

**INFINITELY MANY SMALL SOLUTIONS FOR THE
 p & q -LAPLACIAN PROBLEM WITH CRITICAL SOBOLEV AND
 HARDY EXPONENTS**

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ABSTRACT. In this paper, we study the following p & q -Laplacian problem with critical Sobolev and Hardy exponents

$$\begin{cases} -\Delta_p u - \Delta_q u = \mu \frac{|u|^{p^*(s)-2} u}{|x|^s} + \lambda f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2} \nabla u)$ is the r -Laplacian of u . By using the variational method and concentration-compactness principle, we obtain the existence of infinitely many small solutions for above problem which are the complement of previously known results.

AMS Mathematics Subject Classification : 35J60, 35B33.

Key words and phrases : p & q -Laplacian, Hardy-Sobolev critical exponent, Variational method, Compactness.

1. Introduction

In this paper, we study the following p & q -Laplacian problem with critical Sobolev and Hardy exponents

$$\begin{cases} -\Delta_p u - \Delta_q u = \mu \frac{|u|^{p^*(s)-2} u}{|x|^s} + \lambda f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \Omega, \end{cases} \quad (1)$$

where $1 \leq q < p < N$, $-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplace. μ and λ are two positive parameters, and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is an open bounded domain. $0 \leq s \leq p < N$, $p^*(s) = \frac{(N-s)p}{N-p}$ is the so called Hardy-Sobolev critical exponent. When $s = 0$, $p^*(0) = p^* = \frac{Np}{N-p}$ is the Sobolev critical exponent and if $s = p$,

Received October 27, 2009. Revised November 2, 2009. Accepted November 23, 2009.

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$p^*(p) = p$ is the Hardy critical exponent.

Problem (1) come from a general reaction-diffusion system

$$u_t = \operatorname{div}[D(u)\nabla u] + f(x, u), \quad (2)$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design. In such applications, the function u describes a concentration; the first term on the right hand side of (2) corresponds to diffusion with a diffusion coefficient $D(u)$, whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $f(x, u)$ has a polynomial form with respect to the concentration u .

If $q = p = 2$. Elliptic equations with critical exponent have been considered by many authors since the pioneer work by Brezis and Nirenberg [2] in case $s = 0$ and $f(x, u) = u$. The authors showed that for $N > 4$ and $\lambda \in (0, \lambda_1)$, problem (1) has at least one positive solution. In the sequel, λ_1 denotes the principal eigenvalue of $-\Delta$ on Ω . The same conclusion was proved by Brezis and Nirenberg for $N = 3$ when Ω is a ball and $\lambda \in (\lambda_1/4, \lambda_1)$. In this case, equation (1) has no radial solution when $\lambda \in (0, \lambda_1/4)$. Li and Zou [12] studied problem (1) in case $s = 0$, they obtained the existence theorem of infinitely many solutions of problem (1.1) under suitable hypotheses. It should be noted that the nonlinearity $f(x, u)$ in this paper satisfying fewer conditions than [12]. When a singular potential is concerned, He and Zou [9] proved that the existence infinitely many small solutions under case $\mu \equiv 1$.

If $q = p \neq 2$. Ghoussoub and Yuan [8] obtained the existence of infinitely many nontrivial solutions for Hardy-Sobolev subcritical case and Hardy critical case by establishing Palais-Smale type conditions around appropriate chosen dual sets in bounded domain. Besides, although there are a lot of papers about the singular problems with Hardy-Sobolev critical exponents (see [5, 19]). But there are few results dealing with the case the general form $f(x, t)$. When $f(x, u) = \frac{|u|^{q-2}}{|x|^s}u$, the existence of positive solutions for the equation (1) are obtained in [11]. Chen and Li [4] obtained that the existence of infinitely many solutions by using minimax procedure in the case $f(x, u) = k(x)|u|^{r-2}u$ ($1 < r < \frac{Np}{N-p}$). But they did not give any further information on the sequence of solutions.

If $q \neq p \neq 2$. This case is very interesting and important. Li and Zhang [16] studied the existence of multiple solutions for the nonlinear elliptic problems of p & q -Laplacian type involving the critical Sobolev exponent in case $s = 0$, $\mu \equiv 1$ and $f(x, u) = |u|^{r-2}u$, they obtained infinitely many weak solutions by using Lusternik-Schnirelman's theory for Z_2 -invariant functional. He and Li [15] studied the following p & q -Laplacian type problem

$$\begin{cases} -\Delta_p u - \Delta_q u + m|u|^{p-2}u + n|u|^{q-2}u = f(x, u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases}$$

where $m, n > 0$ and $1 < q < p < N$, they obtained the existence of a nontrivial solution.

Recently, Kajikiya [10] established a critical point theorem related to the symmetric mountain pass lemma and applied to a sublinear elliptic equation. But there are no such results on p - q -Laplacian problem with critical Sobolev and Hardy exponents (1).

Motivated by reasons above, the aim of this paper is to show that the existence of infinitely many solutions of problem (1), and there exists a sequence of infinitely many arbitrarily small solutions converging to zero by using a new version of the symmetric mountain-pass lemma due to Kajikiya [10]. In order to use the symmetric mountain-pass lemma, there are many difficulties. The main one in solving the problem is a lack of compactness which can be illustrated by the fact that the embedding of $H_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is no longer compact. Hence the concentration-compactness principle is used here to overcome the difficulty.

$u \in H_0^{1,p}(\Omega)$ is said to be a solutions of problem (1.1) if u satisfies

$$\int_{\Omega} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla v + |\nabla u|^{q-2} \nabla u \cdot \nabla v - \mu \frac{|u|^{p^*(s)-2} uv}{|x|^s} - \lambda f(x, u)v \right) dx = 0$$

for all $v \in H_0^{1,p}(\Omega)$.

Problem (1.1) is related to the well known Sobolev-Hardy inequalities, which is essentially due to Caffarelli, Kohn and Nirenberg (see [3]),

$$\left(\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^s} dx \right)^{\frac{p}{q}} \leq C_{q,s,p} \int_{\mathbb{R}^N} |\nabla u|^p dx, \quad \forall u \in H_0^{1,p}(\Omega),$$

where $p \leq q \leq p^*$. For sharp constants and extremal functions, see [8]. As $q = s = p$, the above Sobolev inequality becomes the well known Hardy inequality (see[3, 7, 8]),

$$\int_{\Omega} \frac{|u|^p}{|x|^p} dx \leq \frac{1}{\mu} \int_{\Omega} |\nabla u|^p dx, \quad \forall u \in H_0^{1,p}(\Omega). \tag{3}$$

In this paper, we use the norm

$$\|u\| = \|u\|_{H_0^{1,p}(\Omega)} := \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

We denote A_s the best constant of the Hardy-Sobolev inequality, i.e.,

$$A_s(\Omega) := \inf_{u \in H_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx \right)^{\frac{p}{p^*(s)}}}, \tag{4}$$

which is independent of Ω and can be achieved when $0 \leq s \leq p < N$. A_0 is nothing but the best constant in the Sobolev inequality, A_p is the best constant

in Hardy inequality. In this paper, we will use λ_1 , where λ_1 is the first eigenvalue of $-\Delta_p$ according to Dirichlet boundary condition defined as

$$\lambda_1 := \inf_{u \in H_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} |u|^p dx \right)}.$$

The energy functional corresponding to problem (1) is defined as follows,

$$I(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q} \int_{\Omega} |\nabla u|^q dx - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx - \lambda \int_{\Omega} F(x, u) dx,$$

then $I(u)$ is well defined on $H_0^{1,p}(\Omega)$. Standard arguments [17] show that $I(u)$ belongs to $C^1(H_0^{1,p}(\Omega), R)$. The solutions of problem (1) are then the critical points of the functional I .

The main result of this paper is as follows.

Theorem 1. *Suppose that $f(x, u)$ satisfies the following conditions:*

(H₁) $f(x, u) \in C(\Omega \times R, R)$, $f(x, -u) = -f(x, u)$ for all $u \in R$;

(H₂) $\lim_{|u| \rightarrow \infty} \frac{f(x, u)}{|u|^{p^*(s)-1}} = 0$ uniformly for $x \in \Omega$;

(H₃) $\lim_{|u| \rightarrow 0^+} \frac{f(x, u)}{u^{q-1}} = \infty$ uniformly for $x \in \Omega$.

Then for any $\mu > 0$, there exists $\lambda^ > 0$ such that for any $\lambda \in (0, \lambda^*)$, problem (1) has a sequence of non-trivial solutions $\{u_n\}$ and $u_n \rightarrow 0$ as $n \rightarrow \infty$.*

Remark 0.1. *When $p = q = 2, \mu \equiv 1$ and $s = 0$, the authors in [12] proved the existence of infinitely many solutions for (1) under conditions (H₁) – (H₃) and*

(H₄) $\frac{1}{2} f(x, u)u - F(x, u) \geq a - b|u|^{2^*}$ for almost every $x \in \Omega$ and $u \in R$ where

$$F(x, u) = \int_0^u f(x, t) dt, \quad b \geq 0, \quad a \leq 0.$$

But they did not give any further information on the sequence of solutions. In this paper we shall prove that this sequence of solutions may converge to zero.

Remark 0.2. *In this paper, the nonlinearity $f(x, u)$ need not satisfy condition (H₄) as in [12]. Furthermore, we consider more general nonlinearity than is considered in [4, 11, 16]. Hence, we make an improvement of the main results of [4, 9, 11, 12, 16].*

Remark 0.3. *If without the symmetry condition (i.e., $f(x, -u) = -f(x, u)$), we get an existence theorem of at least one nontrivial solution to problem (1) by the same method in this paper.*

Remark 0.4. *There exist many functions $f(x, t)$ satisfying condition (H₁)-(H₃), for example, $f(x, u) = u^{(q-1)/3}$, where $p > q > 1$ and q is even.*

Definition 0.1. A C^1 functional I on Banach space X satisfies the Palais-Smale condition at level c ($(PS)_c$, for short) if every sequence $\{u_n\}$ satisfying

$$I(u_n) \rightarrow c \text{ and } I'(u_n) \rightarrow 0,$$

contains a convergent subsequence.

2. Preliminaries and lemmas

Denote \mathcal{M}^+ as a cone of positive finite Radon measure. Since the proof of the following result is similar to Lions [13, 14] and is an adaptation of lemma by D. Smets [18], we just sketch the proof here.

Lemma 1. Let $0 \leq s \leq p < N$ and $\{u_n\} \subset H_0^{1,p}(\Omega)$ (here Ω is possibly unbounded) be a bounded sequence, going if necessary to subsequence, we may assume that $u_n \rightharpoonup u$ in $H_0^{1,p}(\Omega)$, $|\nabla u_n|^p \rightharpoonup \zeta$ in \mathcal{M}^+ , $\frac{|u_n|^{p^*(s)}}{|x|^s} \rightharpoonup \nu$ in \mathcal{M}^+ . Define

$$\zeta_\infty := \lim_{R \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \int_{\Omega \cap |x| > R} |\nabla u_n|^p dx,$$

$$\nu_\infty := \lim_{R \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \int_{\Omega \cap |x| > R} \frac{|u_n|^{p^*(s)}}{|x|^s} dx.$$

Then there exist a , at most, countable index set J and a collection of points $\{x_j\}$, $j \in J$, in Ω such that

- (i) $\zeta_\infty \geq A_s \nu_\infty^{p/p^*(s)}$;
- (ii) $\nu = \frac{|u|^{p^*(s)}}{|x|^s} + \sum \delta_{x_j} \nu_j$, $\nu_j > 0$, $\zeta = |\nabla u|^p + \sum \delta_{x_j} A_s \nu_j^{p/p^*(s)}$;
- (iii) $\zeta_j \geq A_s \nu_j^{p/p^*(s)}$;
- (iv) $\lim_{n \rightarrow \infty} \int_{\Omega \cap |x| > R} \frac{|u_n|^{p^*(s)}}{|x|^s} dx = \int_{\Omega \cap |x| > R} \frac{|u|^{p^*(s)}}{|x|^s} dx + \|\nu\| + \nu_\infty$.

Under assumption (H_2) , we have

$$f(x, u)u = o\left(\frac{|u|^{p^*(s)}}{|x|^s}\right), \quad F(x, u) = o\left(\frac{|u|^{p^*(s)}}{|x|^s}\right),$$

which means that, for all $\varepsilon > 0$, there exist $a(\varepsilon)$, $b(\varepsilon) > 0$ such that

$$|f(x, u)u| \leq a(\varepsilon) + \varepsilon \frac{|u|^{p^*(s)}}{|x|^s}, \tag{5}$$

$$|F(x, u)| \leq b(\varepsilon) + \varepsilon \frac{|u|^{p^*(s)}}{|x|^s}. \tag{6}$$

Hence,

$$F(x, u) - \frac{1}{p} f(x, u)u \leq c(\varepsilon) + \varepsilon \frac{|u|^{p^*(s)}}{|x|^s}, \tag{7}$$

for some $c(\varepsilon) > 0$.

Lemma 2. *Assume condition (H_2) holds. Then for any $\lambda > 0$, the functional I satisfies the local $(PS)_c$ condition in*

$$c \in \left(-\infty, \frac{(p-s)\mu}{p(N-s)} [\mu^{-1}A_s]^{\frac{N-s}{p-s}} - \lambda c \left(\frac{(p-s)\mu}{2p\lambda(N-s)} \right) |\Omega| \right)$$

in the following sense: if

$$I(u_n) \rightarrow c < \frac{(p-s)\mu}{p(N-s)} [\mu^{-1}A_s]^{\frac{N-s}{p-s}} - \lambda c \left(\frac{(p-s)\mu}{2p\lambda(N-s)} \right) |\Omega|$$

and $I'(u_n) \rightarrow 0$ for some sequence in $H_0^{1,p}(\Omega)$, then $\{u_n\}$ contains a subsequence converging strongly in $H_0^{1,p}(\Omega)$.

Proof. Let $\{u_n\}$ be a sequence in $H_0^{1,p}(\Omega)$ such that

$$\begin{aligned} I(u_n) &= \frac{1}{p} \int_{\Omega} |\nabla u_n|^p dx + \frac{1}{q} \int_{\Omega} |\nabla u_n|^q dx \\ &\quad - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx - \lambda \int_{\Omega} F(x, u_n) dx \\ &= c + o(1), \end{aligned} \tag{8}$$

$$\begin{aligned} \langle I'(u_n), u_n \rangle &= \int_{\Omega} |\nabla u_n|^p dx + \int_{\Omega} |\nabla u_n|^q dx \\ &\quad - \mu \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx - \lambda \int_{\Omega} f(x, u_n) u_n dx \\ &= o(1) \|u_n\|. \end{aligned} \tag{9}$$

By (8) and (9), we have

$$\begin{aligned} &I(u_n) - \frac{1}{p} \langle I'(u_n), u_n \rangle \\ &= \left(\frac{1}{q} - \frac{1}{p} \right) \int_{\Omega} |\nabla u_n|^q dx + \left(\frac{1}{p} - \frac{1}{p^*(s)} \right) \mu \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx \\ &\quad - \lambda \int_{\Omega} \left[F(x, u_n) - \frac{1}{p} f(x, u_n) u_n \right] dx \\ &= c + o(1) \|u_n\|, \end{aligned}$$

i.e.,

$$\frac{(p-s)\mu}{p(N-s)} \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx \leq \lambda \int_{\Omega} \left[F(x, u_n) - \frac{1}{p} f(x, u_n) u_n \right] dx + c + o(1) \|u_n\|,$$

since $p > q$. Then by (7), we have

$$\left(\frac{(p-s)\mu}{p(N-s)} - \lambda \varepsilon \right) \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx \leq \lambda c(\varepsilon) |\Omega| + c + o(1) \|u_n\|.$$

Letting $\varepsilon = (p - s)\mu/2p(N - s)\lambda$, we get

$$\int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx \leq M + o(1)\|u_n\|, \tag{10}$$

where $o(1) \rightarrow 0$ and M is a some positive number. On the other hand, by (6) and (10), we have

$$\begin{aligned} c + o(1)\|u_n\| &= I(u_n) \\ &= \frac{1}{p} \int_{\Omega} |\nabla u_n|^p dx + \frac{1}{q} \int_{\Omega} |\nabla u_n|^q dx \\ &\quad - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx - \lambda \int_{\Omega} F(x, u_n) dx \\ &\geq \frac{1}{p} \|u_n\|^p - \lambda b(\varepsilon)|\Omega| - \left[\frac{\mu}{p^*(s)} + \lambda\varepsilon \right] \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx.. \end{aligned} \tag{11}$$

Thus (10) and (11) imply that $\{u_n\}$ is bounded in $H_0^{1,p}(\Omega)$. Therefore we can assume that $u_n \rightharpoonup u$ in $H_0^{1,p}(\Omega)$, $|\nabla u_n|^p \rightharpoonup \zeta$ in \mathcal{M}^+ , $\frac{|u_n|^{p^*(s)}}{|x|^s} \rightharpoonup \nu$ in \mathcal{M}^+ . Let x_j be a singular point of the measures ζ and ν , define a function $\phi(x) \in C_0^\infty(\Omega)$ such that $\phi(x) = 1$ in $B(x_j, \varepsilon)$, $\phi(x) = 0$ in $\Omega \setminus B(x_j, 2\varepsilon)$ and $|\nabla \phi| \leq 2/\varepsilon$ in Ω . Then $\{\phi u_n\}$ is bounded in $H_0^{1,p}(\Omega)$, Obviously, $\langle I'(u_n), u_n \phi \rangle \rightarrow 0$, i.e.,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left[\int_{\Omega} |\nabla u_n|^p \phi dx + \int_{\Omega} |\nabla u_n|^q \phi dx \right. \\ &\quad \left. - \mu \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} \phi dx - \lambda \int_{\Omega} f(x, u_n) u_n \phi dx \right] \\ &= - \lim_{n \rightarrow \infty} \int_{\Omega} (u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \phi + u_n |\nabla u_n|^{q-2} \nabla u_n \nabla \phi) dx. \end{aligned} \tag{12}$$

On the other hand, by Hölder inequality and boundedness of $\{u_n\}$, we have that

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left| \int_{\Omega} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \phi dx \right| \\ &\leq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left(\int_{\Omega} |u_n|^p |\nabla \phi|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla u_n|^p dx \right)^{\frac{p-1}{p}} \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega} |u|^p |\nabla \phi|^p dx \right)^{\frac{1}{p}} \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\int_{B(x_j, \varepsilon)} |\nabla \phi|^N dx \right)^{\frac{1}{N}} \left(\int_{B(x_j, \varepsilon)} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \\ &\leq C \lim_{\varepsilon \rightarrow 0} \left(\int_{B(x_j, \varepsilon)} |u|^{p^*} dx \right)^{\frac{1}{p^*}} = 0. \end{aligned} \tag{13}$$

Using the same method, we have that

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left| \int_{\Omega} u_n |\nabla u_n|^{q-2} \nabla u_n \nabla \phi dx \right| = 0. \tag{14}$$

From (12)-(14), we get that

$$0 = \lim_{\varepsilon \rightarrow 0} \left[\int_{\Omega} \phi d\zeta + \int_{\Omega} |\nabla u_n|^q \phi dx - \mu \int_{\Omega} \phi d\nu - \lambda \int_{\Omega} f(x, u_n) u_n \phi dx \right],$$

i.e.,

$$0 \geq \lim_{\varepsilon \rightarrow 0} \left[\int_{\Omega} \phi d\zeta - \mu \int_{\Omega} \phi d\nu \right] = \zeta_j - \mu \nu_j.$$

Combing this with Lemma 1, we obtain $\nu_j \geq \mu^{-1} A_s \nu_j^{\frac{p}{p^*(s)}}$. This result implies that

$$\nu_j = 0 \quad \text{or} \quad \nu_j \geq [\mu^{-1} A_s]^{\frac{N-s}{p-s}}.$$

Here we use $\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_n|^q \phi dx = 0$ and $\lim_{\varepsilon \rightarrow 0} \lambda \int_{\Omega} f(x, u_n) u_n \phi dx = 0$. If the second case $\nu_j \geq [\mu^{-1} A_s]^{\frac{N-s}{p-s}}$ holds, for some $j \in J$, then by using Lemma 1 and the Hölder inequality, we have that

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \left(I(u_n) - \frac{1}{p} \langle I'(u_n), u_n \rangle \right) \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{1}{q} - \frac{1}{p} \right) \int_{\Omega} |\nabla u_n|^q dx + \left(\frac{1}{p} - \frac{1}{p^*(s)} \right) \mu \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx \right. \\ &\quad \left. - \lambda \int_{\Omega} \left[F(x, u_n) - \frac{1}{p} f(x, u_n) u_n \right] dx \right] \\ &\geq \frac{(p-s)\mu}{p(N-s)} \int_{\Omega} d\nu - \lambda \int_{\Omega} \left[F(x, u) - \frac{1}{p} f(x, u) u \right] dx \\ &\geq \left(\frac{(p-s)\mu}{p(N-s)} - \lambda \varepsilon \right) \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx + \frac{(p-s)\mu}{p(N-s)} [\mu^{-1} A_s]^{\frac{N-s}{p-s}} - \lambda c(\varepsilon) |\Omega| \\ &\geq \frac{(p-s)\mu}{p(N-s)} [\mu^{-1} A_s]^{\frac{N-s}{p-s}} - \lambda c \left(\frac{(p-s)\mu}{2p\lambda(N-s)} \right) |\Omega|, \end{aligned}$$

where $\varepsilon = (p-s)\mu/2p\lambda(N-s)$. This is impossible. Consequently, $\nu_j = 0$ for all $j \in J$ and hence

$$\int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx \rightarrow \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx.$$

Now $u_n \rightharpoonup u$ in $H_0^{1,p}(\Omega)$ and Brezis-Lieb Lemma [1] implies that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{|u_n - u|^{p^*(s)}}{|x|^s} dx = 0.$$

Thus, we have

$$\begin{aligned}
 o(1)\|u_n\| &= \|u_n\|^p + \int_{\Omega} |\nabla u_n|^q dx - \mu \int_{\Omega} \frac{|u_n|^{p^*(s)}}{|x|^s} dx - \lambda \int_{\Omega} f(x, u_n) u_n dx \\
 &= \|u_n - u\|^p + \|u\|^p + \int_{\Omega} |\nabla u_n|^q dx - \int_{\Omega} |\nabla u|^q dx + \int_{\Omega} |\nabla u|^q dx \\
 &\quad - \mu \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx - \lambda \int_{\Omega} f(x, u) u dx \\
 &= \|u_n - u\|^p + o(1)\|u\|,
 \end{aligned}$$

here we use Brezis-Lieb Lemma [1], $I'(u) = 0$ and Lemma 2.3 of [16]. Thus we prove that $\{u_n\}$ strongly converges to u in $H_0^{1,p}(\Omega)$. \square

3. Existence of a sequence of arbitrarily small solutions

In this section, we prove the existence of infinitely many solutions of (1) which tend to zero. Let X be a Banach space and denote

$$\Sigma := \{A \subset X \setminus \{0\} : A \text{ is closed in } X \text{ and symmetric with respect to the origin}\}.$$

For $A \in \Sigma$, we define genus $\gamma(A)$ as

$$\gamma(A) := \inf\{m \in \mathbb{N} : \exists \varphi \in C(A, \mathbb{R}^m \setminus \{0\}), -\varphi(x) = \varphi(-x)\}.$$

If there is no mapping φ as above for any $m \in \mathbb{N}$, then $\gamma(A) = +\infty$. Let Σ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$. We list some properties of the genus (see [10]).

Proposition 1. *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold.*

- (1) *If there exists an odd continuous mapping from A to B , then $\gamma(A) \leq \gamma(B)$;*
- (2) *If there is an odd homeomorphism from A to B , then $\gamma(A) = \gamma(B)$;*
- (3) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;*
- (4) *Then n -dimensional sphere S^n has a genus of $n + 1$ by the Borsuk-Ulam Theorem;*
- (5) *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $U_\delta(A) \in \Sigma$ and $\gamma(U_\delta(A)) = \gamma(A)$, where $U_\delta(A) = \{x \in X : \|x - A\| \leq \delta\}$.*

The following version of the symmetric mountain-pass lemma is due to Kajikiya [10].

Lemma 3. *Let E be an infinite-dimensional space and $I \in C^1(E, \mathbb{R})$ and suppose the following conditions hold.*

- (C₁) $I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the local Palais-Smale condition, i.e. for some $\bar{c} > 0$, in the case when every sequence $\{u_k\}$ in E satisfying $\lim_{k \rightarrow \infty} I(u_k) = c < \bar{c}$ and $\lim_{k \rightarrow \infty} \|I'(u_k)\|_{E^*} = 0$ has a convergent subsequence;
- (C₂) For each $k \in N$, there exists an $A_k \in \Sigma_k$ such that $\sup_{u \in A_k} I(u) < 0$.

Then either (R₁) or (R₂) below holds.

- (R₁) There exists a sequence $\{u_k\}$ such that $I'(u_k) = 0$, $I(u_k) < 0$ and $\{u_k\}$ converges to zero.
- (R₂) There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_{k \rightarrow \infty} u_k = 0$, $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k \rightarrow \infty} I(v_k) = 0$, and $\{v_k\}$ converges to a non-zero limit.

Remark 0.5. In [10], the functional $I(u)$ is required to satisfy the Palais-Smale condition in global. However, if $I(u)$ satisfies the local Palais-Smale condition with the critical value levels $c \leq 0$, the results of Kajikiya's, i.e., [[10], Theorem 1] remain true.

Remark 0.6. From Lemma 3 we have a sequence $\{u_k\}$ of critical points such that $I(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k \rightarrow \infty} u_k = 0$.

In order to get infinitely many solutions we need some lemmas. Under the assumptions of Theorem 1.1, we take $\varepsilon = \frac{1}{\lambda_1}$ (where λ_1 is given in Section 1), then by the definition of A_s , (6) and Lemma 1, for $\lambda \in (0, \frac{1}{\lambda_1})$ we have

$$\begin{aligned}
 I(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q} \int_{\Omega} |\nabla u|^q dx - \frac{\mu}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx - \lambda \int_{\Omega} F(x, u) dx \\
 &\geq \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\mu + \lambda \varepsilon p^*(s)}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx - \lambda b(\varepsilon) |\Omega| \\
 &\geq \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\mu + p^*(s)}{p^*(s)} A_s^{-p^*(s)/p} \left(\int_{\Omega} (|\nabla u|^p) dx \right)^{\frac{p^*(s)}{p}} - \lambda b \left(\frac{1}{\lambda_1} \right) |\Omega| \\
 &= A \int_{\Omega} |\nabla u|^p dx - B \left(\int_{\Omega} (|\nabla u|^p) dx \right)^{\frac{p^*(s)}{p}} - \lambda C,
 \end{aligned}$$

where

$$A = \frac{1}{p}, \quad B = \frac{\mu + p^*(s)}{p^*(s)} A_s^{-p^*(s)/p}, \quad C = b \left(\frac{1}{\lambda_1} \right) |\Omega|.$$

Let $Q(t) = At^p - Bt^{p^*(s)} - \lambda C$. Then

$$I(u) \geq Q(\|u\|).$$

Furthermore, there exists

$$\lambda_* = \min \left\{ \lambda_1, \frac{A(p-s)}{C(N-s)} \left(\frac{pA}{p^*(s)B} \right)^{p/(p^*(s)-p)} \right\} > 0$$

such that for $\lambda \in (0, \lambda_*)$, $Q(t)$ attains its positive maximum, that is, there exists

$$R_1 = \left(\frac{pA}{p^*(s)B} \right)^{1/(p^*(s)-p)}$$

such that

$$e_1 = Q(R_1) = \max_{t \geq 0} Q(t) > 0.$$

Therefore, for $e_0 \in (0, e_1)$, we may find $R_0 < R_1$ such that $Q(R_0) = e_0$. Now we define

$$\chi(t) = \begin{cases} 1, & 0 \leq t \leq R_0, \\ \frac{At^p - \lambda C - e_1}{Bt^{p^*(s)}}, & t \geq R_1, \\ C^\infty, \chi(t) \in [0, 1], & R_0 \leq t \leq R_1. \end{cases}$$

Then it is easy to see $\chi(t) \in [0, 1]$ and $\chi(t)$ is C^∞ . Let $\varphi(u) = \chi(\|u\|)$ and consider the perturbation of $I(u)$:

$$\begin{aligned} G(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q} \int_{\Omega} |\nabla u|^q dx \\ &\quad - \frac{\mu\varphi(u)}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx - \lambda\varphi(u) \int_{\Omega} F(x, u) dx. \end{aligned} \tag{15}$$

Then

$$\begin{aligned} G(u) &\geq A \int_{\Omega} |\nabla u|^p dx - B\varphi(u) \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{p^*(s)}{p}} - \lambda C \\ &= \bar{Q}(\|u\|), \end{aligned}$$

where $\bar{Q}(t) = At^p - B\chi(t)t^{p^*(s)} - \lambda C$ and

$$\bar{Q}(t) = \begin{cases} Q(t), & t \leq R_0, \\ e_1, & t \geq R_1. \end{cases}$$

From the above arguments, we have the following:

Lemma 4. *Let $G(u)$ is defined as in (15). Then*

- (i) $G \in C^1(H_0^{1,p}(\Omega), R)$ and G is even and bounded from below;
- (ii) If $G(u) < e_0$, then $\bar{Q}(\|u\|) < e_0$, consequently, $\|u\| < R_0$ and $I(u) = G(u)$;
- (iii) There exists λ^* such that, for $\lambda \in (0, \lambda^*)$, G satisfies a local (PS) condition for

$$c < e_0 \in \left(0, \min \left\{ e_1, \frac{(p-s)\mu}{p(N-s)} [\mu^{-1}A_s]^{\frac{N-s}{p-s}} - \lambda c \left(\frac{(p-s)\mu}{2p\lambda(N-s)} \right) |\Omega| \right\} \right).$$

Proof. It is easy to see (i) and (ii). (iii) are consequences of (ii) and Lemma 2. □

Lemma 5. *Assume that (H_3) of Theorem 1 holds. Then for any $k \in N$, there exists $\delta = \delta(k) > 0$ such that $\gamma(\{u \in H_0^{1,p}(\Omega) : G(u) \leq -\delta(k)\} \setminus \{0\}) \geq k$.*

Proof. Firstly, by (H_3) of Theorem 1, for any fixed $u \in H_0^{1,p}(\Omega)$, $u \neq 0$, we have

$$F(x, \rho u) \geq M(\rho)(\rho u)^q \quad \text{with } M(\rho) \rightarrow \infty \text{ as } \rho \rightarrow 0. \tag{16}$$

Secondly, given any $k \in N$, let E_k be a k -dimensional subspace of $H_0^{1,p}(\Omega)$. We take $u \in E_k$ with norm $\|u\| = 1$, for $0 < \rho < R_0$, we have

$$\begin{aligned} G(\rho u) &= I(\rho u) \leq \frac{\rho^p}{p} \int_{\Omega} |\nabla u|^p dx + \frac{\rho^q}{q} \int_{\Omega} |\nabla u|^q dx \\ &\quad - \frac{\mu \rho^{p^*(s)}}{p^*(s)} \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx - \lambda M(\rho) \rho^q \int_{\Omega} |u|^q dx. \end{aligned}$$

Since E_k is a space of finite dimension, all the norms in E_k are equivalent. If we define

$$\begin{aligned} A_k &= \sup \left\{ \int_{\Omega} |\nabla u|^q dx : u \in E_k, \|\nabla u\|_p = 1 \right\} < \infty, \\ B_k &= \inf \left\{ \int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} dx : u \in E_k, \|\nabla u\|_p = 1 \right\} > 0, \\ C_k &= \inf \left\{ \int_{\Omega} |u|^q dx : u \in E_k, \|\nabla u\|_p = 1 \right\} > 0. \end{aligned}$$

From (16) and $p > q$, we have

$$\begin{aligned} G(\rho u) &\leq \frac{\rho^p}{p} + \frac{\rho^q}{q} A_k - \frac{\mu \rho^{p^*(s)}}{p^*(s)} B_k - \lambda M(\rho) \rho^q C_k \\ &\leq \frac{\rho^p}{p} + \rho^q \left(\frac{A_k}{q} - \lambda M(\rho) C_k \right) \\ &= -\delta(k) < 0, \text{ as } \rho \rightarrow 0, \end{aligned}$$

since $\lim_{|\rho| \rightarrow 0} M(\rho) = +\infty$. That is,

$$\{u \in E_k : \|u\| = \rho\} \subset \{u \in H_0^{1,p}(\Omega) : G(u) \leq -\delta(k)\} \setminus \{0\}.$$

This completes the proof. □

Now we give the proof of Theorem 1 as following.

Proof of Theorem 1 Recall that

$$\Sigma_k = \{A \in H_0^{1,p}(\Omega) \setminus \{0\} : A \text{ is closed and } A = -A, \gamma(A) \geq k\}$$

and define

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} G(u).$$

By Lemmas 4 (1) and 5, we know that $-\infty < c_k < 0$. Therefore, assumptions (C_1) and (C_2) of Lemma 3 are satisfied. This means that G has a sequence of solutions $\{u_n\}$ converging to zero. Hence, Theorem 1.1 follows by Lemma 4 (2). □

4. Acknowledgment

The authors of this paper wish to thank the referee for valuable suggestions regarding the original manuscript. The Project is supported by NSFC(10871096), the project of Nanjing Normal University and Project of Graduate Education Innovation of Jiangsu Province(181200000214).

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