ANALYTICAL SOLUTION OF COUPLED RADIATION-CONVECTION DISSIPATIVE NON-GRAY GAS FLOW IN A NON-DARCY POROUS MEDIUM

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ABSTRACT. The homotopy analysis method (HAM) has been applied to develop an analytic solution for the coupled radiation-convection dissipative non-gray gas flow in a non-Darcy porous medium. Results are presented for the surface shear and temperature profiles are presented to illustrate the effect of various parameters appearing in the analytical formulation. The accuracy and convergence of the method is also discussed.

AMS Mathematics Subject Classification : 65M99, 65Z05, 65B10.
Key words and phrases : Homotopy analysis method, Non-gray gas flow, Convergence, Non-Darcy porous medium.

Nomenclature

f	dimensionless stream function
u, v	velocity along x, y directions
g	acceleration due to gravity
$K_{\lambda W}$	absorption coefficient at the wall
$e_{b\lambda}$	Planck function
F	thermal radiation parameter
q_R	radiative heat flux
C_p	specific heat at constant pressure
T	temperature
U_∞	free stream velocity (at the edge of boundary layer)

T_w, T_∞ wall temperature, free stream temperature

Gr   Grashof number$(= g\beta \triangle T L^3/v^2)$

Pr   Prandtl number$(= \mu_f/\alpha_f \rho_f = v_f/\alpha_f)$

Da   Darcy number$(= k/L^2)$

Re   Reynolds number$(= U_0 L/v_f)$

Fs   Forcheimmer number$(= b/L)$

k permeability of the porous medium

L location of the start of the thermal boundary layer

V transpiration velocity

$\mathcal{L}_1, \mathcal{L}_2$ auxiliary linear operators

q embedding parameter

$\hbar$ non-zero auxiliary parameter

$\mathcal{N}_1, \mathcal{N}_2$ non-linear operators

f_m mth-order approximation

Greek symbols

α generalized boundary layer parameter$(= U_0 \delta \delta_x/v_f)$

α_f thermal diffusivity$)(= \kappa/(\rho C_p)_f)$

β coefficient of volume expansion

ε porosity of the medium

θ dimensionless temperature variable

θ_m mth-order approximation

v_f fluid kinematic viscosity$(= \mu_f/\rho_f)$

κ stagnant thermal conductivity of fluid-saturated porous medium

κ_f fluid thermal conductivity

λ thermal conductivity ratio κ_f/κ

μ_f fluid dynamic viscosity

γ transpiration parameter$(= V\delta/v_f)$

δ boundary layer thickness $\sqrt{2\alpha(x - x_0)v_f/U_0}$

η spanwise pseudo-similarity co-ordinate

ξ streamwise pseudo-similarity co-ordinate

ρ_f fluid density

ψ stream function

Subscripts

f   fluid

w properties at the wall

∞ free stream condition

m order of approximation

1. Introduction

Studies of coupled convective-radiative flows in non-porous media began to appear in the 1960s. Cess [1] analyzed the effect of combined convection and

radiation in the boundary layer flow of an absorbing gas flowing over a flat plate. The paper by Habib and Grief [2] reported the results of analytical and numerical studies of heat transfer in forced flow of a non-gray radiating gas. Their study was later extended to natural convective flow [3]. Kubo [4] investigated the stagnation-point flow of a radiating gas in the limit of large optical thickness. On the other hand, Novotny and Kelleher [5] studied the natural convection driven stagnation flow of an absorbing-emitting gas. Mattic [6] focused on coaxial radiative and convective heat transfer in gray and non-gray gases. A perturbation method was used by Tien and Abu-Romia [7] in the differential analysis of radiation interaction with conduction and convection. The problem of combined convection and radiation in an absorbing, emitting, and scattering gas flow in a tube was solved by Azad and Modest [8]. Webb and Viskanta [9] examined the problem of radiation induced buoyancy flow in rectangular enclosures. While the majority of these studies neglected boundary layer effect, Yucel et al. [10] investigated the boundary layer flow of a non-gray radiating fluid. In recent years, the study of coupled convective flows in porous media has become important due to its applicability in nuclear geo-repositories, geothermal systems, and energy storage devices [11, 12, 13]. These studies are extensions of earlier works on non porous media discussed briefly in the preceding paragraph. Takhar and Beg [11] considered the flow of an optically dense fluid in a non-Darcy porous medium and found that the presence of radiation significantly augmented the overall heat transfer process. In a subsequent paper [14], the same authors considered magneto-hydrodynamic radiative-convective flow in a Darcy-Brinkman-Forcheimmer porous medium and obtained numerical solutions using the Kellers box method which is an implicit finite difference scheme. In the present work, we revisit the problem considered by Takhar, Beg, and Kumari [15] and provide a highly accurate analytical solution of the problem using the homotopy analysis method (HAM). A highly accurate and widely used technique for solving nonlinear problems is the homotopy analysis method (HAM) [16, 17, 18, 19], which has been successfully applied to many nonlinear problems in science and engineering [20, 21, 22, 23]. Unlike the perturbation techniques, HAM is independent of any small physical parameters. More importantly, unlike the perturbation and non-perturbation methods, HAM provides a simple way to ensure the convergence of series solution so that one can always get accurate enough approximations even for the strongly nonlinear problems. Furthermore, HAM provides the freedom to choose the so- called auxiliary linear operator so that one can approximate a nonlinear problem more effectively by means of better base functions, as demonstrated by Liao and Tan [18]. The degree of freedom is so large that even the second-order nonlinear two-dimensional Gelfand equation can be solved by means of a 4th-order auxiliary linear operator within the framework of the HAM as shown in [18]. Especially, by means of the HAM, a few new solutions of some nonlinear problems [24, 25] have been achieved which otherwise were not solvable by other analytic methods.

2. Formulation of the problem

Consider the flow of an optically-thin non-gray radiating viscous dissipating gas past a vertical semi-infinite porous plate in a two-dimensional non-Darcy porous medium. The continuity, momentum conservation and energy , under the Boussinesq approximation system, may be written for the two-dimensional (x, y) flow as follows [15]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\rho_f}{\varepsilon^2}\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\mu_f\, u}{k} + \rho_f g\beta\big(T - T_\infty\big) + \frac{\mu'}{\varepsilon}\frac{\partial^2 u}{\partial y^2} - \frac{\rho_f\, bu^2}{k} \tag{2}$$

$$\rho\, C_p\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = k\frac{\partial^2 T}{\partial y^2} + \mu_f\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\partial q_R}{\partial y}. \tag{3}$$

For simplicity, we assume that the porous medium is isotropic and homogeneous and contains no internal heat sources.

The differential radiative flux term $\partial q_R/\partial y$ can be very complex to model and usually, therefore, an algebraic approximations is employed. Cogley et al. [26] introduced a compact and numerically amenable expression. They showed that in the optically thin limit, the fluid is not self-absorbing but will absorb radiation emitted by the confining boundaries (i.e. flat vertical plate in our case) and the radiative flux gradient near equilibrium can be approximated as

$$\frac{\partial q_R}{\partial y} = 4\big(T - T_w\big)\, I, \tag{4}$$

where I is the intensity parameter and is given by the following integral.

$$I = \int_0^\infty K_{\lambda W}\big(\partial e_{b\lambda}/\partial T\big)_w\, d\lambda. \tag{5}$$

Beg et al. [14] introduced the following pseudo-similarity transformations:

$$\psi = U_0\delta(x)f(\xi, \eta) + \int V(x)\, dx \tag{6}$$

$$\eta = \frac{y}{\delta(x)} \tag{7}$$

$$\xi = \xi(x) = \frac{g\,\beta\Delta T\varepsilon^2(x - x_0)}{U_0^2 L^2} \tag{8}$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad F = \frac{4\, IL^2}{\rho C_p v_f}. \tag{9}$$

The transformation leads to the following pseudo-similarity equations.

$$f''' + f''\left(\alpha f + \gamma + 2\alpha\xi\frac{\partial f}{\partial\xi}\right) - 2\alpha\xi\left(f'\frac{\partial^2 f}{\partial\xi\partial\eta} - \theta + \Big(\frac{\mathrm{Re}}{\mathrm{Gr}\,\mathrm{Da}}\Big)f'\right)$$
$$- 2\alpha\left(\frac{\mathrm{Fs}}{\mathrm{Da}}\frac{\mathrm{Re}^2}{\mathrm{Gr}}\right)\xi\big(f'\big)^2 = 0, \tag{10}$$

$$\left(\frac{1}{\text{Pr}}\right)\theta'' + \lambda \text{Ec}\,(f'')^2 + \lambda\theta'\,(\alpha\,f + \gamma) + 2\alpha\lambda\xi\theta'\frac{\partial f}{\partial\xi} - \left(\frac{2\alpha}{\text{Pr}}\frac{\text{Re}}{\text{Gr}}\right)F\xi\theta - 2\alpha\lambda\,\xi\frac{\partial\theta}{\partial\xi}f' = 0 \quad (11)$$

where primes denote differentiation with respect to η.

The appropriate boundary conditions are given by

$$\begin{aligned}
\eta = 0: \qquad & f(\xi) = 0, \quad f'(\xi) = 0, \quad \theta(\xi) = 1, \\
\eta \to \infty: \qquad & f'(\xi) = 1, \quad \theta(\xi) = 0.
\end{aligned} \quad (12)$$

In the next section, we solve the system of non-linear partial differential equations (10)-(12), analytically, using HAM.

3. HAM solution

In view of the boundary conditions (12), $f(\xi, \eta)$ and $\theta(\xi, \eta)$ can be expressed by the set of base functions of the form

$$\left\{\xi^k\eta^j\exp(-n\,\eta)\big|k \geq 0, j \geq 0, n \geq 0\right\} \quad (13)$$

in the form of the following series

$$\begin{aligned}
f(\xi, \eta) &= a_{0,0}^0 + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{n,k}^j\,\eta^k\xi^j\exp(-n\eta), \\
\theta(\xi, \eta) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} b_{n,k}^j\,\eta^k\xi^j\exp(-n\eta),
\end{aligned} \quad (14)$$

in which $a_{n,k}^j$ and $b_{n,k}^j$ are the coefficients. Invoking the so-called *rule of solution expressions* for $f(\xi, \eta)$ and $\theta(\xi, \eta)$ and Eqs. (10)-(12) the initial guesses $f_0(\eta), \theta_0(\eta)$ and linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are

$$f_0(\xi, \eta) = \eta + \exp(-\eta) - 1, \quad (15)$$

$$\theta_0(\xi, \eta) = \exp(-\eta), \quad (16)$$

$$\mathcal{L}_1(f) = \frac{\partial^3 f}{\partial\eta^3} - \frac{\partial f}{\partial\eta}, \quad (17)$$

$$\mathcal{L}_2(f) = \frac{\partial^2 f}{\partial\eta^2} - f. \quad (18)$$

The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ have the following properties:

$$\mathcal{L}_1\Big(c_1 + c_2\exp(-\eta) + c_3\exp(\eta)\Big) = 0, \quad (19)$$

$$\mathcal{L}_2\Big(c_4\exp(-\eta) + c_5\exp(\eta)\Big) = 0. \quad (20)$$

in which $c_i, i = 1, 2, \cdots, 5$ are arbitrary constants.

Let $q \in [0, 1]$ denotes an embedding parameter and $\hbar$ be a non-zero auxiliary parameter. Then we construct the following zeroth order equations

$$(1 - q)\mathcal{L}_1 \left[\hat{f}(\xi, \eta; q) - f_0(\xi, \eta) \right] = q\,\hbar\mathcal{N}_1 \left[\hat{f}(\xi, \eta; q), \hat{\theta}(\xi, \eta; q) \right], \quad (21)$$

$$(1 - q)\mathcal{L}_2 \left[\hat{\theta}(\xi, \eta; q) - \theta_0(\xi, \eta) \right] = q\,\hbar\mathcal{N}_2 \left[\hat{f}(\xi, \eta; q), \hat{\theta}(\xi, \eta; q) \right], \quad (22)$$

subject to the conditions

$$\hat{f}(\xi, 0; q) = 0, \qquad \hat{f}'(\xi, 0; q) = 0, \qquad \hat{f}'(\xi, +\infty; q) = 1, \qquad (23)$$

$$\hat{\theta}(\xi, 0; q) = 1, \qquad \hat{\theta}(\xi, +\infty; q) = 0, \qquad (24)$$

where the non-linear operators are defined as:

$$\mathcal{N}_1 = \frac{\partial^3 \hat{f}(\xi, \eta; q)}{\partial \eta^3} + \frac{\partial^2 \hat{f}(\xi, \eta; q)}{\partial \eta^2} \left(\alpha \hat{f}(\xi, \eta; q) + \gamma + 2\alpha\xi \frac{\partial \hat{f}(\xi, \eta; q)}{\partial \xi} \right)$$

$$- 2\alpha\xi \left(\frac{\partial \hat{f}(\xi, \eta; q)}{\partial \eta} \frac{\partial^2 \hat{f}(\xi, \eta; q)}{\partial \xi \partial \eta} - \hat{\theta}(\xi, \eta; q) + \left(\frac{\text{Re}}{\text{Gr}\,\text{Da}} \right) \frac{\partial \hat{f}(\xi, \eta; q)}{\partial \eta} \right) \qquad (25)$$

$$- 2\alpha \left(\frac{\text{Fs}}{\text{Da}} \frac{\text{Re}^2}{\text{Gr}} \right) \xi \left(\frac{\partial \hat{f}(\xi, \eta; q)}{\partial \xi} \right)^2,$$

$$\mathcal{N}_2 = \left(\frac{1}{\text{Pr}} \right) \frac{\partial^2 \hat{\theta}(\xi, \eta; q)}{\partial \eta^2} + \lambda\text{Ec} \left(\frac{\partial^2 \hat{f}(\xi, \eta; q)}{\partial \eta^2} \right)^2 + \lambda \frac{\partial \hat{\theta}(\xi, \eta; q)}{\partial \eta} \left(\alpha \hat{f}(\xi, \eta; q) + \gamma \right.$$

$$\left. + 2\alpha\xi \frac{\partial \hat{f}(\xi, \eta; q)}{\partial \xi} \right) - \left(\frac{2\alpha}{\text{Pr}} \frac{\text{Re}}{\text{Gr}} \right) F\xi\hat{\theta}(\xi, \eta; q) - 2\alpha\lambda\xi \frac{\partial \hat{f}(\xi, \eta; q)}{\partial \eta} \frac{\partial \hat{\theta}(\xi, \eta; q)}{\partial \xi}. \qquad (26)$$

For $q = 0$ and $q = 1$ we have

$$\hat{f}(\xi, \eta; 0) = f_0(\xi, \eta), \qquad \hat{f}(\xi, \eta; 1) = f(\xi, \eta),$$

$$\hat{\theta}(\xi, \eta; 0) = \theta_0(\xi, \eta), \qquad \hat{\theta}(\xi, \eta; 1) = \theta(\xi, \eta). \qquad (27)$$

Definition. Let ψ be a function of the homotopy-parameter q, then

$$D_m(\psi) = \frac{1}{m!} \frac{\partial^m \psi}{\partial q^m}\bigg|_{q=0}, \qquad (28)$$

is called the mth-order homotopy-derivative of ψ, where $m \geq 0$ is an integer [20].

As q increases from 0 to 1, $\hat{f}(\xi, \eta; q)$ and $\hat{\theta}(\xi, \eta; q)$ vary from $f_0(\xi, \eta)$ and $\theta_0(\xi, \eta)$ to the exact solutions $f(\xi, \eta)$ and $\theta(\xi, \eta)$, respectively. By Taylor's theorem and Eq. (27), we can write

$$\hat{f}(\xi, \eta; q) = f_0(\xi, \eta) + \sum_{m=1}^{\infty} D_m\left(\hat{f}(\xi, \eta; q) \right) q^m, \qquad (29)$$

$$\hat{\theta}(\xi, \eta; q) = \theta_0(\xi, \eta) + \sum_{m=1}^{\infty} D_m\left(\hat{\theta}(\xi, \eta; q) \right) q^m. \qquad (30)$$

The convergence of the series given in Eqs. (29)-(30) strongly depends upon parameter $\hbar$. Therefore $\hbar$ is properly chosen so that the series (29)-(30) are convergent at $q = 1$ and thus by using Eq. (27), one obtains

$$f(\xi, \eta) = f_0(\xi, \eta) + \sum_{m=1}^{\infty} f_m(\xi, \eta), \tag{31}$$

$$\theta(\xi, \eta) = \theta_0(\xi, \eta) + \sum_{m=1}^{\infty} \theta_m(\xi, \eta). \tag{32}$$

Operating on both sides of Eqs. (21)-(22) with D_m, we have the so called mth-order deformation equations

$$\mathcal{L}_1 \Big[f_m(\xi, \eta) - \chi_m f_{m-1}(\xi, \eta) \Big] = \hbar \mathcal{R}_1(\xi, \eta), \tag{33}$$

$$\mathcal{L}_2 \Big[\theta_m(\xi, \eta) - \chi_m \theta_{m-1}(\xi, \eta) \Big] = \hbar \mathcal{R}_2(\xi, \eta), \tag{34}$$

$$f_m(\xi, 0) = f'_m(\xi, 0) = f'_m(\xi, +\infty) = 0, \tag{35}$$

$$\theta_m(\xi, 0) = \theta_m(\xi, +\infty) = 0, \tag{36}$$

where

$$\mathcal{R}_1(\xi, \eta) = \mathcal{D}_{m-1} \left(\mathcal{N}_1 \Big[\hat{f}(\xi, \eta; q), \hat{\theta}(\xi, \eta; q) \Big] \right),$$

$$\mathcal{R}_2(\xi, \eta) = \mathcal{D}_{m-1} \left(\mathcal{N}_2 \Big[\hat{f}(\xi, \eta; q), \hat{\theta}(\xi, \eta; q) \Big] \right),$$

and

$$\chi_m = \begin{cases} 0 & ; m \le 1, \\ 1 & ; m \ge 2. \end{cases}$$

Therefore, we have

$$\begin{aligned} \mathcal{R}_1(\xi, \eta) = {} & f'''_{m-1} + \alpha \sum_{k=0}^{m-1} f_k f''_{m-1-k} + \gamma f''_{m-1} + 2\alpha\xi \sum_{k=0}^{m-1} \left(\frac{\partial f_k}{\partial \xi} f''_{m-1-k} - \frac{\partial f'_k}{\partial \xi} f'_{m-1-k} \right) \\ & + 2\alpha\xi \left(\theta_{m-1} - \left(\frac{\mathrm{Re}}{\mathrm{Gr}\,\mathrm{Da}} \right) f'_{m-1} \right) - 2\alpha \left(\frac{\mathrm{Fs}\,\mathrm{Re}^2}{\mathrm{Da}\,\mathrm{Gr}} \right) \xi \sum_{k=0}^{m-1} f'_k f'_{m-1-k}, \end{aligned} \tag{37}$$

and

$$\begin{aligned} \mathcal{R}_2(\xi, \eta) = {} & \left(\frac{1}{\mathrm{Pr}} \right) \theta''_{m-1} + \lambda \sum_{k=0}^{m-1} \left(\mathrm{Ec} f''_k f''_{m-1-k} + \alpha f_k \theta'_{m-1-k} \right) + \lambda\gamma\theta'_{m-1} \\ & + 2\alpha\lambda\xi \sum_{k=0}^{m-1} \left(\frac{\partial f_k}{\partial \xi} \theta'_{m-1-k} - f'_k \frac{\partial \theta_{m-1-k}}{\partial \xi} \right) - \left(\frac{2\alpha\,\mathrm{Re}}{\mathrm{Pr}\,\mathrm{Gr}} \right) F \xi \theta_{m-1} \end{aligned} \tag{38}$$

The general solutions of Eqs. (33)-(38) can be written as

$$f_m(\xi, \eta) = f^\star(\xi, \eta) + c_1 + c_2 \exp(-\eta) + c_3 \exp(\eta),$$

$$\theta_m(\xi, \eta) = \theta^\star(\xi, \eta) + c_4 \exp(-\eta) + c_5 \exp(\eta),$$

where $f^\star(\xi, \eta)$ and $\theta^\star(\xi, \eta)$ are the particular solutions and the constants are determined by the boundary conditions (35) and (36).

4. Results and discussion

The series in Eqs. (31) and (32) are the solutions of the considered problem if one guarantees the convergency of these series. As pointed out by Liao [19], the convergence and the rate of approximation for the HAM solution strongly depends upon $\hbar$. In order to obtain the admissible value of $\hbar$ for the present problem, the $\hbar$-curves are plotted for 10th-order of approximations. For example, Figs. (1)-(2) demonstrate that the size of the valid region strongly depends on the parameters Ec and Gr. Figure 1 shows the $\hbar$-curves of $\theta''(0)$ for different values of Eckert number (Ec), in $\xi = 0, \alpha = 1, \gamma = 0.1(\text{suction}), \text{Re} = 1, \text{Gr} = 100, \text{Da} = 0.01, \text{Fs} = 0.055, \text{Pr} = 0.733(\text{air}), \lambda = 1$ and F=100 at 10th-order of approximations.

Figure 2 shows the $\hbar$-curves of $f''(0)$ for different values of Grashof number (Gr) in $\xi = 0, \alpha = 1, \gamma = 0.1, \text{Re} = 1, \text{Da} = 0.01, \text{Ec} = 0.25, \text{Fs} = 0.055, \text{Pr} = 0.9(\text{gas}), \lambda = 1$ and F=250 at 10th-order of approximations. The surface shear stress and local heat transfer parameter functions $f''(\xi, 0)$ and $\theta'(\xi, 0)$ (plate surface temperature gradient),have been computed against ξ for different values of the thermofluid parameters F, γ, Ec and α. Additionally, we have plotted the temperature function θ and f' with spanwise co-ordinate η at the edge of the plate, that is, for $\xi = 1$, for different values of F, γ, Pr, Gr and α. The Reynolds number Re is fixed at 1 unless otherwise indicated, and λ has unit value, i.e. the fluid thermal conductivity and fluid-saturated porous continuum thermal conductivity are equal.

Figure 3 (left) and Figure 3 (right) describe the profiles of $f''(\xi, 0)$ and $\theta'(\xi, 0)$ for different values of radiation parameter F, respectively. Pr is taken as 0.733 corresponding to air. Grashof number is fixed at 100. A clear decrease in shear stress $f''(\xi, 0)$ accompanies an increase in F from 0 (no radiative heat transfer contribution, i.e. purely conductive-convective heat transfer) to 250. Conversely, a sharp increase in the magnitude of the surface heat transfer (temperature gradient) $\theta'(\xi, 0)$ occurs as F is increased from 0 to 250. We can see a good agrement between these graphs and Figures 2 and 3 in [15].

Figure 4 (left) and Figure 4 (right) show the effects of the transpiration parameter γ on $f''(\xi, 0)$ and $\theta'(\xi, 0)$ for different values of radiation parameter F, respectively. There is a very good agreement between these plots and Figures 4 and 5 in [15]. In both graphs of Figure 4, Ec has been kept constant at 0.3 (plate cooling), Gr=100, and Darcy parameter $\dfrac{\text{Re}}{\text{Gr}\,\text{Da}} = 1$. The γ parameter simulates the phenomena of lateral mass flux by suction or blowing at the confining boundary which can arise in geothermal systems or other hydrothermal geological systems where lateral flow is directed into or out of the boundary, (see, e.g. [27]).

Figure 5 shows the variation of dimensionless temperature with spanwise pseudo-similarity variable η, at $\xi = 1$ (plate end) for different values of radiation parameter F. As we can see from this figure, increasing F, from 0 to 500, decreases the temperature, all profiles returning asymptotically to zero at $\eta = 5$.

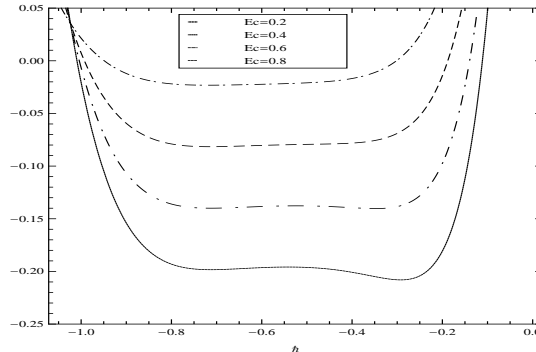FIGURE 1. $\hbar$-curves of $\theta''(0)$ for different values of Ec in $\xi = 0, \alpha = 1, \gamma = 0.1(\text{suction}), \text{Re} = 1, \text{Gr} = 100, \text{Da} = 0.01, \text{Fs} = 0.055, \text{Pr} = 0.733(\text{air}), \lambda = 1$ and F=100 at 10th-order of approximations.

Similarly, Figure 6, shows the variation of dimensionless temperature with span-wise pseudo-similarity variable η, at $\xi = 1$ (plate end) for different values of Grashof number (Gr). As we can see from this figure, increasing Gr, from 0 to 300, decreases the temperature, all profiles returning asymptotically to zero at $\eta \simeq 5$.

Figure 7 (left) shows the variation of dimensionless temperature with spanwise pseudo-similarity variable η, at edge of plate ($\xi = 1$) for different values of parameter α, while Figure 7 (right) shows the values of f' with spanwise pseudo-similarity variable η, at $\xi = 1$ (plate end) for different values of parameter α. In all cases for α the temperature decreases by increasing η, all profiles returning asymptotically to zero at $\eta \simeq 6$. The profiles of f' asymptotically tends to one at $\eta \simeq 8$. Similar conclusions can be drawn from Figures 8 and 9.

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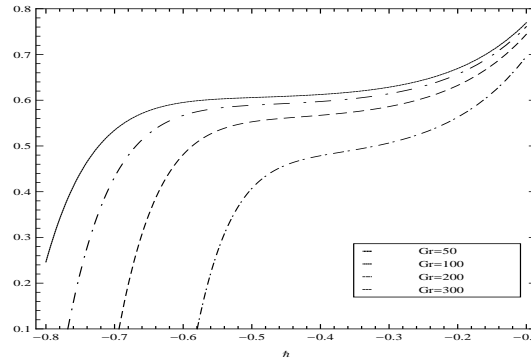
FIGURE 2. $\hbar$-curves of $f''(0)$ for different values of Gr in $\xi = 0, \alpha = 1, \gamma = 0.1, \mathrm{Re} = 1, \mathrm{Da} = 0.01, \mathrm{Ec} = 0.25, \mathrm{Fs} = 0.055, \mathrm{Pr} = 0.9(\mathrm{gas}), \lambda = 1$ and F=250 at 10th-order of approximations.
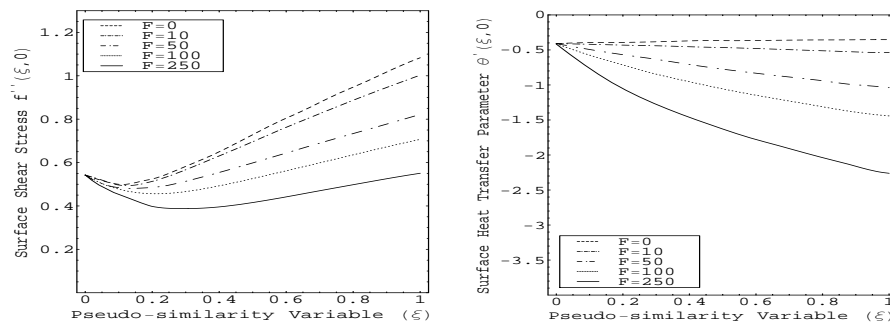

FIGURE 3. Variation of $f''(\xi, 0)$ and $\theta'(\xi, 0)$ with streamwise co-ordinate ξ at the plate surface ($\eta = 0$) for different radiation parameters (F): $\alpha = 1, \gamma = 0.1(\mathrm{suction}), \mathrm{Gr} = 100, \mathrm{Da} = 0.01, \mathrm{Fs} = 0.055, \mathrm{Re} = 1, \mathrm{Pr} = 0.733(\mathrm{air}), \mathrm{Ec} = 0.3, \lambda = 1, \hbar = -0.1$.

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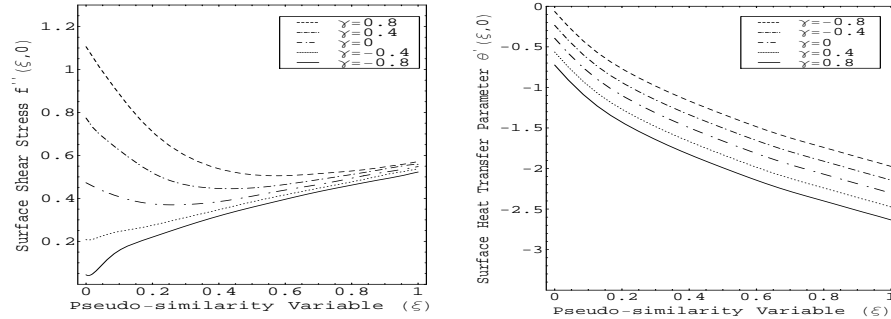
FIGURE 4. Variation of $f''(\xi, 0)$ and $\theta'(\xi, 0)$ with streamwise co-ordinate ξ at the plate surface ($\eta = 0$) for different transpiration parameters (γ): $\alpha = 1, F = 250, Gr = 100, Da = 0.01, Fs = 0.055, Re = 1, Pr = 0.7, Ec = 0.3, \lambda = 1, \hbar = -0.16$.
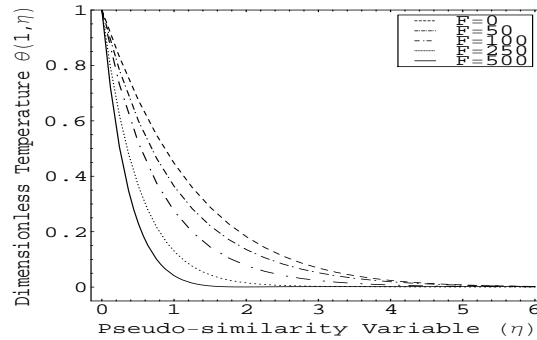
FIGURE 5. Variation of $\theta(1, \eta)$ with spanwise co-ordinate η at edge of plate ($\xi = 1$) for different radiation parameter values (F): $\alpha = 1, \gamma = 0.1(\text{suction}), Gr = 100, Da = 0.01, Fs = 0.055, Re = 1, Pr = 0.733(\text{air}), Ec = 0.3, \lambda = 1, \hbar = -0.1$.

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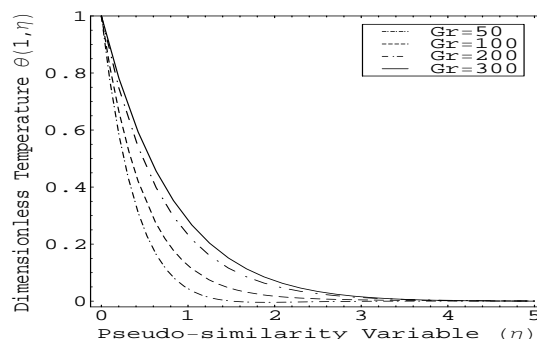
FIGURE 6. Variation of $\theta(1,\eta)$ with spanwise co-ordinate η at edge of plate ($\xi = 1$) for different Grashof number (Gr): $F = 250, \alpha = 1, Da = 0.01, Fs = 0.055, Re = 1, Pr = 0.9, Ec = 0.25, \gamma = 0.1, \lambda = 1, \hbar = -0.1$.
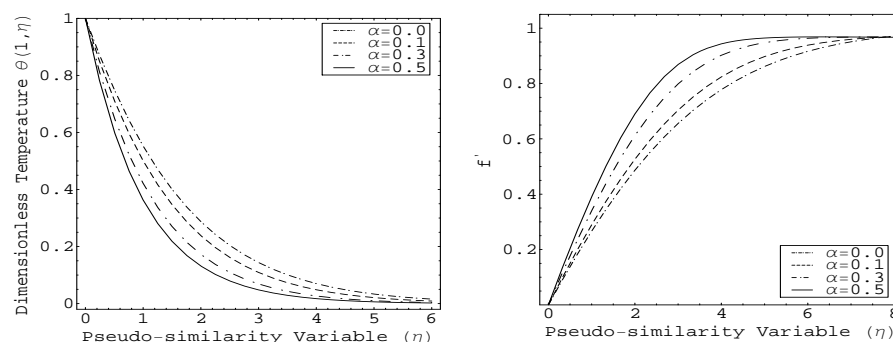

FIGURE 7. Variation of $\theta(1,\eta)$ and $f'(1,\eta)$ with spanwise co-ordinate η at edge of plate ($\xi = 1$) for different parameters (α): $F = 100, Gr = 100, Da = 0.01, Fs = 0.055, Re = 1, Pr = 0.733, Ec = 0.2, \gamma = 0.1, \lambda = 1, \hbar = -0.4$.

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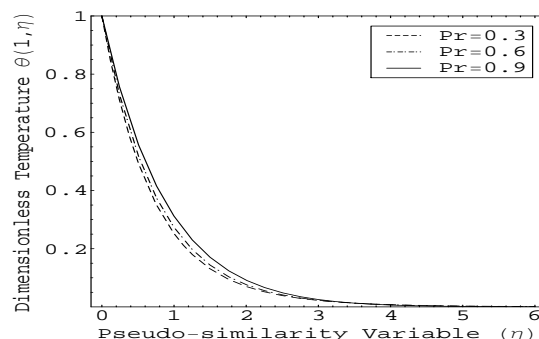FIGURE 8. Variation of $\theta(1, \eta)$ with spanwise co-ordinate η at edge of plate ($\xi = 1$) for different Prandtl numbers (Pr): $F = 100, \mathrm{Gr} = 100, \mathrm{Da} = 0.01, \mathrm{Fs} = 0.055, \mathrm{Re} = 1, \alpha = 1, \mathrm{Ec} = 0.3, \gamma = 0.1, \lambda = 1, \hbar = -0.1$.
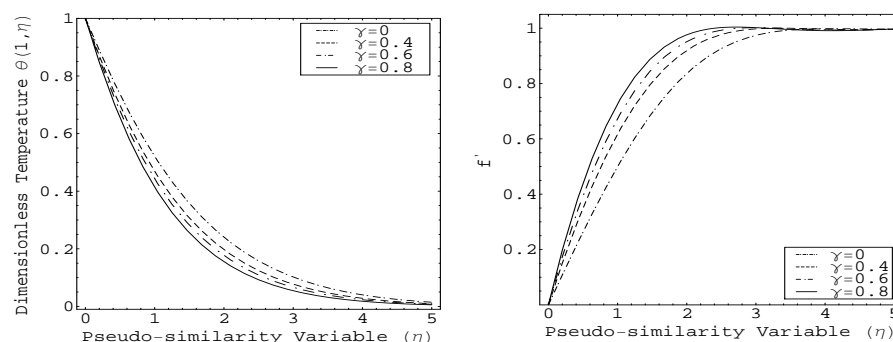

FIGURE 9. Variation of $\theta(1, \eta)$ and $f'(1, \eta)$ with spanwise co-ordinate η at edge of plate ($\xi = 1$) for different transpiration parameters (γ): $\alpha = 1, \mathrm{F} = 0$(no radiation), $\mathrm{Gr} = 100, \mathrm{Da} = 0.01, \mathrm{Fs} = 0.055, \mathrm{Re} = 1, \mathrm{Pr} = 0.7, \mathrm{Ec} = 0.3, \lambda = 1, \hbar = -0.25$.

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