0}^{\infty} \int_0^t (t-\tau)D^2(u_{n-1}(x,\tau))d\tau - \int_0^t (t-\tau)F(u_{n-1}(x,\tau))d\tau \end{aligned} \tag{17}$$

Which is the method of successive approximations. If

$$| u_n(x,t) | < 1$$

then the series solution (17) convergence uniformly.

2.2 Description of the HPM and MHPM

To explain HPM [3,8], we consider the following general nonlinear differential equation:

$$Lu + Nu = f(u) \tag{18}$$

with initial conditions

$$\begin{aligned} &u(x,0) = f(x), \\ &\frac{\partial u(x,t)}{\partial t} |_{t=0} = g(x). \end{aligned}$$

according to HPM, we construct a homotopy which satisfies the following relation

$$H(u,p) = Lu - Lv_0 + p\ Lv_0 + p\ [Nu - f(u)] = 0, \tag{19}$$

where $p \in [0,1]$ is an embedding parameter and v_0 is an arbitrary initial approximation satisfying the given initial conditions.

In HPM, the solution of Eq.(19) is expressed as

$$u(x,t) = u_0(x,t) + p\ u_1(x,t) + p^2\ u_2(x,t) + ... \tag{20}$$

Hence the approximate solution of Eq.(18) can be expressed as a series of the power of p, i.e.

$$u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + ...$$

where,

$$\begin{aligned} &u_0(x,t) = G(x,t),\\ &\vdots\\ &u_m(x,t) = \sum_{k=0}^{m-1} \int_0^t (t-\tau) D^2(u_{m-k-1}(x,\tau))d\tau - F(u_{m-k-1}(x,\tau))d\tau. \end{aligned} \tag{21}$$

To explain MHPM [1,7,10], we consider Eq. (1) as

$$L(u) = u(x,t) - G(x,t) - \int_0^t (t-\tau) D^2(u_{n-1}(x,\tau))d\tau + \int_0^t (t-\tau) F(u_{n-1}(x,\tau))d\tau,$$

where $D^2(u(x,\tau)) = g_1(x)h_1(\tau)$ and $F(u(x,\tau)) = g_2(x)h_2(\tau)$. We can define homotopy $H(u,p,m)$ by

$$H(u,0,m) = f(u), \quad H(u,1,m) = L(u),$$

where, m is an unknown real number and

$$f(u(x,t)) = u(x,t) - G(x,t).$$

Typically we may choose a convex homotopy by $0 \leq p \leq 1$

$$H(u,p,m) = (1-p)f(u) + p\ L(u) + p\ (1-p)[m(g_1(x) + g_2(x))] = 0, \tag{22}$$

where m is called the accelerating parameters, and for $m = 0$ we define $H(u,p,0) = H(u,p)$, which is the standard HPM.

The convex homotopy (22) continuously trace an implicity defined curve from a starting point $H(u(x,t) - f(u), 0, m)$ to a solution function $H(u(x,t), 1, m)$. The embedding parameter p monotonically increase from 0 to 1 as trivial problem $f(u) = 0$ is continuously deformed to original problem $L(u) = 0$.

The MHPM uses the homotopy parameter p as an expanding parameter to obtain

$$v = \sum_{n=0}^{\infty} p^n u_n, \tag{23}$$

when $p \to 1$, Eq. (19) corresponds to the original one and Eq. (23) becomes the approximate solution of Eq. (1), i.e.,

$$u = \lim_{p\to 1} v = \sum_{m=0}^{\infty} u_m,$$

where,

$$
\begin{aligned}
&u_0(x,t) = G(x,t),\\
&u_1(x,t) = \sum_{k=0}^{m-1} \int_0^t (t-\tau) D^2(u_0(x,\tau))d\tau - F(u_0(x,\tau))d\tau - m(g_1(x)+g_2(x)),\\
&u_2(x,t) = \sum_{k=0}^{m-1} \int_0^t (t-\tau) D^2(u_1(x,\tau))d\tau - F(u_1(x,\tau))d\tau + m(g_1(x)+g_2(x)),\\
&\vdots\\
&u_m(x,t) = \sum_{k=0}^{m-1} \int_0^t (t-\tau) D^2(u_{m-1}(x,\tau))d\tau - F(u_{m-1}(x,\tau))d\tau,\ m \geq 3
\end{aligned}
\tag{24}
$$

3. Convergence of the methods

We assume $G(x,t)$ is bounded for all τ, t in $J = [0,T](T \in \mathbb{R})$ and

$$\mid t-\tau \mid \leq M^{'}, \quad \forall\ 0 \leq t, \tau \leq T, M^{'} \in \mathbb{R}.$$

Also, we suppose $D^2(u(x,\tau)) = \frac{d^2}{dx^2}u(x,t)$ and $F(u)$ are Lipschitz continuous with $\mid D^2(u) - D^2(u^*) \mid \leq L_1 \mid u - u^* \mid$, $\mid F(u) - F(u^*) \mid \leq L_2 \mid u - u^* \mid$. We set,

$$\alpha = TM^{'}(L_1 + L_2).$$

Theorem 1. *Let $0 < \alpha < 1$, then nonlinear hyperbolic partial differential equation (3), has a unique solution.*

Proof. Let u and u^* be two different solutions of (3) then

$$
\begin{aligned}
\mid u - u^* \mid =& \mid \int_0^t (t-\tau)\ [D^2(u(x,\tau)) - D^2(u^*(x,\tau))]\ d\tau\\
&- \int_0^t (t-\tau)\ [F(u(x,\tau)) - F(u^*(x,\tau))]\ d\tau \mid\\
&\leq \int_0^t \mid (t-\tau) \mid\mid D^2(u(x,\tau)) - D^2(u^*(x,\tau)) \mid\ d\tau\\
&+ \int_0^t \mid (t-\tau) \mid\mid F(u(x,\tau)) - F(u^*(x,\tau)) \mid\ d\tau\\
&\leq TM^{'}(L_1+L_2)\ \mid u - u^* \mid = \alpha \mid u - u^* \mid
\end{aligned}
$$

From which we get $(1-\alpha) \mid u - u^* \mid \leq 0$. Since $0 < \alpha < 1$, then $\mid u - u^* \mid = 0$. Implies $u = u^*$ and completes the proof. □

Theorem 2. *If the series solution (16) of problem (3) using HAM is convergent then it converges to the exact solution of the problem (3).*

Proof. We assume:

$$u(x,t)=\sum_{m=0}^{\infty}u_m(x,t),\ \ \widehat{D^2}(u(x,t)=\sum_{m=0}^{\infty}D^2(u_m(x,t)),\ \ \widehat{F}(u(x,t)=\sum_{m=0}^{\infty}F(u_m(x,t)),$$

where,

$$\lim_{m\to\infty}u_m(x,t)=0.$$

We can write,

$$\begin{aligned}\sum_{m=1}^{n}[u_m(x,t)-\chi_m u_{m-1}(x,t)]&=u_1+(u_2-u_1)+\ldots+(u_n-u_{n-1})\\&=u_n(x,t).\end{aligned}\tag{25}$$

Hence, from (25)

$$\lim_{n\to\infty}u_n(x,t)=0.\tag{26}$$

So, using (26) and the definition of the nonlinear operator L, we have

$$\sum_{m=1}^{\infty}L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=L[\sum_{m=1}^{\infty}[u_m(x,t)-\chi_m u_{m-1}(x,t)]]=0.$$

Therefore from (12), we can obtain that,

$$\sum_{m=1}^{\infty}L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hH(x,t)\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t))=0.$$

Since $h\neq 0$ and $H(x,t)\neq 0$, we have

$$\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t))=0.\tag{27}$$

By substituting $\Re_{m-1}(u_{m-1}(x,t))$ into the relation (27) and simplifying it , we have

$$\begin{aligned}\sum_{m=1}^{\infty}\Re_{m-1}(u_{m-1}(x,t))&=\sum_{m=1}^{\infty}[u_{m-1}(x,t)-\int_0^t(t-\tau)D^2(u_{m-1}(x,\tau))d\tau\\&\quad+\int_0^t(t-\tau)F(u(_{m-1}x,\tau))d\tau-(1-\chi_m)G(x,t)]\\&=u(x,t)-G(x,t)-\int_0^t(t-\tau)\widehat{D^2}(u_{m-1}(x,\tau))d\tau\\&\quad+\int_0^t(t-\tau)\widehat{F}(u(_{m-1}x,\tau))d\tau].\end{aligned}\tag{28}$$

From (27) and (28), we have

$$u(x,t)=G(x,t)+\int_0^t(t-\tau)\widehat{D^2}(u_{m-1}(x,\tau))d\tau-\int_0^t(t-\tau)\widehat{F}(u_{m-1}(x,\tau))d\tau,$$

Therefore, $u(x,t)$ must be the exact solution. □

Theorem 3. *The series solution* $u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$ *of problem (3) using HPM converges [3].*

Remark 1. *Proving of convergence the MHPM is similar to proving of convergence HPM.*

4. Numerical example

In this section, we compute a numerical example which is solved by the HAM, HPM and MHPM. The program has been provided with Mathematica 6 according to the following algorithm. In this algorithm ε is a given positive value.

Algorithm:

Step 1. Set $n \leftarrow 0$.

Step 2. Calculate the recursive relation (16) for HAM, (21) for HPM and (24) for MHPM.

Step 3. If $\mid u_{n+1} - u_n \mid < \varepsilon$ then go to step 4,
else $n \leftarrow n+1$ and go to step 2.

Step 4. Print $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$ as the approximate of the exact solution.

Lemma 1. *The computational complexity of the HAM is* $O(n)$*, HPM and MHPM are* $O(n^2)$*.*

Proof. The number of computations including division, production, sum and subtraction.

HAM:
In step 2,
$u_0 : 2.$
$u_1 : 5.$
.
.
$u_{n+1} : 5.$

In step 4, the total number of the computations is equal to $u_0 + \sum_{i=1}^{n+1} u_i(x,t) = 5n + 7 = O(n)$.

HPM:
$u_0 : 2.$
$u_1 : 3.$
.
.
$u_{n+1} : 3(n+1).$

In step 4, the total number of the computations is equal to $u_0 + \sum_{i=1}^{n+1} u_i(x,t) = 1 + \sum_{i=1}^{n+1} 3(i+1) = \frac{3}{2}n^2 + \frac{9}{2}n + 5 = O(n^2)$.

MHPM:
In step 2,

$u_0 : 2.$
$u_1 : 5.$
$u_2 : 5.$
.
.
$u_{n+1} : 3(n+1).$
In step 4, the total number of the computations is equal to $u_0 + \sum_{i=1}^{n+1} u_i(x,t) = 1 + 5 + 5 + \sum_{i=3}^{n+1} 3(i+1) = \frac{3}{2}n^2 + \frac{9}{2}n + 6 = O(n^2)$. □

By comparing the results of computational complexity, we see that the number of computations in HAM is less than the number of computations in HPM and MHPM.

Example 1. *Consider the nonlinear Klein-Gordon equation [14]*

$$u_{tt} - u_{xx} = -u^2,$$

subject to initial conditions

$$u(x,0) = 1 + sinx, \quad u_t(x,0) = 0.$$

Table 1. Numerical results for Example 1

x	t=0.1				t=0.2			
	MADM (n=5)	HPM (n=4)	MHPM (n=3)	HAM (n=2)	MADM (n=5)	HPM (n=4)	MHPM (n=3)	HAM (n=2)
0.0	0.994999	0.995833	0.996782	0.998991	0.979999	0.981669	0.988695	0.994452
0.1	1.093291	1.094279	1.094591	1.096279	1.073723	1.079324	1.083532	1.088768
0.2	1.190502	1.190875	1.191375	1.918967	1.166134	1.171348	1.178759	1.184766
0.3	1.285668	1.289843	1.292637	1.298875	1.256326	1.262314	1.270168	1.279378
0.4	1.377844	1.382993	1.386374	1.394356	1.343423	1.349664	1.355784	1.364469

Table 1 shows that, approximate solution of the nonlinear Klein-Gordon equation is convergence with 2 iterations by using the HAM.

5. Conclusion

The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are convergent are rapidly to exact solutions. In this work, the HAM has been successfully employed to obtain the approximate solution to analytical solution of the Klein-Gordon equation. For this purpose, we showed that the HAM is more rapid convergence than the HPM and MHPM. Also, the number of computations in HAM is less than the number of computations in HPM and MHPM.

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