

**POSITIVE SOLUTIONS OF NONLINEAR m -POINT BVP FOR
AN INCREASING HOMEOMORPHISM AND POSITIVE
HOMOMORPHISM ON TIME SCALES**

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ABSTRACT. In this paper, by using fixed point theorems in cones, the existence of positive solutions is considered for nonlinear m -point boundary value problem for the following second-order dynamic equations on time scales

$$\begin{aligned}(\phi(u^\Delta))^\nabla + a(t)f(t, u(t)) &= 0, \quad t \in (0, T), \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi(u^\Delta(T)) &= \sum_{i=1}^{m-2} b_i \phi(u^\Delta(\xi_i)),\end{aligned}$$

where $\phi : R \rightarrow R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$. In [27], we obtained the existence results of the above problem for an increasing homeomorphism and positive homomorphism with sign changing nonlinearity. The purpose of this paper is to supplement with a result in the case when the nonlinear term f is non-negative, and the most point we must point out for readers is that there is only the p -Laplacian case for increasing homeomorphism and positive homomorphism due to the sign restriction. As an application, one example to demonstrate our results are given.

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1. Introduction

A time scale \mathbf{T} is a nonempty closed subset of \mathbb{R} . We make the blanket assumption that $0, T$ are points in \mathbf{T} . By an interval $(0, T)$, we always mean the intersection of the real interval $(0, T)$ with the given time scale, that is $(0, T) \cap \mathbf{T}$.

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In this paper, we will be concerned with the existence of positive solutions for the following dynamic equations on time scales:

$$(\phi(u^\Delta))^\nabla + a(t)f(t, u(t)) = 0, \quad t \in (0, T), \quad (1.1)$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \phi(u^\Delta(\xi_i)), \quad (1.2)$$

where $\phi : R \rightarrow R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$.

A projection $\phi : R \rightarrow R$ is called an increasing homeomorphism and positive homomorphism, if the following conditions are satisfied:

- (i) if $x \leq y$, then $\phi(x) \leq \phi(y)$, $\forall x, y \in R$;
- (ii) ϕ is a continuous bijection and its inverse mapping is also continuous;
- (iii) $\phi(xy) = \phi(x)\phi(y)$, $\forall x, y \in R_+ = [0, +\infty)$.

We will assume that the following conditions are satisfied throughout this paper:

$$(H_1) \quad 0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T), \quad a_i, b_i \in [0, +\infty) \text{ satisfy } 0 < \sum_{i=1}^{m-2} a_i < 1, \\ \text{and } \sum_{i=1}^{m-2} b_i < 1;$$

$$(H_2) \quad a(t) \in C_{ld}((0, T), [0, +\infty)) \text{ and there exists } t_0 \in (\xi_{m-2}, T), \text{ such that } a(t_0) > 0;$$

$$(H_3) \quad f \in C([0, T] \times [0, +\infty), [0, +\infty)). \text{ (The } \Delta\text{-derivative and the } \nabla\text{-derivative in (1.1), (1.2) and the } C_{ld} \text{ space in (H}_2\text{) are defined in Section 2.)}$$

For the existence problems of positive solutions of boundary value problems on time scales, some authors have obtained many results in the recent years, see [4, 12, 13, 24] and the references therein. Recently, there has been much attention paid to the existence of positive solutions for second-order nonlinear boundary value problems on time scales, for examples, see [3, 9, 14, 25] and references therein. At the same time, multipoint nonlinear boundary value problems with p -Laplacian operators on time scales have also been studied extensively in the literature, for details, see [4, 12, 13, 18, 21-23] and references therein. But to the best of our knowledge, few people considered the second-order dynamic equations of increasing homeomorphism and positive homomorphism on time scales.

Feng et. al. [10] discussed the following multipoint boundary-value problem with one dimensional p -Laplacian:

$$(\phi_p(u'))' + q(t)f(t, u) = 0, \quad t \in (0, 1), \quad (1.3)$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{1.4}$$

They obtained sufficient conditions for the existence of multiple positive solutions for the above boundary value problem by using a fixed point theorem in a cone.

D.Ma, Z.Du and W.Ge [19] have obtained the existence of monotone positive solutions for the following BVP:

$$(\phi(u'))' + a(t)f(t, u(t)) = 0, \quad t \in (0, 1), \tag{1.5}$$

$$u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{1.6}$$

The main tool is the monotone iterative technique.

The present work is motivated by papers [16, 17, 26]. Very recently, Yang and Xiao [26] studied the existence of multiple positive solutions for the following multipoint BVP:

$$(\phi(x'(t)))' + q(t)f(t, x(t), x'(t)) = 0, \quad t \in (0, 1), \tag{1.7}$$

$$x(0) = \sum_{i=1}^{n-2} \alpha_i x(\xi_i), \quad \phi(x'(1)) = \sum_{i=1}^{n-2} \beta_i \phi(x'(\xi_i)), \tag{1.8}$$

where $\phi : R \rightarrow R$ is an odd, increasing homeomorphism from R to R . By using the fixed point theorems, they obtained new results on the existence of at least three positive solutions of above boundary value problem.

On the one hand, the purpose of our paper is to supplement with a proof in the case when the nonlinear term f is nonnegative. The proof is quite similar to that of our previous paper [27]. On the other hand, the first author would like to point out that there is only the p -Laplacian case for increasing homeomorphism and positive homomorphism due to the sign restriction, this point was proposed by professor Jeff Webb. This is the main motivation for us to write down the present paper. We also point out that when $\mathbf{T} = R$, $p = 2$, (1.1) and (1.2) becomes a boundary value problem of differential equations and just is the problem considered in [20]. Our main results extend and include the main results of [16, 17, 20].

The rest of the paper is arranged as follows. We state some basic time scale definitions and prove several preliminary results in Section 2, Section 3 is devoted to the existence of positive solution of (1.1) and (1.2), the main tool being the fixed point theorem in cone. At the end of the paper, we will give one simple example which illustrate that our work is true.

2. Preliminaries and some Lemmas

For convenience, we list the following definitions which can be found in [1, 5, 7, 8, 27].

Definition 2.1. A time scale \mathbf{T} is a nonempty closed subset of real numbers R . For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and backward jump operator ρ , respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T},$$

$$\rho(r) = \sup\{\tau \in \mathbf{T} \mid \tau < r\} \in \mathbf{T}.$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If \mathbf{T} has a right scattered minimum m , define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If \mathbf{T} has a left scattered maximum M , define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 2.2. For $f : \mathbf{T} \rightarrow R$ and $t \in \mathbf{T}^k$, the delta derivative of f at the point t is defined to be the number $f^\Delta(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$.

For $f : \mathbf{T} \rightarrow R$ and $t \in \mathbf{T}_k$, the nabla derivative of f at t is the number $f^\nabla(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e. *ld*-continuous), if f is continuous at each left-dense point in \mathbf{T} and its right-sided limit exists at each right-dense point in \mathbf{T} .

Definition 2.4. If $G^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = G(b) - G(a).$$

If $F^\nabla(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear BVP

$$(\phi(u^\Delta))^\nabla + h(t) = 0, \quad t \in (0, T), \quad (2.1)$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \phi(u^\Delta(\xi_i)). \tag{2.2}$$

Lemma 2.1(see [27]). *If $\sum_{i=1}^{m-2} a_i \neq 1$ and $\sum_{i=1}^{m-2} b_i \neq 1$, then for $h \in C_{ld}[0, T]$ the BVP (2.1) and (2.2) has the unique solution*

$$u(t) = \int_0^t \phi^{-1} \left(\int_s^T h(\tau) \nabla \tau - A \right) \Delta s + B, \tag{2.3}$$

where

$$A = - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T h(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i}, \quad B = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T h(\tau) \nabla \tau - A \right) \Delta s}{1 - \sum_{i=1}^{m-2} a_i}.$$

Lemma 2.2(see [27]). *Assume (H_1) holds, For $h \in C_{ld}[0, T]$ and $h \geq 0$, then the unique solution u of (2.1) and (2.2) satisfies*

$$u(t) \geq 0, \quad \text{for } t \in [0, T].$$

Lemma 2.3(see [27]). *Assume (H_1) holds, if $h \in C_{ld}[0, T]$ and $h \geq 0$, then the unique solution u of (2.1) and (2.2) satisfies*

$$\inf_{t \in [0, T]} u(t) \geq \gamma \|u\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i \xi_i}{\left(1 - \sum_{i=1}^{m-2} a_i\right) T + \sum_{i=1}^{m-2} a_i \xi_i}, \quad \|u\| = \max_{t \in [0, T]} |u(t)|.$$

Let the norm on $C_{ld}[0, T]$ be the maximum norm. Then the $C_{ld}[0, T]$ is a Banach space. It is easy to see that the BVP (1.1) and (1.2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator equation

$$(Au)(t) = \int_0^t \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \Delta s + \tilde{B},$$

where

$$\tilde{A} = - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) f(\tau, u(\tau)) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i},$$

$$\tilde{B} = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \Delta s}{1 - \sum_{i=1}^{m-2} a_i}.$$

Denote

$$K = \{u | u \in C_{ld}[0, T], u(t) \geq 0, \inf_{t \in [0, T]} u(t) \geq \gamma \|u\|\},$$

where γ is the same as in Lemma 2.3. It is obvious that K is a cone in $C_{ld}[0, T]$. By Lemma 2.3, $A(K) \subset K$. So by applying Arzela-Ascoli theorem on time scales [2], we can obtain that $A(K)$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [6], it is easy to prove that A is continuous. Hence, $A : K \rightarrow K$ is completely continuous.

Lemma 2.4.(see [11]) *Let K be a cone in a Banach space X . Let D be an open bounded subset of X with $D_K = D \cap K \neq \emptyset$ and $\overline{D}_K \neq K$. Assume that $A : \overline{D}_K \rightarrow K$ is a completely continuous map such that $x \neq Ax$ for $x \in \partial D_K$. Then the following results hold:*

- (1) *If $\|Ax\| \leq \|x\|$, $x \in \partial D_K$, then $i(A, D_K, K) = 1$;*
- (2) *If there exists $x_0 \in K \setminus \{\theta\}$ such that $x \neq Ax + \lambda x_0$, for all $x \in \partial D_K$ and all $\lambda > 0$, then $i(A, D_K, K) = 0$;*
- (3) *Let U_K be open in X such that $\overline{U}_K \subset D_K$. If $i(A, D_K, K) = 1$ and $i(A, U_K, K) = 0$, then A has a fixed point in $D_K \setminus \overline{U}_K$.*

The same results holds, if $i(A, D_K, K) = 0$ and $i(A, U_K, K) = 1$, we define

$$K_\rho = \{u(t) \in K : \|u\| < \rho\}, \quad \Omega_\rho = \{u(t) \in K : \min_{0 \leq t \leq T} u(t) < \gamma \rho\}.$$

Lemma 2.5.(see [15]) Ω_ρ defined above has the following properties:

- (a) $K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$;
- (b) Ω_ρ is open relative to K ;
- (c) $x \in \partial\Omega_\rho$ if and only if $\min_{0 \leq t \leq T} x(t) = \gamma\rho$
- (d) *If $x \in \partial\Omega_\rho$, then $\gamma\rho \leq x(t) \leq \rho$ for $t \in [0, T]$.*

Now, for the convenience, we introduce the following notations. Let

$$\varphi(s) = \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right),$$

$$f_{\gamma\rho}^\rho = \min \left\{ \min_{0 \leq t \leq T} \frac{f(t, u)}{\phi(\rho)} : u \in [\gamma\rho, \rho] \right\}, \quad f_0^\rho = \max \left\{ \max_{0 \leq t \leq T} \frac{f(t, u)}{\phi(\rho)} : u \in [0, \rho] \right\},$$

$$f^\alpha = \limsup_{u \rightarrow \alpha} \max_{0 \leq t \leq T} \frac{f(t, u)}{\phi(u)}, \quad f_\alpha = \liminf_{u \rightarrow \alpha} \min_{0 \leq t \leq T} \frac{f(t, u)}{\phi(u)}, \quad (\alpha := \infty \text{ or } 0^+),$$

$$m = \left\{ \int_0^T \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s + \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right\}^{-1}, \tag{2.4}$$

$$M = \left\{ \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right\}^{-1}. \tag{2.5}$$

Lemma 2.6. *If f satisfies the following conditions*

$$f_0^p \leq \phi(m) \text{ and } u \neq Au, \text{ for } u \in \partial K_\rho, \tag{2.6}$$

then $i(A, K_\rho, K) = 1$.

Proof. By (2.4) and (2.6), we have for $\forall u \in \partial K_\rho$,

$$\begin{aligned} & \int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \\ &= \int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) f(\tau, u(\tau)) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \\ &\leq \phi(\rho) \phi(m) \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right], \end{aligned}$$

so that

$$\begin{aligned} \varphi(s) &= \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \\ &\leq \rho m \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right]. \end{aligned}$$

Therefore, by (2.4), we have

$$\begin{aligned}
 \| Au \| &\leq \int_0^T \varphi(s) \Delta s + \tilde{B} \\
 &= \int_0^T \varphi(s) \Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s) \Delta s}{1 - \sum_{i=1}^{m-2} a_i} \\
 &\leq \rho m \left\{ \int_0^T \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right. \\
 &\quad \left. + \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right\} \\
 &= \rho = \|u\|.
 \end{aligned}$$

This implies that $\|Au\| \leq \|u\|$ for $u \in \partial K_\rho$. By Lemma 2.4(1), we have

$$i(A, K_\rho, K) = 1.$$

Lemma 2.7. *If f satisfies the following conditions*

$$f_{\gamma\rho}^\rho \geq \phi(M\gamma) \text{ and } u \neq Au \text{ for } u \in \partial\Omega_\rho, \tag{2.7}$$

then $i(A, \Omega_\rho, K) = 0$.

Proof. Let $e(t) \equiv 1$, for $t \in [0, T]$; then $e \in \partial K_1$. We claim that $u \neq Au + \lambda e$ for $u \in \partial \Omega_\rho$, and $\lambda > 0$. In fact, if not, there exist $u_0 \in \partial\Omega$, and $\lambda_0 > 0$ such that $u_0 = Au_0 + \lambda_0 e$.

By (2.5) and (2.7), we have for $t \in [0, T]$,

$$\begin{aligned} & \int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau - \tilde{A} \\ &= \int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \\ &\geq \phi(\rho)\phi(M\gamma) \left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \right], \end{aligned}$$

so that

$$\begin{aligned} \varphi(s) &= \phi^{-1} \left(\int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau - \tilde{A} \right) \\ &\geq \rho M\gamma\phi^{-1} \left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \right]. \end{aligned}$$

Applying (2.5), it follows that

$$\begin{aligned} u_0(t) &= Au_0(t) + \lambda_0 e(t) \\ &\geq \tilde{B} + \lambda_0 = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s)\Delta s}{1 - \sum_{i=1}^{m-2} a_i} + \lambda_0 \\ &\geq \gamma\rho M \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \phi^{-1} \left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s + \lambda_0 \\ &= \gamma\rho + \lambda_0. \end{aligned}$$

This implies that $\gamma\rho \geq \gamma\rho + \lambda_0$, a contradiction. Hence, by Lemma 2.4 (2), it follows that

$$i(A, \Omega_\rho, K) = 0.$$

3. Existence theorems of positive solutions

Theorem 3.1. Assume (H_1) , (H_2) and (H_3) hold, and assume that one of the following conditions hold:

(H_4) There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \gamma\rho_2$ such that

$$f_0^{\rho_1} \leq \phi(m), f_{\gamma\rho_2}^{\rho_2} \geq \phi(M\gamma);$$

(H_5) There exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 < \rho_2$ such that

$$f_0^{\rho_2} \leq \phi(m), f_{\gamma\rho_1}^{\rho_1} \geq \phi(M\gamma).$$

Then (1.1), (1.2) has a positive solution.

Proof. Assume that (H_4) holds. We show that A has a fixed point u_1 in $\Omega_{\rho_2} \setminus \bar{K}_{\rho_1}$. By Lemma 2.6, we have that

$$i(A, K_{\rho_1}, K) = 1.$$

By Lemma 2.7, we have that

$$i(A, \Omega_{\rho_2}, K) = 0.$$

By Lemma 2.5 (a) and $\rho_1 < \gamma\rho_2$, we have $\bar{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 2.4(3) that A has a fixed point u_1 in $\Omega_{\rho_2} \setminus \bar{K}_{\rho_1}$. The proof is similar when H_5 holds, and we omit it here. The proof is complete.

As a special case of Theorem 3.1, we obtain the following result:

Corollary 3.1. Assume (H_1) , (H_2) and (H_3) holds, and suppose that one of the following conditions holds:

(H_6) $0 \leq f^0 < \phi(m)$ and $\phi(M) < f_\infty \leq \infty$.

(H_7) $0 \leq f^\infty < \phi(m)$ and $\phi(M) < f_0 \leq \infty$.

Then (1.1), (1.2) has a positive solution.

Theorem 3.2. Assume (H_1) , (H_2) and (H_3) hold, and suppose that one of the following conditions holds:

(H_8) There exist $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ with $\rho_1 < \gamma\rho_2$ and $\rho_2 < \rho_3$ such that

$$f_0^{\rho_1} \leq \phi(m), f_{\gamma\rho_2}^{\rho_2} \geq \phi(M\gamma), u \neq Au, \forall u \in \partial\Omega_{\rho_2}, \text{ and } f_0^{\rho_3} \leq \phi(m);$$

(H_9) There exist $\rho_1, \rho_2, \rho_3 \in (0, +\infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3$ such that

$$f_0^{\rho_2} \leq \phi(m), f_{\gamma\rho_1}^{\rho_1} \geq \phi(M\gamma), u \neq Au, \forall u \in \partial K_{\rho_2}, \text{ and } f_{\gamma\rho_3}^{\rho_3} \geq \phi(M\gamma).$$

Then (1.1), (1.2) has two positive solutions. Moreover, if $(H_8)f_0^{\rho_1} \leq \phi(m)$ is replaced by $f_0^{\rho_1} < \phi(m)$, then (1.1), (1.2) has a third positive solution $u_3 \in K_{\rho_1}$.

Proof. Assume that (H_8) holds. We show that either A has a fixed point u_1 in ∂K_{ρ_1} or $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. If $u \neq Au$ for $u \in \partial K_{\rho_1} \cup \partial K_{\rho_3}$. by Lemma 2.6 and 2.7, we have that

$$i(A, K_{\rho_1}, K) = 1, \quad i(A, K_{\rho_3}, K) = 1, \quad i(A, \Omega_{\rho_2}, K) = 0.$$

By Lemma 2.5 (a) and $\rho_1 < \gamma\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 2.4 (3) that A has a fixed point u_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$. Similarly, A has a fixed point in $K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$. The proof is similar when (H_9) holds and we omit it here. The proof is complete.

As a special case of Theorem 3.2, we obtain the following result:

Corollary 3.2. *Assume $(H_1), (H_2)$ and (H_3) holds, if there exist $\rho > 0$ such that one of the following conditions holds:*

(H_{10}) $0 \leq f^0 < \phi(m)$, $f_{\gamma\rho}^{\rho} \geq \phi(M\gamma)$, $u \neq Au$, $\forall u \in \partial\Omega_{\rho}$ and $0 \leq f^{\infty} < \phi(m)$;
 (H_{11}) $\phi(m) < f_0 \leq \infty$, $f_0^{\rho} \leq \phi(m)$, $u \neq Au$, $\forall u \in \partial K_{\rho}$ and $\phi(M) < f_{\infty} \leq \infty$.
 Then (1.1), (1.2) has two positive solutions.

Remark 3.1. If $\mathbf{T} = R$, $(0, T) = (0, 1)$, $p = 2$. Theorem 3.1 and 3.2 improve Theorem 3.1 in [20].

4. Applications

In this section, we present one simple examples to explain our results.

Example 4.1. Let $\mathbf{T} = \{(\frac{1}{2})^n : n \in N\} \cup \{1\}$, $T = 1$. Consider the following BVP on time scales

$$(\phi(u^{\Delta}))^{\nabla} + f(t, u(t)) = 0, \quad t \in (0, T), \tag{4.1}$$

$$u(0) = \frac{1}{4}u(\frac{1}{3}), \quad \phi(u^{\Delta}(T)) = \frac{1}{2}\phi(u^{\Delta}(\frac{1}{3})), \tag{4.2}$$

where

$$\phi(u) = \begin{cases} u^3, & u \leq 0, \\ u^2, & u > 0, \end{cases} \quad f(t, u) = \frac{1}{81}(1+t)(u(t))^{20}, \quad (t, u) \in [0, 1] \times [0, +\infty).$$

It is easy to check that $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous. In this case, $a(t) \equiv 1$, $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{2}$, $\xi_1 = \frac{1}{3}$, it follows from a direct calculation

that

$$\begin{aligned}
 m &= \left\{ \int_0^T \phi^{-1} \left[(T-s) + \frac{b_1(T-\xi_1)}{1-b_1} \right] \Delta s \right. \\
 &\quad \left. + \frac{a_1}{1-a_1} \int_0^{\xi_1} \phi^{-1} \left[(T-s) + \frac{b_1(T-\xi_1)}{1-b_1} \right] \Delta s \right\}^{-1} \\
 &= \left[\int_0^1 \left(1-s + \frac{\frac{1}{2}(1-\frac{1}{3})}{1-\frac{1}{2}} \right)^{\frac{1}{2}} ds + \frac{\frac{1}{4}}{1-\frac{1}{4}} \int_0^{\frac{1}{3}} \left(1-s + \frac{\frac{1}{2}(1-\frac{1}{3})}{1-\frac{1}{2}} \right)^{\frac{1}{2}} ds \right]^{-1} \\
 &= \left[\int_0^1 \left(\frac{5}{3} - s \right)^{\frac{1}{2}} ds + \frac{1}{3} \int_0^{\frac{1}{3}} \left(\frac{5}{3} - s \right)^{\frac{1}{2}} ds \right]^{-1} \approx 0.8281,
 \end{aligned}$$

$$\begin{aligned}
 M &= \left\{ \frac{a_1}{1-a_1} \int_0^{\xi_1} \phi^{-1} \left[(T-s) + \frac{b_1(T-\xi_1)}{1-b_1} \right] \Delta s \right\}^{-1} \\
 &= \left[\frac{\frac{1}{4}}{1-\frac{1}{4}} \int_0^{\frac{1}{3}} \left(\frac{5}{3} - s \right)^{\frac{1}{2}} ds \right]^{-1} \\
 &\approx 7.3523,
 \end{aligned}$$

$$\gamma = \frac{a_1 \xi_1}{(1-a_1)T + a_1 \xi_1} = \frac{\frac{1}{4} \cdot \frac{1}{3}}{(1-\frac{1}{4}) \cdot 1 + \frac{1}{4} \cdot \frac{1}{3}} = \frac{1}{10}.$$

Choose $\rho_1 = 1$, $\rho_2 = 20$, it is easy to check that $1 = \rho_1 < \gamma \rho_2 = \frac{1}{10} \times 20 = 2$,

$$\begin{aligned}
 f_0^{\rho_1} &= \max \left\{ \max_{0 \leq t \leq 1} \frac{\frac{1}{81} \cdot (1+t) \cdot u^{20}}{1^2} : u \in [0, 1] \right\} \\
 &= \frac{\frac{1}{81} \cdot (1+1) \cdot 1^{20}}{1^2} = \frac{2}{81} \\
 &\leq \phi(m) = m^2 = (0.8281)^2,
 \end{aligned}$$

$$\begin{aligned}
 f_{\gamma \rho_2}^{\rho_2} &= \min \left\{ \min_{0 \leq t \leq 1} \frac{\frac{1}{81} \cdot (1+t) \cdot u^{20}}{20^2} : u \in [2, 20] \right\} \\
 &= \frac{\frac{1}{81} \cdot 1 \cdot 2^{20}}{20^2} = \frac{2^{20}}{81 \cdot 20^2} \approx 32.3635 \\
 &\geq \phi(M\gamma) = (M\gamma)^2 = (7.3523 \cdot \frac{1}{10})^2 \approx 0.5405.
 \end{aligned}$$

It follows that f satisfies the conditions (H_4) of Theorem 3.1, then problem (4.1) and (4.2) has at least a positive solution.

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