J. Appl. Math. & Informatics Vol. **28**(2010), No. 5 - 6, pp. 1101 - 1116
Website: http://www.kcam.biz

# ON THE DEGREE OF APPROXIMATION FOR BIVARIATE LUPAŞ TYPE OPERATORS

NAOKANT DEO

Abstract. The aim of this paper is to give some simultaneous approximation properties as well as differential properties, Voronovskaya type theorem, several asymptotic formulae for the partial derivative and the degree of approximation for two dimensional Lupaş type operators.

AMS Mathematics Subject Classification : 41A25, 41A28, 41A35.
*Key word and phrases* : Lupaş operators, Degree of approximation, Voronovskaya type theorem.

## 1. Introduction

In [8], Lupaş type operators are defined as:

$$(B_n f)(x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n+1}\right), \tag{1}$$

where

$$b_{n,k}(x) = \frac{1}{\beta(k+1,n)}\frac{x^k}{(1+x)^{n+k+1}}, \qquad x \in [0,\infty) \equiv R_0,\ n \in N := \{1,2,...\},$$

and $\beta(v+1,n)$ denotes as:

$$\beta(v+1,n) = \frac{\Gamma(v+1)\Gamma n}{\Gamma(v+n+1)},$$

in polynomial weighted spaces $C_p$, such type operators were examined in [2]. Author [4] has given Durrmeyer variant of these operators as:

$$(D_n f)(x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t) f(t)\,dt, \tag{2}$$

Received November 17, 2009. Revised November 24, 2009. Accepted December 20, 2009.

and studied some direct results of these operators. Now in this paper, we consider bivariate Lupaş type operators as:

$$B_{n,n}(f;x,y) = \frac{1}{n^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}(x)b_{n,l}(y)f\left(\frac{k}{n+1},\frac{l}{n+1}\right) \tag{3}$$

where

$$n \in N,\ (x,y) \in [0,\infty)\times[0,\infty) \equiv R_0^2 \ \text{ and } \ f \in C([0,\infty)\times[0,\infty)).$$

Now we propose Durrmeyer variant of (3) as:

$$Z_n^{[i,j]}(f;x,y) = \frac{1}{n^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}^{(i)}(x)b_{n,l}^{(j)}(y)\int_0^{\infty}\int_0^{\infty} b_{n,k}(s)b_{n,l}(t)f(s,t)dsdt. \tag{4}$$

In particular, if $f^{(0)}(x) = f(x)$ then the meanings of $b_n^{[i,0]}(f;x,y)$ and $b_n^{[0,j]}(f;x,y)$ are clear. By $H[0,\infty)^2$, we denote the class of all measurable functions defined on $[0,\infty)$ satisfying

$$\int_0^{\infty}\int_0^{\infty}\frac{|f(s)|\,|f(t)|}{\{(1+s)(1+t)\}^{n+1}}dsdt < \infty, \ \text{ for some positive integer n.}$$

Similar type operators were studied by several researchers (see e.g. [3], [5], [7], [10], [11], [12]). In the present paper, we study the degree of approximation and Voronovskaya type theorem the for two dimensional Lupaş type operators $B_{n,n}(f,x,y)$ and at the end of this paper, we obtain the properties of simultaneous approximation and several asymptotic formulas for the partial derivative of these operators.

Throughout this paper $\phi(x) = x(1+x)$ and $\mathscr{C}_k(g,h),\ k = 1,2,...,$ will denote positive constants depending only on parameters $g,h$.

## 2.Auxiliary Results

In this section we give some notational convention, definitions and lemmas, which will be needed to prove our main results, given in section 3.

The space $C_p,\ p \in N_0 := \{0,1,2,...\}$, is associated with the weighted function

$$\omega_0(x) := 1,\ \omega_p(x) := (1+x^p)^{-1} \quad \text{if } \ p \geq 1,\ x \in R_0 \tag{5}$$

and consists of all real-valued functions $f$, continuous on $R_0$ and such that $w_p f$ is uniformly continuous and bounded on $R_0$. The norm on $C_p$ is defined by the formula

$$\|f\|_p \equiv \|f(.)\|_p := \sup_{x\in R_0}\omega_p\,|f(x)|\,.$$

For given $p,q \in N_0$, we define the weighted function $\omega_{p,q}$ on $R_0^2$ as

$$\omega_{p,q}(x,y) := \omega_p(x)\omega_q(y), \quad (x,y) \in R_0^2 := R_0 \times R_0, \tag{6}$$

and the weighted space $C_{p,q}$ of all real-valued functions $f$ continuous on $R_0^2$ for which $\omega_{p,q}f$ is uniformly continues and bounded on $R_0^2$. The norm on $C_{p,q}$ is defined by the formula

$$\|f\|_{p,q} \equiv \|f(.,.)\|_{p,q} := \sup_{(x,y)\in R_0^2} \omega_{p,q}(x,y)\,|f(x,y)|\,. \tag{7}$$

The modulus of continuity of $f \in C_{p,q}$ we define as usual by the formula

$$\omega(f, C_{p,q}; t, s) := \sup_{0\le h\le t,\ 0\le \delta\le s} \|\Delta_{h,\delta} f(.,.)\|_{p,q}\,, \quad t, s \ge 0, \tag{8}$$

where $\Delta_{h,\delta}f(x,y) := f(x+h, y+\delta) - f(x,y)$ for $h,\delta \in R_0$ and $(x,y) \in R_0^2$. Let $C_{p,q}^m$ be the set of all the functions $f \in C_{p,q}$ having partial derivatives $\frac{\partial^k f}{\partial x^s \partial y^{k-s}} \in C_{p,q}$, for fixed $m \in N$, $p,q \in N_0$ and $k = 1,2,...,m$.

**Lemma 2.1.** *Suppose $n \in N$ and $x \in R_0$, then it is easily verified from (1)*

$$B_n(1;x) = 1,$$
$$B_n\left((t-x);x\right) = 0,$$
$$B_n\left((t-x)^2;x\right) = \frac{\phi(x)}{n+1}.$$

**Lemma 2.2.** [8] *For all $(x,y) \in R_0^2$ and $n \in N$, we get*

$$B_{n,n}(1;x,y) = 1, \tag{9}$$

*Moreover, if $f \in C_{p,q}$ and $f(x,y) = f_1(x)f_2(y)$ for all $(x,y) \in R_0^2$ and for $n \in N$, then we obtain*

$$B_{n,n}(f(t,v);x,y) = B_n(f_1(t);x)B_n(f_2(v);y). \tag{10}$$

**Lemma 2.3.** [1] *For ever fixed $x_0 \in R_0$ there exists a positive constant $\mathscr{C}_1(x)$ such that*

$$B_n\left((t-x_0)^4;x_0\right) \le \mathscr{C}_1(x_0)(n+1)^{-2}, \quad \text{for } n \in N,$$

*Now for every $p \in N_0$ there exist positive constants $\mathscr{C}_k(p)$, $k = 2,3$, such that*

$$\omega_p(x)B_n\left(\frac{1}{\omega_p(t)};x\right) \le \mathscr{C}_2(p),$$
$$\omega_p(x)B_n\left(\frac{(t-x)^2}{\omega_p(t)};x\right) \le \mathscr{C}_3(p).\frac{\phi(x)}{n+1},$$

*for all $x_0 \in R_0$ and $n \in N$.*

**Lemma 2.4.** *For every fixed $p \in N_0$ there exists positive constants $\mathscr{C}_4(p)$ such that for all $x \in R_0$, $n \in N$, we have*

$$\omega_p(x)\sum_{k=0}^{\infty}\left|\frac{1}{n}\frac{d}{dx}b_{n,k}(x)\right|\left(\omega_p\left(\frac{k}{n+1}\right)\right)^{-1} \le (n+1)\mathscr{C}_4(p) \tag{11}$$

*Proof.* Applying (1) and (5), we get

$$\sum_{k=0}^{\infty}\left|\frac{1}{n}\frac{d}{dx}b_{n,k}(x)\right|\left(\omega_p\left(\frac{k}{n+1}\right)\right)^{-1}$$
$$\leq \frac{1}{n}\frac{(n+1)}{(1+x)}\left\{\sum_{k=0}^{\infty}b_{n,k}(x)\left(\omega_p\left(\frac{k}{n+1}\right)\right)^{-1}+\sum_{k=0}^{\infty}b_{n+1,k}(x)\left(\omega_p\left(\frac{k+1}{n+1}\right)\right)^{-1}\right\}$$
$$\leq (n+1)\left\{B_n\left(\frac{1}{\omega_p(t)};x\right)+B_{n+1}\left(\frac{1}{\omega_p(t)};x\right)\right\},$$

this lead to (11) by above Lemma. This complete the proof of Lemma 2.4. □

**Lemma 2.5.** *For every fixed $p,q\in N_0$ there exist two positive constants $\mathscr{C}_5(p,q)$ and $\mathscr{C}_6(p,q)$, such that*

$$\left\|B_{n,n}\left(\frac{1}{\omega_{p,q}(t,v)};x;.,.\right)\right\|_{p,q}\leq \mathscr{C}_5(p,q),\quad \text{for } n\in N. \tag{12}$$

*Moreover for every $f\in C_{p,q}$ and for all $n\in N$ we have*

$$\|B_{n,n}(f;.,.)\|_{p,q}\leq \mathscr{C}_5(p,q)\,\|f\|_{p,q}\,, \tag{13}$$

$$\left\|\frac{\partial}{\partial x}B_{n,n}(f;x,y)\right\|_{p,q}\leq (n+1)\mathscr{C}_6(p,q)\,\|f\|_{p,q}\,, \tag{14}$$

$$\left\|\frac{\partial}{\partial y}B_{n,n}(f;x,y)\right\|_{p,q}\leq (n+1)\mathscr{C}_6(p,q)\,\|f\|_{p,q}\,. \tag{15}$$

*Thus, $B_{n,n}$ is a linear operators from the space $C_{p,q}$ into $C^1_{p,q}$.*

*Proof.* From (5), (6) and (10) we get for $(x,y)\in R_0^2$ and $n\in N$

$$\omega_{p,q}(x,y)B_{n,n}\left(\frac{1}{\omega_{p,q}(t,v)};x,y\right)=\left(\omega_p(x)B_n\left(\frac{1}{\omega_p(t)};x\right)\right)\left(\omega_q(y)B_n\left(\frac{1}{\omega_q(v)};y\right)\right)$$

Using Lemma 2.3 and from (7), we obtain (12).

Now from (6), (7) and (3) we get for $f\in C_{p,q}$ and for all $n\in N$

$$\|B_{n,n}(f;.,.)\|_{p,q}\leq \|f\|_{p,q}\left\|B_{n,n}\left(\frac{1}{\omega_{p,q}(t,v)};.,.\right)\right\|_{p,q},$$

which by (12) implies (13). Moreover, we obtain from (3)

$$\left|\frac{\partial}{\partial x}B_{n,n}(f;x,y)\right|\leq \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\left|\frac{1}{n}\frac{d}{dx}b_{n,l}(x)\right|\frac{1}{n}b_{n,k}(y)\left|f\left(\frac{l}{n+1},\frac{k}{n+1}\right)\right|,$$

which implies, from (5) to (7) and Lemma 2.3 and Lemma 2.4, we get

$$\omega_{p,q}(x,y)\left|\frac{\partial}{\partial x}B_{n,n}(f;x,y)\right| \leq \|f\|_{p,q}\frac{1}{n}\left\{\omega_p(x)\sum_{l=0}^{\infty}\left|b'_{n,l}(x)\right|\left(\omega_p\left(\frac{l}{n}\right)\right)^{-1}\right\}\omega_q(y)B_n\left(\frac{1}{\omega_q(v)};y\right)$$
$$\leq (n+1)\mathscr{C}_2(p)\mathscr{C}_4(p)\|f\|_{p,q},$$

which leads (14). The proof of (15) is analogous. □

**Lemma 2.6.** [6] *For $m \in N \cup \{0\}$, if*

$$T_{n,m}(x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\left(\frac{k}{n+1} - x\right)^m,$$

*then*

$$(n+1)T_{n,m+1}(x) = x(1+x)\left[T'_{n,m}(x) + mT_{n,m-1}(x)\right].$$

*Consequently, we have*

(i) $T_{n,m}(x)$ *is a polynomial in $x$ of degree $\leq m$.*
(ii) $T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, *where $\lfloor\gamma\rfloor$ denotes the integral part of $\gamma$.*

**Lemma 2.7.** *There exists the polynomials $q_{c,h,r}(x)$ independent of $n$ and $k$ such that*

$$x^r(1+x)^r\frac{d^r}{dx^r}b_{n.k}(x) = \sum_{\substack{2c+h\leq r\\ c,h\geq 0}}(n+1)^c\{k-(n+1)x\}^h q_{c,h,r}(x)b_{n.k}(x). \quad (16)$$

*Proof.* The proof of this lemma follows along the lines of [9, Theorem 1.8.1, p. 26]. □

**Lemma 2.8.** [4] *For $i, r \in N \cup \{0\}$ and $n \in N$, if*

$$U_{n,r,i}(x) = \frac{1}{(n+r)}\sum_{k=0}^{\infty} b_{n+r,k}(x)\int_0^{\infty} b_{n-r,k+r}(s)(s-x)^i dt$$

*then, we have following recurrence relation:*

$$(n-i-r-1)U_{n,r,i+1}(x) = x(1+x)\big[U'_{n,r,i}(x) + 2iU_{n,r,i-1}(x)$$
$$+ (1+2x)(r+i+1)U_{n,r,i}(x)\big]. \quad (17)$$

*where $n > i+r+1$. Consequently,*

(i): *we have*

$$U_{n,r,0}(x) = 1,\ \ U_{n,r,1}(x) = \frac{(1+2x)(1+r)}{(n-r-1)}$$

*and*

$$U_{n,r,2}(x) = \frac{x(1+x)2n}{(n-r-1)(n-r-2)} + \frac{(1+2x)^2(r+1)(r+2)}{(n-r-1)(n-r-2)}$$

(ii): *for all* $x \in [0,\infty)$, *we get*

$$U_{n,r,i}(x) = O\Big(n^{-\left[\frac{i+1}{2}\right]}\Big),$$

*where* $\lfloor \alpha \rfloor$ *stands for the integral part of* $\alpha$.

**Lemma 2.9.** [4] *Suppose that*

$$\begin{aligned} S_{n,r_1,r_2,i,j}(x,y) &= \frac{1}{(n+r_1)(n+r_2)} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n+r_1,k}(x) b_{n+r_2,l}(y) \\ &\quad \cdot \int_0^\infty \int_0^\infty b_{n-r_1,k+r_1}(s) b_{n-r_2,l+r_2}(t)\,(s-x)^i\,(t-y)^j\,dsdt \\ &= U_{n,r_1,i}(x).U_{n,r_2,j}(y), \end{aligned} \tag{18}$$

*then we obtain the following results by Lemma 2.8*

$$\begin{aligned} &S_{n,r_1,r_2,0,0}(x,y) = 1,\ S_{n,r_1,r_2,i,j}(x,y) = O\left(n^{-\left(\left[\frac{i+1}{2}\right]+\left[\frac{j+1}{2}\right]\right)}\right), \\ &S_{n,r_1,r_2,1,0}(x,y) = \frac{(r_1+1)(1+2x)}{(n-r_1-1)},\ S_{n,r_1,r_2,0,1}(x,y) = \frac{(r_2+1)(1+2y)}{(n-r_2-1)}, \\ &S_{n,r_1,r_2,1,1}(x,y) = \frac{(r_1+1)(r_2+1)(1+2x)(1+2y)}{(n-r_1-1)(n-r_2-1)}, \\ &S_{n,r_1,r_2,2,0}(x,y) = \frac{2(n-1)x(1+x)+(r_1+1)(r_1+2)(1+2x)^2}{(n-r_1-1)(n-r_1-2)}, \\ &S_{n,r_1,r_2,0,2}(x,y) = \frac{2(n-1)y(1+y)+(r_2+1)(r_2+2)(1+2y)^2}{(n-r_2-1)(n-r_2-2)}. \end{aligned}$$

**Lemma 2.10.** [4] *If* $f(x,y)$ *exist* $r_1+r_2$ *derivatives on* $[0,\infty)$, *then we obtain*

$$\begin{aligned} Z_n^{[r_1,r_2]}(f;x,y) &= C(n,r_1,r_2) \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n+r_1,k}(s) b_{n+r_2,l}(t) \\ &\quad \cdot \int_0^\infty \int_0^\infty b_{n-r_1,k+r_1}(s) b_{n-r_2,l+r_2}(t) \frac{\partial^{r_1+r_2}}{\partial s^{r_1}\partial t^{r_2}} f(s,t)\,dsdt, \end{aligned}$$

*where*

$$C(n,r_1,r_2) = \frac{(n-r_1-1)!(n-r_2-1)!(n+r_1-1)!(n+r_2-1)!}{\{n!(n-1)!\}^2}.$$

## 3.Main Results

In this section we shall give total four theorems, first two theorems on the degree of approximation of functions by $B_{n,n}$ and the last two theorems are Voronovskaya type theorems.

**Theorem 3.1.** *Suppose that* $f \in C^1_{p,q}$ *with fixed* $p,q \in N_0$, *then there exists a positive constant* $\mathscr{C}_7 = \mathscr{C}_7(p,q)$ *such that for all* $(x,y) \in R^2_+$ *and* $n \in N$

$$\omega_{p,q}(x,y)\left|B_{n,n}(f;x,y)-f(x,y)\right| \leq \mathscr{C}_7\left\{\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\sqrt{\frac{\phi(x)}{n+1}}+\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\sqrt{\frac{\phi(y)}{n+1}}\right\}. \tag{19}$$

*Proof.* Let $(x,y)\in R_+^2$ be a fixed point then for $f\in C_{p,q}^1$, we have

$$f(t,v)-f(x,y)=\int_x^t \frac{\partial}{\partial u}f(u,v)du+\int_y^v \frac{\partial}{\partial z}f(x,z)dz, \quad (t,v)\in R_0^2. \tag{20}$$

From (9), we obtain

$$B_{n,n}\left(f(t,v);x,y\right)-f(x,y)=B_{n,n}\left(\int_x^t \frac{\partial}{\partial u}f(u,v)du;x,y\right)+B_{n,n}\left(\int_y^v \frac{\partial}{\partial z}f(x,z)dz;x,y\right) \tag{21}$$

From (5) to (7), we get

$$\begin{aligned}\left|\int_x^t \frac{\partial}{\partial u}f(u,v)du\right| &\leq \left\|\frac{\partial f}{\partial x}\right\|_{p,q}\left|\int_x^t \frac{du}{\omega_{p,q}(u,v)}\right|\\ &\leq \left\|\frac{\partial f}{\partial x}\right\|_{p,q}\left(\frac{1}{\omega_{p,q}(t,v)}+\frac{1}{\omega_{p,q}(x,v)}\right)|t-x|,\end{aligned}$$

and similar manner, we obtain

$$\left|\int_y^v \frac{\partial}{\partial x}f(x,z)dz\right| \leq \left\|\frac{\partial f}{\partial y}\right\|_{p,q}\left(\frac{1}{\omega_{p,q}(x,v)}+\frac{1}{\omega_{p,q}(x,y)}\right)|v-y|.$$

Using these inequalities and from (10), we have for all $m,n\in N$

$$\begin{aligned}&\omega_{p,q}(x,y)\left|B_{n,n}\left(\int_x^t \frac{\partial}{\partial u}f(u,v)du;x,y\right)\right|\\ &\leq \omega_{p,q}(x,y)B_{n,n}\left(\left|\int_x^t \frac{\partial}{\partial u}f(u,v)du\right|;x,y\right)\\ &\leq \left\|\frac{\partial f}{\partial x}\right\|_{p,q}\omega_{p,q}(x,y)\left\{B_{n,n}\left(\frac{|t-x|}{\omega_{p,q}(t,v)};x,y\right)+B_{n,n}\left(\frac{|t-x|}{\omega_{p,q}(x,v)};x,y\right)\right\}\\ &=\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\omega_q(y)B_n\left(\frac{1}{\omega_q(v)};y\right)\left\{\omega_p(x)B_n\left(\frac{|t-x|}{\omega_p(t)};x\right)+B_n(|t-x|;x)\right\},\end{aligned}$$

again by similar manner, we obtain the following

$$\begin{aligned}&\omega_{p,q}(x,y)\left|B_{n,n}\left(\int_y^v \frac{\partial}{\partial z}f(x,z)dz;x,y\right)\right|\\ &=\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\omega_p(x)B_n\left(\frac{1}{\omega_p(z)};x\right)\left\{\omega_q(y)B_n\left(\frac{|v-y|}{\omega_q(v)};y\right)+B_n(|v-y|;y)\right\},\end{aligned}$$

Applying Hölder inequality, Lemma 2.1 and Lemma 2.3, we have

$$B_n(|t-x|\,;x) \leq \left\{B_n\left((t-x)^2;x\right)B_n(1;x)\right\}^{1/2} \leq \sqrt{\frac{\phi(x)}{n+1}},$$

and

$$\omega_p(x)B_n\left(\frac{|t-x|}{\omega_p(t)};x\right) \leq \omega_p(x)\left\{B_n\left(\frac{(t-x)^2}{\omega_p(t)};x\right)B_n\left(\frac{1}{\omega_p(t)};x\right)\right\}^{1/2} \leq \mathscr{C}_8(p)\sqrt{\frac{\phi(x)}{n+1}}.$$

Similarly, for $n \in N$, we get

$$B_n(|v-y|\,;y) \leq \sqrt{\frac{\phi(y)}{n+1}},$$

$$\omega_p(y)B_n\left(\frac{|v-y|}{\omega_q(v)};y\right) \leq \mathscr{C}_9(q)\sqrt{\frac{\phi(y)}{n+1}},$$

Combining these estimations, we derive from (21)

$$\omega_{p,q}(x,y)\left|B_{n,n}\left(f(t,v);x,y\right)-f(x,y)\right| \leq \mathscr{C}_{10}\left[\left\|\frac{\partial f}{\partial x}\right\|_{p,q}\sqrt{\frac{\phi(x)}{n+1}}+\left\|\frac{\partial f}{\partial y}\right\|_{p,q}\sqrt{\frac{\phi(y)}{n+1}}\right],$$

for all $m,n \in N$, where $\mathscr{C}_{10} = \mathscr{C}_{10}(p,q) = \text{constant} > 0$. This ends the proof of (19). □

**Theorem 3.2.** *Suppose that $f \in C_{p,q}$, $p,q \in N_0$, then there exists a positive constant $\mathscr{C}_{11} = \mathscr{C}_{11}(p,q)$ such that for all $(x,y) \in R_+^2$ and $n \in N$*

$$\omega_{p,q}(x,y)\left|B_{n,n}(f;x,y)-f(x,y)\right| \leq \mathscr{C}_{11}\omega\left(f,C_{p,q};\sqrt{\frac{\phi(x)}{n+1}},\sqrt{\frac{\phi(y)}{n+1}}\right). \quad (22)$$

*Proof.* Applying the Steklov function $f_{h,\delta}$ for $f \in C_{p,q}$

$$f_{h,\delta}(x,y) := \frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+z)dz, \quad (x,y) \in R_0^2\; h,\delta > 0. \quad (23)$$

Using (23), we get

$$\begin{aligned}
f_{h,\delta}(x,y)-f(x,y) &= \frac{1}{h\delta}\int_0^h du\int_0^\delta \Delta_{u,z}f(x,y)dz,\\
\frac{\partial}{\partial x}f_{h,\delta}(x,y) &= \frac{1}{h\delta}\int_0^\delta(\Delta_{h,z}f(x,y)-\Delta_{0,z}f(x,y))dz,\\
\frac{\partial}{\partial y}f_{h,\delta}(x,y) &= \frac{1}{h\delta}\int_0^h(\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y))du.
\end{aligned}$$

This implies that $f_{h,\delta} \in C_{p,q}^1$ for $f \in C_{p,q}$ and $h,\delta > 0$. Moreover, from (7) and (8), we obtain

$$\|f_{h,\delta}-f\|_{p,q} \leq \omega\left(f,C_{p,q};h,\delta\right), \quad (24)$$

$$\left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{p,q} \leq 2h^{-1} \omega \left(f, C_{p,q}; h, \delta\right), \tag{25}$$

$$\left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{p,q} \leq \delta^{-1} \omega \left(f, C_{p,q}; h, \delta\right), \tag{26}$$

for all $h, \delta > 0$. Thus

$$\begin{aligned} &\omega_{p,q}(x,y) \left| B_{n,n}(f(t,v);x,y) - f(x,y) \right| \\ &\leq \omega_{p,q}(x,y) \big\{ \left| B_{n,n}\left(f(t,v) - f_{h,\delta}(t,v);x,y\right) \right| + \left| B_{n,n}\left(f_{h,\delta}(t,v);x,y\right) - f_{h,\delta}(x,y) \right| \\ &+ \left| f_{h,\delta}(x,y) - f(x,y) \right| \big\} := T_1 + T_2 + T_3 \end{aligned}$$

From (7), (13) and (24), we get

$$T_1 \leq \left\| B_{n,n}\left(f - f_{h,\delta}; .,.\right) \right\|_{p,q} \leq \mathscr{C}_5 \left\| f - f_{h,\delta} \right\|_{p,q} \leq \mathscr{C}_5 \ \omega\left(f, C_{p,q}; h, \delta\right),$$

$$T_3 \leq \omega\left(f, C_{p,q}; h, \delta\right).$$

Using Theorem 3.1 and (25) and (26), we obtain

$$\begin{aligned} T_2 &\leq \mathscr{C}_7 \left\{ \left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{p,q} \sqrt{\frac{\phi(x)}{n+1}} + \left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{p,q} \sqrt{\frac{\phi(y)}{n+1}} \right\} \\ &\leq 2\mathscr{C}_7 \ \omega\left(f, C_{p,q}; h, \delta\right) \left\{ h^{-1} \sqrt{\frac{\phi(x)}{n+1}} + \delta^{-1} \sqrt{\frac{\phi(y)}{n+1}} \right\}. \end{aligned}$$

From the above we deduce that there exists a positive constant $\mathscr{C}_{12} \equiv \mathscr{C}_{12}(p,q)$ such that

$$\omega_{p,q}(x,y) \left| B_{n,n}(f;x,y) - f(x,y) \right| \leq \mathscr{C}_{12} \ \omega\left(f, C_{p,q}; h, \delta\right) \left\{ 1 + h^{-1} \sqrt{\frac{\phi(x)}{n+1}} + \delta^{-1} \sqrt{\frac{\phi(y)}{n+1}} \right\}, \tag{27}$$

for $m, n \in N$ and $h, \delta > 0$. Now, for $m, n \in N$ setting $h = \sqrt{\frac{\phi(x)}{n+1}}$ and $\delta = \sqrt{\frac{\phi(y)}{n+1}}$ to (27), we obtain (22). □

**Corollary 3.3.** *Suppose that $f \in C_{p,q},\ p, q \in N_0$, then for every $x, y \in R_+^2$ we get*

$$\lim_{n \to \infty} B_{n,n}(f;x,y) = f(x,y). \tag{28}$$

*Moreover, the assertion hold uniformly on every rectangle $0 \leq x \leq a, 0 \leq y \leq b$.*

**Theorem 3.4.** *Suppose that $f \in C_{p,q}^2$ with some $p, q \in N_0$, then for every $(x,y) \in R_+^2$*

$$\lim_{n \to \infty} n \left\{ B_{n,n}(f;x,y) - f(x,y) \right\} = \frac{\phi(x)}{2} \frac{\partial^2}{\partial x^2} f(x,y) + \frac{\phi(y)}{2} \frac{\partial^2}{\partial y^2} f(x,y)(x,y). \tag{29}$$

*Proof.* Let $(x,y)$ be a fixed point in $R_+^2$. Applying the Taylor formula for $f \in C_{p,q}^2$, we get

$$\begin{aligned} f(t,v) &= f(x,y) + \frac{\partial}{\partial x} f(x,y)(t-x) + \frac{\partial}{\partial y} f(x,y)(v-y) \\ &+ \frac{1}{2}\left\{ \frac{\partial^2}{\partial x^2} f(x,y)(t-x)^2 + 2\frac{\partial^2}{\partial x \partial y} f(x,y)(t-x)(v-y) + \frac{\partial^2}{\partial y^2} f(x,y)(v-y)^2 \right\} \\ &+ \chi(t,v;x,y)\sqrt{(t-x)^4+(v-y)^4}, \end{aligned}$$

for $(t,v) \in R_0^2$, where $\chi(.,.;x,y) \equiv \chi(.,.) \in C_{p,q}$ and $\chi(x,y) = 0$. Hence, we get

$$\begin{aligned} B_{n,n}\left(f(t,v);x,y\right) &= f(x,y) + \frac{\partial}{\partial x} f(x,y) B_n(t-x;x) + \frac{\partial}{\partial y} f(x,y) B_n(v-y;y) \\ &+ \frac{1}{2}\left\{ \frac{\partial^2}{\partial x^2} f(x,y) B_n\left((t-x)^2;x\right) + 2\frac{\partial^2}{\partial x \partial y} f(x,y) B_n(t-x;x) B_n(v-y;y) \right. \\ &\left. + \frac{\partial^2}{\partial y^2} f(x,y) B_n\left((v-y)^2;y\right) \right\} + B_{n,n}\left(\chi(t,v)\sqrt{(t-x)^4+(v-y)^4};x,y\right). \end{aligned} \tag{30}$$

Applying the Hölder inequality, we obtain

$$\begin{aligned} &\left| B_{n,n}\left(\chi(t,v)\sqrt{(t-x)^4+(v-y)^4};x,y\right)\right| \\ &\leq \left\{B_{n,n}\left(\chi^2(t,v);x,y\right)\right\}^{1/2}\left\{B_{n,n}\left((t-x)^4+(v-y)^4;x,y\right)\right\}^{1/2} \\ &= \left\{B_{n,n}\left(\chi^2(t,v);x,y\right)\right\}^{1/2}\left\{B_n\left((t-x)^4;x\right)+B_n\left((v-y)^4;y\right)\right\}^{1/2}. \end{aligned} \tag{31}$$

From Corollary 3.3 it follows that

$$\lim_{n\to\infty} B_{n,n}\left(\chi^2(t,v);x,y\right) = \chi^2(x,y) = 0. \tag{32}$$

By Lemma 2.3 and (32), we get from (31)

$$\lim_{n\to\infty} nB_{n,n}\left(\chi(t,v)\sqrt{(t-x)^4+(v-y)^4};x,y\right) = 0. \tag{33}$$

By using Lemma 2.1 and (33), we reach (29), from (30). $\square$

**Theorem 3.5.** *Suppose that $f \in C_{p,q}^1$ with some $p,q \in N_0$, then for every $(x,y) \in R_+^2$*

$$\lim_{n\to\infty} \frac{\partial}{\partial x} B_{n,n}(f;x,y) = \frac{\partial}{\partial x} f(x,y), \tag{34}$$

$$\lim_{n\to\infty} \frac{\partial}{\partial y} B_{n,n}(f;x,y) = \frac{\partial}{\partial y} f(x,y). \tag{35}$$

*Proof.* Let $(x,y) \in R_+^2$ be a fixed point and $n \in N$, then from (1) and (3), we get

$$\frac{\partial}{\partial x} B_{n,n}(f;x,y) = -\frac{1}{n^2}\left(\frac{n+1}{1+x}\right)\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} b_{n,l}(x) b_{n,k}(y) f\left(\frac{l}{n+1},\frac{k}{n+1}\right)$$

$$+\frac{1}{n^2}\{x(1+x)\}^{-1}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} l b_{n,l}(x) b_{n,k}(y) f\left(\frac{l}{n+1},\frac{k}{n+1}\right)$$
$$=-\left(\frac{n+1}{1+x}\right)B_{n,n}(f(t,v);x,y)+\frac{(n+1)}{x(1+x)}B_{n,n}(tf(t,v);x,y).$$

Applying the Taylor formula for $f \in C^1_{p,q}$, we get

$$f(t,v)=f(x,y)+\frac{\partial}{\partial x}f(x,y)(t-x)+\frac{\partial}{\partial y}f(x,y)(v-y)$$
$$+\zeta(t,v;x,y)\sqrt{(t-x)^2+(v-y)^2}, \quad \text{for } (t,v)\in R_0^2, \tag{36}$$

where $\zeta(.,.;x,y)\equiv\zeta(.,.)\in C_{p,q}$ and $\zeta(x,y)=0$. From (9), (10) and Lemma 2.1 we obtain

$$\frac{\partial}{\partial x}B_{n,n}(f(t,v);x,y)=-\left(\frac{n+1}{1+x}\right)\left\{f(x,y)+\frac{\partial}{\partial x}f(x,y)B_n(t-x;x)+\frac{\partial}{\partial y}f(x,y)B_n(v-y;y)\right.$$
$$\left.+B_{n,n}\left(\zeta(t,v)\sqrt{(t-x)^2+(v-y)^2};x,y\right)\right\}$$
$$+\left\{\frac{n+1}{x(1+x)}\right\}\left\{f(x,y)B_n(t;x)+\frac{\partial}{\partial x}f(x,y)B_n(t(t-x;x))\right.$$
$$+\frac{\partial}{\partial y}f(x,y)B_n(t;x)B_n(v-y;y)$$
$$\left.+B_{n,n}\left(t\zeta(t,v)\sqrt{(t-x)^2+(v-y)^2};x,y\right)\right\}, \quad n\in N.$$

From Lemma 2.1 we get for $x\in R_0$ and $n\in N$

$$B_n(t;x)=x,$$
$$B_n(t(t-x);x)=B_n\left((t-x)^2;x\right)+xB_n(t-x;x)=\frac{x(1+x)}{n+1}.$$

Consequently, we get

$$\frac{\partial}{\partial x}B_{n,n}(f(t,v);x,y)$$
$$=\frac{\partial}{\partial x}f(x,y)+\frac{n+1}{x(1+x)}B_{n,n}\left(\zeta(t,v)(t-x)\sqrt{(t-x)^2+(v-y)^2};x,y\right). \tag{37}$$

Applying the Hölder inequality and from Lemma 2.1, Lemma 2.2 and Lemma 2.3, we get

$$\left|B_{n,n}\left(\zeta(t,v)(t-x)\sqrt{(t-x)^2+(v-y)^2};x,y\right)\right|$$
$$\le\left\{B_{n,n}\left(\zeta^2(t,v);x,y\right)\right\}^{\frac12}\left\{B_n\left((t-x)^4;x\right)+B_n\left((t-x)^2;x\right)B_n\left((v-y)^2;y\right)\right\}^{\frac12}$$
$$\le\mathscr{C}_{13}(x,y)(n+1)^{-1}\left\{B_{n,n}\left(\zeta^2(t,v);x,y\right)\right\}^{\frac12}$$

Corollary 3.3 yields

$$\lim_{n\to\infty} nB_{n,n}\left(\zeta^2(t,v);x,y\right)=\chi^2(x,y)=0.$$

and therefore

$$\lim_{n\to\infty} nB_{n,n}\left(\zeta(t,v)(t-x)\sqrt{(t-x)^2+(v-y)^2};x,y\right)=0. \tag{38}$$

By (37) and (38), we leads to (34). The proof of (35) is identical. □

**Theorem 3.6.** *Suppose $f\in H[0,\infty)^2$ and is bounded on every finite subinterval of $[0,\infty)$. If $f^{(r+2)}$ exists at a fixed point $x\in[0,\infty)$ and $\left|\frac{\partial^{r+2}}{\partial x^j\partial y^{r+2-j}}f(x,y)\right|\le \mu x^\rho y^\eta$, $(x\to\infty,\ y\to\infty)$; $j=1,...,r+2$ for some $\rho,\eta\ge 0$, then we get*

$$\begin{aligned}\lim_{n\to\infty}\left[Z_n^{*[r,0]}(f;x,y)-\frac{\partial^r}{\partial x^r}f(x,y)\right]&=(1+2y)\frac{\partial^{r+1}}{\partial x^r\partial y}f(x,y)+(1+2x)(r+1)\frac{\partial^{r+1}}{\partial x^{r+1}}f(x,y)\\&\quad+y(1+y)\frac{\partial^{r+2}}{\partial x^r\partial y^2}f(x,y)+x(1+x)\frac{\partial^{r+2}}{\partial x^{r+2}}f(x,y)\end{aligned} \tag{39}$$

*and*

$$\begin{aligned}\lim_{n\to\infty}\left[Z_n^{*[0,r]}(f;x,y)-\frac{\partial^r}{\partial y^r}f(x,y)\right]&=(1+2x)\frac{\partial^{r+1}}{\partial y^r\partial x}f(x,y)+(1+2y)(r+1)\frac{\partial^{r+1}}{\partial y^{r+1}}f(x,y)\\&\quad+x(1+x)\frac{\partial^{r+2}}{\partial y^r\partial x^2}f(x,y)+y(1+y)\frac{\partial^{r+2}}{\partial y^{r+2}}f(x,y).\end{aligned} \tag{40}$$

*Proof.* Using Taylor's expansion of $f(s,t)$ as in [12], we have

$$f(s,t)=\sum_{d=0}^{r+2}\sum_{i+j=d}\frac{1}{i!j!}\left(\frac{\partial^d}{\partial x^i\partial y^j}f(x,y)\right)(s-x)^i(t-y)^j+\sum_{i+j=r+2}\varepsilon(s,t,x,y)(s-x)^i(t-y)^j.$$

where $\varepsilon(s,t,x,y)\to 0$ as $s\to x$, $t\to y$ and $\varepsilon(s,t,x,y)\le\mu(s-x)^\rho(t-y)^\eta$ as $s\to\infty$ , $x\to\infty$ for some $\rho,\eta>0$ then

$$\begin{aligned}n\left[Z_n^{*[r,0]}(f;x,y)-\frac{\partial^r}{\partial x^r}f(x,y)\right]&=n\sum_{d=0}^{r+2}\sum_{i+j=d}\frac{1}{i!j!}\left(\frac{\partial^d}{\partial x^i\partial y^j}f(x,y)\right)Z_n^{*[r,0]}\left((s-x)^i(t-y)^j;x,y\right)\\&\quad+n\sum_{i+j=r+2}Z_n^{*[r,0]}\left(\varepsilon(s,t,x,y)(s-x)^i(t-y)^j;x,y\right)-n\frac{\partial^r}{\partial x^r}f(x,y)\\&=Q_1+Q_2-n\frac{\partial^r}{\partial x^r}f(x,y).\end{aligned}$$

From Lemma 2.10, we get

$$\begin{aligned}Q_1&=n\sum_{d=0}^{r+2}\sum_{i+j=d}\frac{1}{i!j!}\frac{\partial^d}{\partial x^i\partial y^j}f(x,y)\frac{1}{n(n+r)}\sum_{k=0}^\infty\sum_{l=0}^\infty b_{n+r,k}(x)b_{n,l}(y)\\&\qquad\cdot\int_0^\infty\int_0^\infty b_{n-r,k+r}(s)b_{n,l}(t)\frac{\partial^r}{\partial s^r}\left((s-x)^i(t-y)^j\right)dsdt\\&=\frac{n}{r!}\frac{\partial^r}{\partial x^r}f(x,y)\frac{1}{n(n+r)}\sum_{k=0}^\infty\sum_{l=0}^\infty b_{n+r,k}(x)b_{n,l}(y)\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_0^\infty \int_0^\infty b_{n-r,k+r}(s) b_{n,l}(t) r! dsdt \\
&+ \frac{n}{r!1!} \frac{\partial^{r+1}}{\partial x^r \partial y} f(x,y) \frac{1}{n(n+r)} \sum_{k=0}^\infty \sum_{l=0}^\infty b_{n+r,k}(x) b_{n,l}(y) \\
&\quad \cdot \int_0^\infty \int_0^\infty b_{n-r,k+r}(s) b_{n,l}(t) r!(t-y) dsdt \\
&+ \frac{n}{(r+1)!} \frac{\partial^{r+1}}{\partial x^{r+1}} f(x,y) \frac{1}{n(n+r)} \sum_{k=0}^\infty \sum_{l=0}^\infty b_{n+r,k}(x) b_{n,l}(y) \\
&\quad \cdot \int_0^\infty \int_0^\infty b_{n-r,k+r}(s) b_{n,l}(t) \frac{\partial^r}{\partial s^r} (s-x)^{r+1} dsdt \\
&+ \frac{n}{r!2!} \frac{\partial^{r+2}}{\partial x^r \partial y^2} f(x,y) \frac{1}{n(n+r)} \sum_{k=0}^\infty \sum_{l=0}^\infty b_{n+r,k}(x) b_{n,l}(y) \\
&\quad \cdot \int_0^\infty \int_0^\infty b_{n-r,k+r}(s) b_{n,l}(t) r!(t-y)^2 dsdt \\
&+ \frac{n}{(r+1)!} \frac{\partial^{r+2}}{\partial x^{r+1} \partial y} f(x,y) \frac{1}{n(n+r)} \sum_{k=0}^\infty \sum_{l=0}^\infty b_{n+r,k}(x) b_{n,l}(y) \\
&\quad \cdot \int_0^\infty \int_0^\infty b_{n-r,k+r}(s) b_{n,l}(t) \frac{\partial^r}{\partial s^r} \left\{ (s-x)^{r+1}(t-y) \right\} dsdt \\
&+ \frac{n}{(r+2)!} \frac{\partial^{r+2}}{\partial x^{r+2}} f(x,y) \frac{1}{n(n+r)} \sum_{k=0}^\infty \sum_{l=0}^\infty b_{n+r,k}(x) b_{n,l}(y) \\
&\quad \cdot \int_0^\infty \int_0^\infty b_{n-r,k+r}(s) b_{n,l}(t) \frac{\partial^r}{\partial s^r} (s-x)^{r+2} dsdt \\
&= n \frac{\partial^r}{\partial x^r} f(x,y) S_{n,r,0,0,0}(x,y) + n \frac{\partial^{r+1}}{\partial x^r \partial y} f(x,y) S_{n,r,0,0,1}(x,y) \\
&\quad + n \frac{\partial^{r+1}}{\partial x^{r+1}} f(x,y) S_{n,r,0,1,0}(x,y) + \frac{n}{2} \frac{\partial^{r+2}}{\partial x^r \partial y^2} f(x,y) S_{n,r,0,0,2}(x,y) \\
&\quad + n \frac{\partial^{r+2}}{\partial x^{r+1} \partial y} f(x,y) S_{n,r,0,1,1}(x,y) + \frac{n}{2} \frac{\partial^{r+2}}{\partial x^{r+2}} f(x,y) S_{n,r,0,2,0}(x,y) \\
&= n \frac{\partial^r}{\partial x^r} f(x,y) + \frac{n(1+2y)}{(n-1)} \frac{\partial^{r+1}}{\partial x^r \partial y} f(x,y) + \frac{n(r+1)(1+2x)}{(n-r-1)} \frac{\partial^{r+1}}{\partial x^{r+1}} f(x,y) \\
&\quad + \frac{n\{(n-1)y(1+y) + (1+2y)^2\}}{(n-1)(n-2)} \frac{\partial^{r+2}}{\partial x^r \partial y^2} f(x,y) \\
&\quad + \frac{n(r+1)(1+2x)(1+2y)}{(n-1)(n-r-1)} \frac{\partial^{r+2}}{\partial x^{r+1} \partial y} f(x,y) \\
&\quad + \frac{n\{2(n-1)x(1+x) + (r+1)(r+2)(1+2x)^2\}}{2(n-r-1)(n-r-2)} \frac{\partial^{r+2}}{\partial x^{r+2}} f(x,y), \quad \text{(by Lemma 2.9).}
\end{aligned}
$$

In order to prove the theorem, it is sufficient to show that

$$
E_n \cong x^r(1+x)^r Q_2 = \frac{1}{n} \sum_{i+j=r+2} \sum_{k=0}^\infty \sum_{l=0}^\infty x^r(1+x)^r b_{n,k}^{(r)}(x) b_{n,l}(y)
$$

$$.\int_0^\infty\int_0^\infty b_{n,k}(s)b_{n,l}(t)\varepsilon(s,t,x,y)(s-x)^i(t-y)^j dsdt \to 0 \text{ as } (n\to\infty).$$

Using Lemma 2.7, we get

$$|E_n| \le \frac{1}{n}\sum_{i+j=r+2}\sum_{k=0}^\infty\sum_{l=0}^\infty\sum_{2c+h\le r}(n+1)^c\Big|k-(n+1)x\Big|^h|q_{c,h,r}(x)|b_{n,k}(x)b_{n,l}(y)$$
$$.\int_0^\infty\int_0^\infty b_{n,k}(s)p_{n,l}(t)|\varepsilon(s,t,x,y)||s-x|^i|t-y|^j dsdt$$
$$\le n\sum_{\substack{2c+h\le r\\ c,h\ge 0}}\sup|q_{c,h,r}(x)|\sum_{i+j=r+2}\sum_{2c+h\le r}(n+1)^c\sum_{k=0}^\infty\sum_{l=0}^\infty\left((b_{n,k}(x)b_{n,l}(y))^{\frac12}\Big|k-(n+1)x\Big|^h\right)$$

$$.(b_{n,k}(x)b_{n,l}(y))^{\frac12}\frac{1}{n^2}\int_0^\infty\int_0^\infty b_{n,k}(s)b_{n,l}(t)|\varepsilon(s,t,x,y)||s-x|^i|t-y|^j dsdt$$
$$\le n\sum_{\substack{2c+h\le r\\ c,h\ge 0}}\sup|q_{c,h,r}(x)|.\sum_{i+j=r+2}\sum_{2c+h\le r}(n+1)^c\left[\sum_{k=0}^\infty\sum_{l=0}^\infty b_{n,k}(x)b_{n,l}(y)\{k-(n+1)x\}^{2h}\right]^{\frac12}$$
$$.\left[\sum_{k=0}^\infty\sum_{l=0}^\infty b_{n,k}(x)b_{n,l}(y)\left\{\frac{1}{n^2}\int_0^\infty\int_0^\infty b_{n,k}(s)b_{n,l}(t)|\varepsilon(s,t,x,y)||s-x|^i|t-y|^j dsdt\right\}^2\right]^{\frac12}.$$

From Lemma 2.6, we get

$$\begin{aligned}\sum_{k=0}^\infty\sum_{l=0}^\infty b_{n,k}(x)b_{n,l}(y)\{k-(n+1)x\}^{2h} &= (n+1)^{2h}\sum_{k=0}^\infty b_{n,k}(x)\left(\frac{k}{n+1}-x\right)^{2h}\\ &= (n+1)^{2h}O\left(n^{-\left[\frac{2h+1}{2}\right]}\right) = (n+1)^{2h}O\left(n^{-h}\right)\\ &= n^h\left(1+\frac1n\right)^{2h}O(1)\end{aligned}$$

and let

$$\tau_n = \sum_{k=0}^\infty\sum_{l=0}^\infty b_{n,k}(x)b_{n,l}(y)\left[\frac{1}{n^2}\int_0^\infty\int_0^\infty b_{n,k}(s)b_{n,l}(t)|\varepsilon(s,t,x,y)(s-x)^i(t-y)^j|dsdt\right]^2.$$

Therefore, we obtain

$$|E_n| \le n\sum_{\substack{2c+h\le r\\ c,h\ge 0}}\sup|q_{c,h,r}(x)|.\sum_{i+j=r+2}\sum_{2c+h\le r}(n+1)^c\left(n^h\left(1+\frac1n\right)^{2h}O(1)\right)^{\frac12}(\tau_n)^{\frac12}.$$

Now

$$\begin{aligned}&\left[\frac{1}{n^2}\int_0^\infty\int_0^\infty b_{n,k}(s)b_{n,l}(t)|\varepsilon(s,t,x,y)(s-x)^i(t-y)^j|dsdt\right]^2\\ &\le \frac{1}{n^2}\int_0^\infty\int_0^\infty b_{n,k}(s)b_{n,l}(t)dsdt.\frac{1}{n^2}\int_0^\infty\int_0^\infty b_{n,k}(s)b_{n,l}(t)\varepsilon^2(s,t,x,y)(s-x)^{2i}(t-y)^{2j}dsdt\\ &= \frac{1}{n^2}\int_0^\infty\int_0^\infty b_{n,k}(s)b_{n,l}(t)\varepsilon^2(s,t,x,y)(s-x)^{2i}(t-y)^{2j}dsdt\\ &= \frac{1}{n^2}\left[\int_{(s-x)^2+(t-y)^2\le\delta^2}+\int_{(s-x)^2+(t-y)^2>\delta^2}\right]b_{n,k}(s)b_{n,l}(t)\varepsilon^2(s,t,x,y)(s-x)^{2i}(t-y)^{2j}dsdt.\end{aligned}$$

For a given $\varepsilon>0$, there exists a $\delta>0$ such that $|\varepsilon(s,t,x,y)|<\varepsilon$ whenever $(s-x)^2+(t-y)^2\le\delta^2$. For $(s-x)^2+(t-y)^2>\delta^2$, we obtain $|\varepsilon(s,t,x,y)|<$

$K(s-x)^{\rho}(t-y)^{\eta}$.

$$\tau_n \leq \varepsilon^2 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}(x)b_{n,l}(y)\frac{1}{n^2}\int_0^{\infty}\int_0^{\infty} b_{n,k}(s)b_{n,l}(t)(s-x)^{2i}(t-y)^{2j}dsdt$$

$$+K\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}(x)b_{n,l}(y)\frac{1}{n^2}\int_{(s-x)^2+(t-x)^2>\delta^2}\frac{(s-x)^2+(t-y)^2}{\delta^2}$$

$$.(s-x)^{2(i+\rho)}(t-y)^{2(j+\eta)}b_{n,k}(s)b_{n,l}(t)dsdt$$

$$=\varepsilon^2 O\left(n^{-\left(\left[\frac{2i+1}{2}\right]+\left[\frac{2j+1}{2}\right]\right)}\right)+\frac{1}{\delta^2}O\left(n^{-\left(\left[\frac{2i+2\rho+2+1}{2}\right]+\left[\frac{2j+2\eta+1}{2}\right]\right)}\right)$$

$$+\frac{1}{\delta^2}O\left(n^{-\left(\left[\frac{2i+2\rho+1}{2}\right]+\left[\frac{2j+2\eta+2+1}{2}\right]\right)}\right)$$

$$=\varepsilon^2 O\left(n^{-(i+j)}\right)+\frac{1}{\delta^2}O\left(n^{-(i+j)}n^{-\left(\left[\frac{2\rho+1}{2}\right]+1+\left[\frac{2\eta+1}{2}\right]\right)}\right)$$

$$+\frac{1}{\delta^2}O\left(n^{-(i+j)}n^{-\left(\left[\frac{2\rho+1}{2}\right]+1+\left[\frac{2\eta+1}{2}\right]\right)}\right)$$

$$=O\left(n^{-(i+j)}\right)\left(\varepsilon^2+\frac{2}{\delta^2}n^{-W}\right),\quad \text{where } W=\left[\frac{2\rho+1}{2}\right]+1+\left[\frac{2\eta+1}{2}\right]>0.$$

Thus, we get

$$|E_n| \leq n\sum_{\substack{2c+h\leq r\\ c,h\geq 0}}\sup|q_{c,h,r}(x)|\sum_{i+j=r+2}\sum_{2c+h\leq r}(n+1)^c\left(n^h\left(1+\frac{1}{n}\right)^{2h}O(1)\right)^{\frac{1}{2}}$$

$$.\left[O\left(n^{-(i+j)}\right)\left(\varepsilon^2+\frac{2}{\delta^2}n^{-W}\right)\right]^{\frac{1}{2}}$$

$$\leq n\sum_{\substack{2c+h\leq r\\ c,h\geq 0}}\sup|q_{c,h,r}(x)|\sum_{2c+h\leq r} n^{\frac{2c+h}{2}}\left(1+\frac{1}{n}\right)^{c+h}O(1)$$

$$.\sum_{i+j=r+2}\left[O\left(n^{-(i+j)}\right)\left(\varepsilon^2+\frac{2}{\delta^2}n^{-W}\right)\right]^{\frac{1}{2}}$$

$$=O(1)n^{\frac{r+2}{2}}n^{-\frac{r+2}{2}}\left(1+\frac{1}{n}\right)^{r-c}\left(\varepsilon^2+\frac{2}{\delta^2}n^{-W}\right)^{\frac{1}{2}}$$

$$=O(1)\left(1+\frac{1}{n}\right)^{r-c}\left(\varepsilon^2+\frac{2}{\delta^2}n^{-W}\right)^{\frac{1}{2}}\to 0,\quad \text{as } n\to\infty.$$

This completes the proof of (39) and the proof of (40) is identical. $\square$

## Acknowledgment

The author is extremely thankful to the referee for his valuable comments and suggestions, leading to a better presentation of the paper. The author is also thankful to Professor Chin-Hong Park for special attention.

## References

1. V. A. Baskakov, An example of a sequence of linear positive operators in the space of continuous functions, *DAN, **113(1957)**,* 249-251 (in Russian).
2. M. Becker, *Global approximation theorems for Szász-Mirakyan and Baskakov operators in polynomial weight spaces,* Indiana Univ. Math. J., **27**(1) (1978), 127-142.
3. F. Cao and C. Ding and Z. Xu, *On multivariate Baskakov operators*, J. Math. Anal. Appl. **307** (2005), 274-291.
4. N. Deo, *Direct result on the Durrmeyer variant of Beta operators,* Southeast Asian Bull. Math., **32**(2008), 283-290.
5. F. Guo, *On convergence rate of approximation for two-dimensional Baskakov operators*, Anal. Theory Appl. **19** (2003), 273-279.
6. V. Gupta and A. Ahmad, *Simultaneous approximation by modified Lupaş operators,* Instanbul Uni. Fen. Fak. Mat. Der., **54**(1995), 11-22.
7. M. Gurdek, L. Rempulska and M. Skorupka, *The Baskakov operators for functions of two variables,* Collect. Math., **50**(3) (1999), 289-302.
8. V. Gupta and N. Deo, *On the rate of convergence for bivariate Lupaş operators,* General Math. **13**(3) (2005), 107-114.
9. G. G. Lorentz, *Bernstein Polynomials,* University of Toronto Press, Toronto, 1953.
10. L. Rempulska and M. Skorupka, *On strong approximation of functions of one and two variables by certain operator,* Fasc. Math., **35**(2005), 115-133.
11. L. Rempulska and M. Skorupka, *Approximation of differentiable functions of two variables by Baskakov operators,* 5-th International Conference APLIMAT, (2006), 315-319.
12. Q. Shen and J. Xu, *Simultaneous approximation by multivariate Szâsz-Durrmeyer operator,* J. of Hubei Uni. (NSE)(in Chinese), **21**(2)(1999), 107-111.

**Naokant Deo** received M.Sc. from Seoul National University and Ph.D at University of Minnesota. Since 1992 he has been at Chungnam National University. His research interests include numerical optimization and biological computation.

Department of Applied Mathematics, Delhi College of Engineering, Bawana Road, Delhi-110042, India.

e-mail: dr_naokant_deo@yahoo.com