J. Appl. Math. & Informatics Vol. **28**(2010), No. 5 - 6, pp. 1117-1130
Website: http://www.kcam.biz

NECESSARY CONDITION AND SUFFICIENT CONDITION FOR THE WAVELET FRAMES IN $L^2(R^n)$

GUOCHANG WU* AND RUI ZHANG

ABSTRACT. The main goal for this paper is consider the necessary conditions and sufficient conditions of wavelet frames in higher dimensions with an arbitrary expanding matrix dilation. At first, we give a necessary condition of wavelet frame in $L^2(R^n)$, which generalizes the univariate results of Shi from one dimension with an arbitrary real number $a(a > 1)$ dilation to higher dimension with an arbitrary expansive matrix dilation. Secondly, we deduce a necessary condition for wavelet frames in $L^2(R^n)$ when the function ψ satisfies some property of the decay. For the case $n = 1$, we obtain a corollary which has weaker condition comparing with existing result.

AMS Mathematics Subject Classification : 42A38, 42C40.
Key words and phrases : Wavelet frame, Expansive matrix dilation, The property of the decay.

1. Introduction

Frames were first introduced by Duffin and Schaeffer [1] in the context of nonharmonic Fourier series. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of Daubechies, Grossmann, and Meyer [2]. Since then, the theory of frames began to be more widely studied. Traditionally, frames have been used in signal processing, image processing, data compression, and sampling theory. Recently, frames are also used to mitigate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission [3], [4], and to design high-rate constellation with full diversity in multiple-antenna code design [5]. We refer to the monograph of Daubechies [6] or the research-tutorial [7] for basic properties of frames.

Wavelets were introduced relatively recently, in the beginning of the 1980s. They attracted considerable interest from the mathematical community and

Received October 31, 2009. Accepted May 26, 2010. *Corresponding author.

from members of many diverse disciplines in which wavelets had promising applications. Daubechies, Grossman and Meyer([2]) combined the theory of the continuous wavelet transform with the theory of frames to define wavelet frames for $L^2(R)$. In 1990, Daubechies[8] obtained the first result on the necessary conditions for affine frames, and then in 1993, Chui and Shi[9] obtained an improved result. After about ten years, Casazza and Christensen [10] established a stronger condition which also works for wavelet frame. Recently, Shi and his co-authors [11],[12] introduced their results as the necessary conditions and sufficient conditions of wavelet frames.

However, few people consider the necessary conditions and sufficient conditions of wavelet frames in higher dimensions. The main goal for this paper is consider this issue. We establish a necessary condition and a necessary condition for wavelet frames in $L^2(R^n)$. At first, we give a necessary condition of wavelet frame in $L^2(R^n)$ with matrix dilations of the form $(Df)(x) = \sqrt{q} f(Ax)$, where A is an arbitrary expanding $n \times n$ matrix with integer coefficients and $q = |detA|$. Our result is a generalization of some facts of the papers in [8,9]. Though we follow [9] as a blueprint, it is well known that the situation in higher dimensions is so complex that we have to recur to some special matrices to solve problem. Secondly, we deduce a necessary condition for wavelet frames in $L^2(R^n)$ when the function ψ satisfies some conditions of the decay. For the case $n = 1$, we obtain a corollary which has weaker condition comparing with Theorem 3.3.2 in [8]. And we borrow some thoughts and technique in [13] and [14].

Let us now describe the organization of the material that follows. Section 2 is of a preliminary character: it contains various results on matrices belonging to the class E_n and some facts about a frame wavelet. In Section 3, we establish a necessary condition and a necessary condition for wavelet frames in $L^2(R^n)$.

2. Preliminaries

Let us now establish some basic notations.

For $x = (x_1, x_2, \cdots, x_n) \in R^n$, define $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. We denote by T^n the n-dimensional torus. By $L^p(T^n)$ we denote the space of all Z^n-periodic functions f (i.e., f is 1-periodic in each variable) such that $\int_{T^n} |f(x)|^p dx < +\infty$.

We use the Fourier transform in the form

$$\hat{f}(\omega) = \int_{R^n} f(x) e^{-2\pi i <x,\omega>} dx, \tag{2.1}$$

where $< \cdot, \cdot >$ denotes the standard inner product in R^n. Sometimes, $\hat{f}(\omega)$ is defined by $\mathcal{F}f$.

The Lebesgue measure of a set $S \subseteq R^n$ will be denoted by $|S|$. When measurable sets X and Y are equal up to a set of measure zero, we write $X \doteq Y$.

Let E_n denote the set of all $n \times n$ expanding matrices with real entries. The expanding matrices mean that all eigenvalues have magnitude greater than 1. For $A \in E_n$, we denote by A^* the transpose of A. It is obvious that $A^* \in E_n$.

We first present the concept of A-adic vector which was introduced by Shi and Li[13]:

$$\Lambda(A,\, B) := \{\alpha \in R^n :\ \exists (j,m) \in Z \times B^{*-1}(Z^n),\, \alpha = A^{*-j}m\} \tag{2.2}$$

and

$$I_{A,\, B}(\alpha) := \{(j,m) \in Z \times B^{*-1}(Z^n) :\ \alpha = A^{*-j}m\}. \tag{2.3}$$

Here, $\Lambda(A,\, B)$ is thought of as the set of all A-adic vectors relative to the lattice $B^{*-1}(Z^n)$, i.e., the set of representatives of the equivalence classes of $Z \times B^{*-1}(Z^n)$ with respect to the equivalence relation defined by

$$(j,m) \sim (j',m') \ \text{ if and only if } \ \alpha = A^{*-j}m = A^{*-j'}m'. \tag{2.4}$$

Also, the set $I_{A,\, B}(\alpha)$ is the set of points of $Z \times B^{*-1}(Z^n)$ in the equivalence class of $\alpha \in \Lambda(A,\, B)$.

Thus, we can define

$$\Delta_\alpha(\omega) := \sum_{(j,m)\in I_{A,\, B}(\alpha)} \hat{\psi}(A^{*j}\omega)\overline{\hat{\psi}(A^{*j}(\omega + A^{*-j}m))}. \tag{2.5}$$

Let us recall the definition of frame.

Definition 2.1. Let H be a separable Hilbert space. A sequence $\{f_i\}_{i\in N}$ of elements of H is a frame for H if there exist constants $0 < C \le D < \infty$ such that for all $f \in H$,

$$C\|f\|^2 \le \sum_{i=1}^{\infty} |< f, f_i >|^2 \le D\|f\|^2. \tag{2.6}$$

The numbers C, D are called lower and upper frame bounds, respectively(the largest C and the smallest D for which (2.6) holds are the optimal frame bounds). Those sequences which satisfy only the upper inequality in (2.6) are called Bessel sequences. A frame is tight if $C = D$. If $C = D = 1$, it is called a Parseval frame.

Let T_f denote the synthesis operator of $f = \{f_i\}_{i\in N}$, i.e., $T_f(c) = \sum_i c_i f_i$ for each sequence of scalars $c = (c_i)_{i\in N}$. Then the frame operator $Sh = T_f T_f^*(h)$ associated with $\{f_i\}_{i\in N}$ is a bounded, invertible, and positive operator mapping of H on itself. This provides the reconstruction formula

$$h = \sum_{i=1}^{\infty} < h, \tilde{f}_i > f_i = \sum_{i=1}^{\infty} < h, f_i > \tilde{f}_i, \forall\ h \in H. \tag{2.7}$$

where $\tilde{f}_i = S^{-1}f_i$. The family $\{\tilde{f}_i\}_{i\in N}$ is also a frame for H, called the canonical dual frame of $\{f_i\}_{i\in N}$. If $\{g_i\}_{i\in N}$ is any sequence in H which satisfies

$$h = \sum_{i=1}^{\infty} < h, g_i > f_i = \sum_{i=1}^{\infty} < h, f_i > g_i,\ \forall\ h \in H, \tag{2.8}$$

it is called an alternate dual frame of $\{f_i\}_{i\in N}$.

In this paper, we will work with two families of unitary operators on $L^2(R^n)$. The first one consists of all translation operators $T_{Bk}: L^2(R)^n \to L^2(R^n)$, $k \in Z^n$, defined by $(T_{Bk}f)(x) = f(x - Bk)$, where B is an $n \times n$ non-singular matrix. The second one consists of all integer powers of the dilation operator $D_A : L^2(R^n) \to L^2(R^n)$ defined by $(Df)(x) = q^{\frac{1}{2}} f(Ax)$ with $A \in E_n$ and $q =: |detA|$.

Let us now fix an arbitrary matrix $A \in E_n$ and let B be an $n\times n$ non-singular matrix. For a function $\psi \in L^2(R^n)$, we will consider the affine system Ψ defined by

$$\Psi = \left\{\psi_{j,\ k}(x) \quad | \quad \psi_{j,\ k}(x) = q^{\frac{j}{2}}\psi(A^j x - Bk), \quad j \in Z,\ k \in Z^n\right\}. \tag{2.9}$$

Then, we will give the definitions of the wavelet frame and the frame wavelet.

Definition 2.2. We say that the affine system Ψ defined by (2.9) is a wavelet frame if it is a frame for $L^2(R^n)$. Then, the function ψ is called a frame wavelet.

In order to prove theorems to be presented in next section, we need the following lemmas.

Lemma 2.1 *Suppose that $\{f_k\}_{k=1}^{+\infty}$ is a family of elements in a Hilbert space H such that there exist constants $0 < C \le D < +\infty$ satisfying (2.6) for all f belonging to a dense subset D of H. Then, the same inequalities (2.6) are true for all $f \in H$; that is, $\{f_k\}_{k=1}^{+\infty}$ is a frame for H.*

For proof of Lemma 2.1, people can refer to the book[6].

In view of Lemma 2.1, we will consider the following set of functions:

$$D = \left\{f \in L^2(R^n):\ \hat{f} \in L^\infty(R^n) \text{ and } \hat{f} \text{ has compact support in } R^n \setminus \{0\}\right\}. \tag{2.10}$$

The following result is well known.

Lemma 2.2 *D is a dense subset $L^2(R^n)$.*

In general, the constants used in the following section are different each other. But, for the sake of simplicity, we notate by the symbol C.

3. Main Results

We firstly give some existing results in real line R.

Let a and b be the real numbers with $a > 1, b > 0, \psi \in L^2(R)$, and the system $\psi_{j,k}(x) := \{a^{\frac{j}{2}}\psi(a^j x - kb)\}_{j,k\in Z}$ be a wavelet frame. In 1990, Daubechies[8] proved that if the system $\psi_{j,k}(x)$ forms a wavelet frame in $L^2(R)$ with bounds C and D, then

$$bC\ln a \le \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega \le bD\ln a \tag{3.1}$$

and

$$bC\ln a \le \int_{-\infty}^{0} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega \le Db\ln a. \tag{3.2}$$

In 1993, C. K. Chui and X. L. Shi[9] established the following improvement if $\psi(x)$ is a frame wavelet:

$$bC \le \sum_{j\in Z} |\hat{\psi}(a^j\omega)|^2 \le bD. \tag{3.3}$$

Motivating by the fundament works in [9], we will give a necessary condition of wavelet frame for higher dimension with an arbitrary expansive matrix dilation in the following.

Theorem 3.1 *Suppose that* $\{\psi_{j,k}(x) = q^{\frac{j}{2}}\psi(A^jx - Bk)\}_{j\in Z,\, k\in Z^n}$ *is a wavelet frame for with bounds* C *and* D*, then*

$$pC \le \sum_{j\in Z} |\hat{\psi}(A^{*j}\omega)|^2 \le pD, \tag{3.4}$$

where $q = |detA|$ *and* $p = |detB|$.

Proof. Let $\hat{f} \in C_c(R)$ and $\hat{f}$ has compact support.

By the commutator relations for the Fourier transform and operators, we have

$$\begin{aligned}
&\sum_{j\in Z}\sum_{m\in Z^n} |< f,\, D_{A^j}T_{Bm}\psi >|^2 \\
=& \sum_{j\in Z}\sum_{m\in Z^n} |< \mathcal{F}f,\, \mathcal{F}D_{A^j}T_{Bm}\psi >|^2 \\
=& \sum_{j\in Z}\sum_{m\in Z^n} |< \hat{f},\, D_{A^{-j}}E_{-Bm}\hat{\psi} >|^2 \\
=& \sum_{j\in Z}\sum_{m\in Z^n} |< \hat{f},\, E_{-A^jBm}D_{A^{-j}}\hat{\psi} >|^2 \\
=& \sum_{j\in Z} q^{-j} \sum_{m\in Z^n} |\int_{R^n} \hat{f}(\omega)\overline{\hat{\psi}(A^{*-j}\omega)} e^{2\pi i A^{-j}Bm\omega} d\omega|^2 \\
=& \sum_{j\in Z} q^{j} \sum_{m\in Z^n} |\int_{R^n} \hat{f}(A^{*j}\omega)\overline{\hat{\psi}(\omega)} e^{2\pi i Bm\omega} d\omega|^2
\end{aligned} \tag{3.5}$$

where we change variables by $\omega' = A^{-j}\omega$ in the last equality.

We assert:

$$\begin{aligned} & \sum_{j\in Z}\sum_{m\in Z}|\int_{R^n}\hat{f}(A^{*j}\omega)\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega|^2 \\ = & \sum_{j\in Z}\frac{q^j}{p}\int_{B^{*-1}([0,1]^n)}|\sum_{s\in Z^n}\hat{f}(A^{*j}(\omega+B^{*-1}s))\bar{\hat{\psi}}(\omega+B^{*-1}s)|^2d\omega. \end{aligned} \tag{3.6}$$

Fix $j\in Z$, we have

$$\begin{aligned} & \int_{B^{*-1}([0,1]^n)}\sum_{s\in Z^n}|\hat{f}(A^{*j}(\omega+B^{*-1}s))\bar{\hat{\psi}}(\omega+B^{*-1}s)|d\omega \\ = & \sum_{s\in Z^n}\int_{B^{*-1}([0,1]^n)}|\hat{f}(A^{*j}(\omega+B^{*-1}s))\bar{\hat{\psi}}(\omega+B^{*-1}s)|d\omega \\ = & \sum_{s\in Z^n}\int_{B^{*-1}s+B^{*-1}([0,1]^n)}|\hat{f}(A^{*j}(\omega))\bar{\hat{\psi}}(\omega)|d\omega \\ = & \int_{R^n}|\hat{f}(A^{*j}(\omega))\bar{\hat{\psi}}(\omega)|d\omega \\ \le & (\int_{R^n}|\hat{f}(A^{*j}(\omega))|^2d\omega)^{\frac{1}{2}}(\int_{R^n}|\bar{\hat{\psi}}(\omega)|^2d\omega)^{\frac{1}{2}} \\ < & \infty, \end{aligned} \tag{3.7}$$

where the fourth inequality is obtained by using Cauchy-Schwarz's inequality.

Thus we can define a function $F_j : R \to C$ by

$$F_j(\omega)=\sum_{s\in Z^n}\hat{f}(A^{*j}(\omega+B^{*-1}s))\bar{\hat{\psi}}(\omega+B^{*-1}s),\ a.e.\ \omega. \tag{3.8}$$

$F_j(\omega)$ is $B^{*-1}T^n$-periodic, and the above argument gives that

$$F_j(\omega)\in L^1(B^{*-1}[0,1]^n).$$

In fact, we even have $F_j(\omega)\in L^2(B^{*-1}[0,1]^n)$. To see this, we first see that

$$|F_j(\omega)|^2\le\sum_{s\in Z^n}|\hat{f}(A^{*j}(\omega+B^{*-1}s))|^2\sum_{s\in Z^n}|\hat{\psi}(\omega+B^{*-1}s)|^2. \tag{3.9}$$

Since $\hat{f}\in C_c(R)$, the function $\omega\to\sum_{s\in Z^n}|\hat{f}(A^{*j}(\omega+B^{*-1}s))|^2$ is bounded. According to above argument, we easily get $F_j(x)\in L^2(B^{*-1}[0,1]^n)$ and the following equality

$$\int_{R^n}\hat{f}(A^{*j}\omega)\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega=\int_{B^{*-1}([0,1]^n)}F_j(\omega)e^{2\pi iBm\omega}d\omega \tag{3.10}$$

holds.

Parseval's equality shows that

$$\sum_{m\in Z^n}|\int_{B^{*-1}([0,1]^n)}F_j(\omega)e^{2\pi iBm\omega}d\omega|^2=\frac{1}{p}\int_{B^{*-1}([0,1]^n)}|F_j(\omega)|^2d\omega; \tag{3.11}$$

Combining (3.10),(3.11) and the definition of $F_j(\omega)$, we obtain that

$$\begin{aligned}&\sum_{m\in Z}|\int_{R^n}\hat{f}(A^{*j}\omega)\overline{\hat{\psi}(\omega)}e^{2\pi iBm\omega}d\omega|^2\\ =\ &\frac{q^j}{p}\int_{B^{*-1}([0,1]^n)}|\sum_{s\in Z^n}\hat{f}(A^{*j}(\omega+B^{*-1}s))\bar{\hat{\psi}}(\omega+B^{*-1}s)|^2d\omega.\end{aligned}\tag{3.12}$$

So, we obtain (3.6). Thus, we complete the assertion.

Choose $\omega_0\in R$ to be Lebesgue point of the function $\sum_{j\in Z}\hat{\psi}(A^{*j}\omega)|^2$. Letting $B(\epsilon)$ denote the ball of radius $\epsilon>0$ about the origin and ϵ be sufficiently small, define f_ϵ by

$$\hat{f}_\epsilon(\omega)=\frac{1}{\sqrt{|B(\epsilon)|}}\chi_{B(\epsilon)}(\omega-\omega_0).\tag{3.13}$$

Therefore, we obtain

$$\|f_\epsilon\|^2=\|\hat{f}_\epsilon\|^2=1.\tag{3.14}$$

Thus, we have

$$\sum_{j\le M}\hat{\psi}(A^{*j}\omega)|^2=\int_{|\omega-\omega_0|<\epsilon}\frac{1}{|B(\epsilon)|}\sum_{j\le M}\hat{\psi}(A^{*j}\omega)|^2d\omega.\tag{3.15}$$

From the definition of f, we have

$$\begin{aligned}&\int_{|\omega-\omega_0|<\epsilon}\frac{1}{|B(\epsilon)|}\sum_{j\le M}\hat{\psi}(A^{*j}\omega)|^2d\omega\\ =\ &\sum_{|j|\le M}\int_{\omega_0+B^{*-1}([0,1]^n)}|\hat{f}_\epsilon(\omega)|^2|\hat{\psi}(A^{*-j}\omega)|^2d\omega\\ =\ &\sum_{|j|\le M}q^j\int_{B^{*-1}([0,1]^n)}|\sum_{s\in Z^n}\hat{f}_\epsilon(A^{*j}(\omega+B^{*-1}s))\bar{\hat{\psi}}(\omega+B^{*-1}s)|^2d\omega\\ =\ &p\sum_{|j|\le M}\sum_{k\in Z^n}|<f_\epsilon,\,D_{A^j}T_{Bk}\psi>|^2\\ \le\ &pD,\end{aligned}\tag{3.16}$$

where the third equality is obtained by changing variables $\omega'=A^{-j}\omega$ and the fourth equality is obtained by (3.6).

Let $\epsilon\to 0$, and let $M\to\infty$, using the definition of Lebesgue point, we get

$$\sum_{j\in Z}|\hat{\psi}(A^{*j}\omega_0)|^2\le pD.\tag{3.17}$$

Let $\omega_0\in R$ to be Lebesgue point of the function $\sum_{j\in Z}\hat{\psi}(A^{*j}\omega)|^2$.

Fix $M\in N$. Let $Q_\delta(\omega_0)$ denote the ball of radius $\delta=q^{-M}$ about the center ω_0 and M be sufficiently large.

Define f_δ by

$$\hat{f}_\delta(\omega)=(q^{-M})^{-\frac{1}{2}}\chi_{Q_\delta(\omega_0)}(\omega).\tag{3.18}$$

In the same way, we have $\|f_\delta\|^2 = \|\hat{f}_\delta\|^2 = 1$.

Then, by similar proof of (3.16), we have

$$\begin{aligned} & \frac{1}{q^{-M}} \int_{Q_\delta(\omega_0)} \sum_{j>-M} |\hat{\psi}(A^{*j}\omega)|^2 d\omega \\ = \quad & \sum_{j>-M} q^j \int_{\omega_0+B^{*-1}([0,1]^n)} |\sum_{s\in Z^n} \hat{f}_\delta(A^{*j}(\omega+B^{*-1}s))\bar{\hat{\psi}}(\omega+B^{*-1}s)|^2 d\omega \\ = \quad & p \sum_{j>-M} \sum_{k\in Z^n} |< f_\delta, D_{A^j}T_{Bk}\psi >|^2 \\ = \quad & p\sum_{j\in Z} \sum_{k\in Z^n} |< f_\delta, D_{A^j}T_{Bk}\psi >|^2 - p \sum_{j\le -M} \sum_{k\in Z^n} |< f_\delta, D_{A^j}T_{Bk}\psi >|^2 \\ \ge \quad & pC - \frac{1}{q^{-M}} \sum_{j\le -M} \int_{Q_\delta(\omega_0)} |\hat{\psi}(A^{*j}\omega)|^2 d\omega. \end{aligned} \tag{3.19}$$

Therefore, let $M \to \infty$, we obtain

$$\frac{1}{q^{-M}} \sum_{j\le -M} \int_{Q_\delta(\omega_0)} |\hat{\psi}(A^{*-j}\omega)|^2 d\omega \to 0. \tag{3.20}$$

From (3.19) and (3.20), let $M \to \infty$, we obtain

$$\sum_{j\in Z} |\hat{\psi}(A^{*j}\omega_0)|^2 \ge pC. \tag{3.21}$$

Comparing with (3.17) and (3.21), we have (3.4).

Therefore, we have completed the proof of Theorem 2.1. □

Not all choices for ψ, A, B lead to the system

$$\{\psi_{j,k}(x) = q^{\frac{j}{2}}\psi(A^j x - Bk)\}_{j\in Z,\, k\in Z^n}$$

to be a wavelet frame, even if ψ satisfies (3.4).

In this section, we will derive a sufficient condition of wavelet frame with an arbitrary matrix dilation when the function ψ satisfies some conditions of the decay and estimate the corresponding frame bounds. For the case $n = 1$, we obtain a corollary which has weaker condition comparing with existing result.

Theorem 3.2 *Let the matrix $A \in E_n$ and B be a $n \times n$ non-singular matrix. Suppose that the function $\psi \in L^2(R^n)$ and satisfies $|\hat{\psi}(\omega)| \le C|\omega|^\delta(1 + |\omega|)^{-\gamma}(\delta > 0, \gamma > \delta + \frac{1}{2})$ and*

$$\inf_{1\le|\omega|\le q} \sum_{j\in Z} |\hat{\psi}(A^{*j}\omega)|^2 > 0. \tag{3.22}$$

Then, the system $\{\psi_{j,k}(x) = q^{\frac{j}{2}}\psi(A^j x - Bk)\}_{j\in Z,\, k\in Z^n}$ is a wavelet frame with frame bounds C, D defined by

$$C = \frac{1}{p}\{\inf_{1\le|\omega|\le q} \sum_{j\in Z} |\hat{\psi}(A^{*j}\omega)|^2 - \sum_{k\ne 0} [\Delta_\alpha(\omega)\Delta_{-\alpha}(\omega)]^{\frac{1}{2}}\}, \tag{3.23}$$

$$D=\frac{1}{p}\{\sup_{1\le|\omega|\le q}\sum_{j\in Z}|\hat{\psi}(A^{*j}\omega)|^2+\sum_{k\neq 0}[\Delta_\alpha(\omega)\Delta_{-\alpha}(\omega)]^{\frac{1}{2}}\}, \tag{3.24}$$

where $q=|detA|$ *and* $p=|detB|$.

Proof. By Lemma 2.1 and Lemma 2.2, it suffices to show that Theorem 3.2 holds for all $f\in D$.

To do this, we need to estimate $\sum_{j\in Z}\sum_{k\in Z^n}|<f,\,D_{A^j}T_{Bk}\psi>|^2$.

Because $f\in D$, the number of k is finite, so (3.6) and the Fourier transform inversion formula imply that

$$\begin{aligned}
&\sum_{j\in Z}\sum_{k\in Z^n}|<f,\,D_{A^j}T_{Bk}\psi>|^2\\
=\;&\sum_{j\in Z}\frac{q^j}{p}\int_{B^{*-1}([0,1]^n)}|\sum_{s\in Z^n}\hat{f}(A^{*j}(\omega+B^{*-1}s))\bar{\hat{\psi}}(\omega+B^{*-1}s)|^2d\omega\\
=\;&\sum_{j\in Z}\frac{q^j}{p}\int_{B^{*-1}([0,1]^n)}\sum_{s\in Z^n}\hat{f}(A^{*j}(\omega+B^{*-1}s))\bar{\hat{\psi}}(\omega+B^{*-1}s)\\
&\sum_{m\in Z^n}\bar{\hat{f}}(A^{*j}(\omega+B^{*-1}m))\hat{\psi}(\omega+B^{*-1}m)d\omega\\
=\;&\sum_{j\in Z}\frac{q^j}{p}\int_{R^n}\overline{\hat{f}(A^{*j}\omega)}\hat{\psi}(\omega)\left[\sum_{s\in Z^n}\hat{f}(A^{*j}(\omega+B^{*-1}s)))\bar{\hat{\psi}}(\omega+B^{*-1}s)\right]d\omega.
\end{aligned} \tag{3.25}$$

Then, by (3.25) and changing variables $\omega'=A^{*j}\omega$, we can write

$$\begin{aligned}
&\sum_{j\in Z}\sum_{k\in Z^n}|<f,\,D_{A^j}T_{Bk}\psi>|^2\\
=\;&\frac{1}{p}\sum_{j\in Z}\sum_{s\in Z^n}\int_{R^n}\overline{\hat{f}(\omega)}\hat{\psi}(A^{*-j}\omega)\hat{f}(\omega+A^{*j}B^{*-1}s))\bar{\hat{\psi}}(A^{*-j}\omega+B^{*-1}s)d\omega\\
=\;&Q_1+Q_2,
\end{aligned} \tag{3.26}$$

where,

$$Q_1=\frac{1}{p}\sum_{j\in Z}\int_{R^n}|\hat{f}(\omega)\hat{\psi}(A^{*-j}\omega)|^2d\omega \tag{3.27}$$

and

$$Q_2=\frac{1}{p}\sum_{j\in Z}\sum_{s\in Z^n\setminus\{0\}}\int_{R^n}\overline{\hat{f}(\omega)}\hat{\psi}(A^{*-j}\omega)[\hat{f}(\omega+A^{*j}B^{*-1}s))\bar{\hat{\psi}}(A^{*-j}\omega+B^{*-1}s)]d\omega. \tag{3.28}$$

Thus, by the notations of (2.2),(2.3) and (2.5), we can rearrange the series Q_2 as

$$\begin{aligned} Q_2 &= \frac{1}{p}\sum_{\alpha\in\Lambda(A,\,B)\setminus\{0\}}\int_{R^n}\overline{\hat{f}(\omega)}\hat{f}(\omega+\alpha))\left(\sum_{(j,m)\in I_{A,\,B}(\alpha)}\hat{\psi}(A^{*j}\omega)\bar{\hat{\psi}}(A^{*j}(\omega+\alpha))\right)d\omega \\ &= \frac{1}{p}\sum_{\alpha\in\Lambda(A,\,B)\setminus\{0\}}\int_{R^n}\overline{\hat{f}(\omega)}\hat{f}(\omega+\alpha))\Delta_\alpha(\omega)d\omega. \end{aligned} \tag{3.29}$$

According to Hölder's inequality and (3.29), we have

$$\begin{aligned} |Q_2| &\le \frac{1}{p}\sum_{\alpha\in\Lambda(A,\,B)\setminus\{0\}}\int_{R^n}\left(|\bar{\hat{f}}(\omega)|\sqrt{|\Delta_\alpha(\omega)|})(|\hat{f}(\omega+\alpha)|\sqrt{|\Delta_\alpha(\omega)|}\right)d\omega \\ &\le \frac{1}{p}\sum_{\alpha\in\Lambda(A,\,B)\setminus\{0\}}\left[\int_{R^n}|\bar{\hat{f}}(\omega)|^2|\Delta_\alpha(\omega)|d\omega\int_{R^n}|\hat{f}(\omega+\alpha)|^2|\Delta_\alpha(\omega)|d\omega\right]^{\frac{1}{2}}. \end{aligned} \tag{3.30}$$

Therefore, by (3.30) and Cauchy-Schwarz's inequality,

$$\begin{aligned} |Q_2| \le\ & \frac{1}{p}\sqrt{\int_{R^n}|\hat{f}(\omega)|^2\sum_{\alpha\in\Lambda(A,\,B)\setminus\{0\}}|\Delta_\alpha(\omega)|d\omega} \\ & \times\sqrt{\sum_{\alpha\in\Lambda(A,\,B)\setminus\{0\}}\int_{R^n}|\hat{f}(\omega+\alpha)|^2|\Delta_\alpha(\omega)|d\omega.} \end{aligned} \tag{3.31}$$

Combining with (3.31) and the following facts:

$$\int_{R^n}|\hat{f}(\omega+\alpha)|^2|\Delta_\alpha(\omega)|d\omega = \int_{R^n}|\hat{f}(\omega)|^2|\Delta_\alpha(\omega-\alpha)|d\omega \tag{3.32}$$

and

$$\begin{aligned} \Delta_\alpha(\omega-\alpha) &= \sum_{(j,m)\in I_{A,\,B}(\alpha)}\hat{\psi}(A^{*j}\omega-\alpha)\bar{\hat{\psi}}(A^{*j}\omega) \\ &= \overline{\sum_{(j,m)\in I_{A,\,B}(\alpha)}\hat{\psi}(A^{*j}\omega)\bar{\hat{\psi}}(A^{*j}\omega-m)} \\ &= \overline{\Delta_{-\alpha}(\omega)}, \end{aligned} \tag{3.33}$$

we obtain

$$|Q_2| \le \frac{1}{p}\|f\|^2\sum_{k\ne 0}[\Delta_\alpha(\omega)\Delta_{-\alpha}(\omega)]^{\frac{1}{2}}, \tag{3.34}$$

where $\Delta_\alpha(\omega)$ is defined by (2.5).

From (3.26), (3.27) and (3.8), we see that

$$\begin{aligned} & \inf_{f\ne 0}\|f\|^{-2}\sum_{j\in Z}\sum_{k\in Z^n}|<f,\,D_{A^j}T_{Bk}\psi>|^2 \\ \ge\ & \frac{1}{p}\{\inf_{\omega\in R}\sum_{j\in Z}|\hat{\psi}(A^{*j}\omega)|^2-\sum_{k\ne 0}[\Delta_\alpha(\omega)\Delta_{-\alpha}(\omega)]^{\frac{1}{2}}\}, \end{aligned} \tag{3.35}$$

$$\begin{aligned} & \sup_{f\neq 0} \|f\|^{-2} \sum_{j\in Z}\sum_{k\in Z^n} |< f,\, D_{A^j}T_{Bk}\psi >|^2 \\ \leq \; & \frac{1}{p}\{\sup_{\omega\in R}\sum_{j\in Z}|\hat{\psi}(A^{*j}\omega)|^2 + \sum_{k\neq 0}[\Delta_\alpha(\omega)\Delta_{-\alpha}(\omega)]^{\frac{1}{2}}\}. \end{aligned} \tag{3.36}$$

If the right-hand sides of (3.35) and (3.36) are strictly positive and bounded, then the system $\{\psi_{j,k}(x) = q^{\frac{j}{2}}\psi(A^j x - Bk)\}_{j\in Z,\, k\in Z^n}$ constitute a frame, and (3.23) give a lower bound A and (3.24) gives an upper bound B.

To make this work, it suffices that, for all $1 \leq |\omega| \leq q$ (other values of ω can be reduced to this range by multiplication with a suitable factor A^j),

$$0 < C_1 \leq \sum_{j\in Z}|\hat{\psi}(A^{*j}\omega)|^2 \leq C_2 < \infty; \tag{3.37}$$

moreover, $\Delta_\alpha(\omega) = \sum_{j\in Z}|\hat{\psi}(A^{*j}\omega)||\hat{\psi}(A^{*j}(\omega+\alpha))|$ should have sufficient decay at ∞, i.e. $\Delta_\alpha(\omega)$ converges.

Noticing that (2.24) implies that lower frame bound holds. Because $|\hat{\psi}(\omega)| \leq C|\omega|^\alpha(1+|\omega|^2)^{-\gamma}$, $\delta > 0, \gamma > \delta + \frac{1}{2}$, this condition implies both boundedness of $\sum_{j\in Z}|\hat{\psi}(A^{*j}\omega)|^2$ and the decay of $\Delta_\alpha(\omega)$:

From the convergence of geometric series, we have

$$\begin{aligned} \sum_{j\in Z}|\hat{\psi}(A^{*j}\omega)|^2 \;& \leq \; \sup_{1\leq|\omega|\leq q}\sum_{j\in Z}|\hat{\psi}(A^{*j}\omega)|^2 \\ & \leq \; C^2q^{2\delta}\left[\sum_{j=-\infty}^{0} q^{2j\delta} + \sum_{j=1}^{+\infty} q^{2j\delta}(1+q^{2j})^{-\gamma}\right] \\ & \leq \; C^2q^{2\delta}\left[\sum_{j=0}^{+\infty} q^{-2j\delta} + \sum_{j=1}^{+\infty} 1+q^{2j})^{\delta}(1+q^{2j})^{-\gamma}\right] \\ & \leq \; C^2q^{2\delta}\left[\sum_{j=0}^{+\infty} q^{-2j\delta} + \sum_{j=1}^{+\infty}(1+q^{2j})^{-1}\right] \\ & < \; +\infty. \end{aligned} \tag{3.38}$$

Furthermore, let $\beta = A^{*j}\alpha = B^{*-1}k\,(k \in Z^n)$, then, we can obtain

$$\begin{aligned}
\Delta_\alpha(\omega) &= \sum_{(j,m)\in I_{A,\,B}(\alpha)} \hat{\psi}(A^{*j}\omega)\overline{\hat{\psi}(A^{*j}(\omega+\alpha))} \\
&= \sup_{1\le|\omega|\le q}\sum_{j\in Z}\hat{\psi}(A^{*j}\omega)\bar{\hat{\psi}}(A^{*j}\omega+\beta) \\
&\le C^2\sup_{1\le|\omega|\le q}\sum_{j\in Z}|A^{*j}\omega|^\delta(1+|A^{*j}\omega|^2)^{-\gamma}|A^{*j}\omega+\beta|^\delta(1+|A^{*j}\omega+\beta|^2)^{-\gamma} \\
&\le C^2\sup_{1\le|\omega|\le q}\left\{\sum_{j=-\infty}^{-1}|A^{*j}\omega|^\delta(1+|A^{*j}\omega|^2)^{-\gamma}|A^{*j}\omega+\beta|^\delta(1+|A^{*j}\omega+\beta|^2)^{-\gamma}\right. \\
&\quad\left.+\sum_{j=0}^{+\infty}|A^{*j}\omega|^\delta(1+|A^{*j}\omega|^2)^{-\gamma}|A^{*j}\omega+\beta|^\delta(1+|A^{*j}\omega+\beta|^2)^{-\gamma}\right\} \\
&\le C^2\left\{q^\delta\sum_{j=-\infty}^{-1}q^{j\delta}(1+|A^{*j}\omega+\beta|^2)^{-(\gamma-\delta)}\right. \\
&\quad\left.+\sum_{j=0}^{+\infty}[(1+|A^{*j}\omega|^2)(1+|A^{*j}\omega+\beta|^2)]^{-(\gamma-\delta)}\right\}.
\end{aligned} \tag{3.39}$$

If $|\beta|\ge 2$, we have $|A^{*j}\omega+\beta|\ge|\beta|-1\ge\frac{|\beta|}{2}$, hence $1+|A^{*j}\omega+\beta|^2)^{-1}\le 4(1+|\beta|^2)^{-1}$. For $|\beta|\le 2, 1+|A^{*j}\omega+\beta|^2)^{-1}\le 1\le 5(1+|\beta|^2)^{-1}$.

For any $\beta\in R^n$ and $1\le|\omega|\le q$, we have

$$(1+|A^{*j}\omega+\beta|^2)^{-(\gamma-\delta)}\le C(1+|\beta|^2)^{-(\gamma-\delta)}. \tag{3.40}$$

Thus, by (3.40) and the convergence of geometric series, we obtain

$$\begin{aligned}
&\sum_{j=-\infty}^{-1}q^{j\delta}(1+|A^{*j}\omega+\beta|^2)^{-(\gamma-\delta)} \\
\le\ & C(1+|\beta|^2)^{-(\gamma-\delta)}\sum_{j=-\infty}^{-1}q^{j\delta} \\
\le\ & C(1+|\beta|^2)^{-(\gamma-\delta)}.
\end{aligned} \tag{3.41}$$

Again using (3.40) and the convergence of geometric series, we have

$$\begin{aligned}
&\sum_{j=0}^{+\infty}[(1+|A^{*j}\omega|^2)(1+|A^{*j}\omega+\beta|^2)]^{-(\gamma-\delta)} \\
\le\ & C(1+|\beta|^2)^{-(\gamma-\delta)}\sum_{j=0}^{+\infty}(1+|A^{*j}\omega|^2)^{-(\gamma-\delta)} \\
\le\ & C(1+|\beta|^2)^{-(\gamma-\delta)}\sum_{j=0}^{+\infty}q^{-2j(\gamma-\delta)} \\
\le\ & C(1+|\beta|^2)^{-(\gamma-\delta)}.
\end{aligned} \tag{3.42}$$

Combining with (3.39), (3.41) and (3.42), we have

$$\Delta_\alpha(\omega) \le C(1+|\beta|^2)^{-(\gamma-\delta)}. \tag{3.43}$$

By (3.43) and the convergence of famous p-series, we have

$$\begin{aligned} &\sum_{k\neq 0}[\Delta_\alpha(\omega)\Delta_{-\alpha}(\omega)]^{\frac{1}{2}} \\ \le & \sum_{k\neq 0} C(1+|\beta|^2)^{-(\gamma-\delta)} \\ \le & \sum_{k\neq 0} C(1+\frac{1}{p^2}|k|^2)^{-(\gamma-\delta)} \\ \le & \sum_{k_1\in Z} C|k_1|^{2-(\gamma-\delta)} \\ < & \ \infty, \end{aligned} \tag{3.44}$$

where $k=(k_1,k_2,\cdots,k_n)$.

Then, the system $\{\psi_{j,k}(x)=q^{\frac{j}{2}}\psi(A^jx-Bk)\}_{j\in Z,\,k\in Z^n}$ is a wavelet frame with frame bounds C, D defined by (3.23) and (3.24).

Thus, we have completed the proof. □

For the case $n=1$, we obtain the following corollary.

Corollary 3.3 *Let the number a, b satisfy $a>1$ and $b>0$. Suppose that the function $\psi\in L^2(R)$ and satisfies $|\hat{\psi}(\omega)|\le C|\omega|^\delta(1+|\omega|)^{-\gamma}(\delta>0, \gamma>\delta+\frac{1}{2})$ and*

$$\inf_{1\le|\omega|\le q}\sum_{j\in Z}|\hat{\psi}(a^j\omega)|^2>0. \tag{3.45}$$

Then, the system $\{\psi_{j,k}(x)=a^{\frac{j}{2}}\psi(a^jx-bk)\}_{j\in Z,\,k\in Z}$ constitutes a wavelet frame with frame bounds C, D defined by

$$C=\frac{1}{b}\{\inf_{1\le|\omega|\le a}\sum_{j\in Z}|\hat{\psi}(a^j\omega)|^2-\sum_{k\neq 0}[\beta(\frac{k}{b})\beta(-\frac{k}{b})]^{\frac{1}{2}}\}, \tag{3.46}$$

$$D=\frac{1}{b}\{\sup_{1\le|\omega|\le a}\sum_{j\in Z}|\hat{\psi}(a^j\omega)|^2+\sum_{k\neq 0}[\beta(\frac{k}{b})\beta(-\frac{k}{b})]^{\frac{1}{2}}\}, \tag{3.47}$$

where $\beta(s):=\sup_{\omega\in R}\sum_{j\in Z}|\hat{\psi}(a^j\omega)||\hat{\psi}(a^j\omega+s)|$.

Remark. If the inequality $\gamma>\delta+\frac{1}{2}$ in Corollary 3.3 is replaced by $\gamma>\delta+1$ and other conditions remain unchanged, we get Theorem 3.3.2 in [8], so our Corollary 3.3 has weaker condition comparing with existing result.

4. Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 60774041) and the Natural Science Foundation for the Education Department of Henan Province of China (No. 2010A110002).

References

1. R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341-366.
2. I. Daubechies, A. Groddmann and Y. Mayer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271-1283.
3. P. G. Casazza and J. Kovačević, *Equal-norm tight frames with erasures*, Adv. Comput. Math. **18** (2003), 387-430.
4. V. K. Goyal, J. Kovačević and J. A. Kelner, *Quantized frames expansions with erasures*, Appl. Comput. Harmon. Anal. **10** (2001), 203-233.
5. B. Hassibi, B. Hochwald, A. Shokrollahi and W. Sweldens, *Representation theory for high-rate multiple-antenna code design*, IEEE Trans. Inform. Theory **47** (2001), 2335-2367.
6. I. Daubechies, *Ten Lectures on Wavelets, in: CBMS-NSF Regional Conference Series in Applied Mathematics*, vol. 61, SIAM, Philadelphia, 1992.
7. R. Young, *An introduction to nonharmonic Fourier series*, Academic Press, New York, 1980.
8. I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory **36** (1990), 961-1005.
9. C. K. Chui and X. L. Shi, *Inequalities of Littlewood-Paley type for frames and wavelets*, SIAM J. Math. Anal. **24** (1993), 263-277.
10. P. G. Casazza and O. Christensen, *Weyl-Heisenberg frames for subspaces of $L^2(R)$*, Proc. Amer. Math. Soc. **129** (2001), 145-154.
11. X. L. Shi and Q. L. Shi, *A new criterion of affine frame*, Chin. Ann. of Math.(Series A) **26** (2005): 257-262.
12. X. L Shi. and F. Chen, *Necessary conditions and sucient conditions of affine frame*, Science in China (Series A) **48** (2005), 1369-1378.
13. D. F. Li and X. L. Shi, *A sufficient condition for affine frames with matrix dilation*, Analysis in Theory and Applications **25** (2009), 66-174.
14. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhauser, Boston, 2002.

Guochang Wu received his B.Sc. (Mathematics) from Henan University, M.Sc. (Mathematics) and Ph.D. (Mathematics) at Xi'an Jiaotong University under the direction of Prof. Zhengxing Cheng. Since 2008 he has teached at Henan University of Finance and Economics. His research interests focus on wavelet theory and sampling theorem.

Department of Mathematics and Information Science, Henan University of Finance and Economics, Zhengzhou 450002, P. R. China
e-mail: archang-0111@163.com

Rui Zhang received her B.Sc. (Mathematics) from Beijing Normal University and her M.Sc. (Mathematics) at Henan University. Since 1984 she has worked at Henan University of Finance and Economics. Her research interests focus on wavelet theory and actuarial economy.

Department of Mathematics and Information Science, Henan University of Finance and Economics, Zhengzhou 450002, P. R. China
e-mail: zhrui64652002@yahoo.com.cn