HYBRID DIFFERENCE SCHEMES FOR SINGULARLY PERTURBED PROBLEM OF MIXED TYPE WITH DISCONTINUOUS SOURCE TERM

R. MYTHILI PRIYADHARSHINI, N. RAMANUJAM* AND T. VALANARASU

ABSTRACT. We consider a mixed type singularly perturbed one dimensional elliptic problem with discontinuous source term. The domain under consideration is partitioned into two subdomains. A convection-diffusion and a reaction-diffusion type equations are posed on the first and second subdomains respectively. Two hybrid difference schemes on Shishkin mesh are constructed and we prove that the schemes are almost second order convergence in the maximum norm independent of the diffusion parameter. Error bounds for the numerical solution and its numerical derivative are established. Numerical results are presented which support the theoretical results.

AMS Mathematics Subject Classification : 65L10, CR G1.7
Key words and phrases : Singular perturbation problems, mixed type, elliptic problem, Shishkin mesh, hybrid scheme, scaled derivative.

1. INTRODUCTION

Many physical processes connected with non-uniform transitions are described by differential equations with large and/or small parameter(s). Singular Perturbation Problems (SPPs) are differential equations with a small positive parameter multiplying the highest derivative term. These problems arise in several branches of applied mathematics, including fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, gas porous electrodes theory etc. Examples of SPPs include the Navier-Stokes equation of fluid flow at high Reynolds number, the equation governing flow in porous media, the drift-diffusion equation of semi-conductor devices, physics and mathematical models of liquid crystal material and the convection-diffusion and reaction-diffusion equations to mention but a few [2, 3].

Such equations typically exhibit solutions with layers, which cause severe computational difficulties for standard numerical methods. Consequently a variety of different numerical strategies have been devoted ([1]-[3] and the references are therein) to the construction and analysis of accurate numerical methods for SPPs. Recently, authors [7] - [11] have considered SPPs for second/third order Ordinary Differential Equations (ODEs) with discontinuous source term and/or discontinuous convection coefficient. Due to the discontinuity at one or more points in the interior domain, this gives raise an interior layer in the solution of the problem, in addition to the boundary layer at the outflow boundary point. Therefore these types of SPPs have to be dealt separately and carefully. So often the main objective in the investigation of heat and mass transfer processes is to determine derivatives for small values of the parameter for example if it is necessary to find skin friction and/or heat and diffusion fluxes in problems of flow around some body for large Reynolds and Peclet numbers. Hence we obtain numerical approximations not only to the solution but also to its scaled first derivative [12] - [13].

There are two broad classes of interest within singularly perturbed problems: problems of convection-diffusion type and problems of reaction-diffusion type. In [6], the author have analyzed an inverse-monotone finite volume method on Shishkin mesh for a one dimensional singularly perturbed elliptic problem with discontinuous source term and established an almost second-order global pointwise convergence. Our objective in this paper is to propose two hybrid finite difference schemes to approximate solution and its scaled first derivative of a one dimensional singularly perturbed elliptic problem with discontinuous source term. Here, a convection-diffusion and a reaction-diffusion type equations are considered in the first and second subdomains of the whole domain ($\Omega = (0, 1)$) respectively. A single discontinuity is assumed to occur at a point $d \in \Omega$. The solution of this problem has a boundary layer at $x = 1$ and interior layers with different widths at $x = d$. It is convenient to introduce the notation $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$ and to denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$.

In this article we consider the following class of problems:

$$\begin{cases} L^- u \equiv -\varepsilon u'' + a(x)u' + b(x)u = f(x), \ x \in \Omega^-, \\ L^+ u \equiv -\varepsilon u'' + c(x)u = f(x), \ x \in \Omega^+, \end{cases} \tag{1}$$

$$u(0) = p, \quad [u(d)] = [u'(d)] = 0, \quad u(1) = q, \tag{2}$$

where $\varepsilon \ (0 < \varepsilon << 1)$ is a singular perturbation parameter, $a(x), b(x)$ and $c(x)$ are sufficiently smooth functions on $\bar{\Omega}^-$ and $\bar{\Omega}^+$ respectively and $a(x) \geq \alpha > 0, \ b(x) \geq 0, \ c(x) \geq \gamma > 0$. It is assumed that f is sufficiently smooth function in $\Omega^- \cup \Omega^+ \cup \{0, 1\}$; the left and right limit of f and their derivatives are assumed to exist at $x = d$. The function f ia assumed to have simple discontinuity at $x = d$. Hence the solution u of (1)-(2) does not necessarily have a continuous second derivative at the point d, that is, u does not belong to the class of

functions $C^2(\Omega)$. Hence the class of functions, where u belongs to it, is taken as $C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$. Further the SPPs (1)-(2) has a unique solution $u \in Y \equiv C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [6].

Through out this paper, C denotes a generic constant is independent of the singular perturbation parameter ε and the dimension of the discrete problem N. Let $y : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the supremum norm $\| y \| = \sup_{x \in D} |y(x)|$.

Assumption: We shall assume that $\varepsilon \leq CN^{-1}$ throughout the paper as is generally the case in practice for discretization of convection-dominated problem [4].

2. Preliminaries

For the sake of completeness, we now reproduce the following theorems from [6, §2] for the above problem. Also we derive a cubic spline difference scheme for a convection-diffusion and a reaction-diffusion type equations.

Theorem 1. (Maximum Principle) *Suppose that $u \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ satisfies*

$$u(0) \geq 0, \quad u(1) \geq 0,$$
$$L^- u(x) \geq 0, \quad x \in \Omega^-, \quad L^+ u(x) \geq 0, \quad x \in \Omega^+,$$

and

$$[u'](d) \leq 0. \quad Then \quad u(x) \geq 0, \forall x \in \bar{\Omega}.$$

Theorem 2. (Stability result) *If $u \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$, then*

$$\| u \|_{\bar{\Omega}} \leq C \max\{|u(0)|, |u(1)|, \| L^- u \|_{\Omega^-}, \| L u^+ \|_{\Omega^+}\}.$$

The sharper bounds on the derivatives of the solution are obtained by decomposing the solution as $u = v + w$, where $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3$, where v_i, $i = 0, 1, ..., 3$ are defined as in [6]. Thus the smooth component $v \in C^0(\Omega)$ is the solution of

$$\begin{cases} L^- v(x) = f(x), & x \in \Omega^-, \\ L^+ v(x) = f(x), & x \in \Omega^+, \end{cases} \tag{3}$$

$$v(0) = u(0), \; [v'](d) = [v_0'](d) + \varepsilon[v_1'](d) + \varepsilon^2[v_2'](d), \; v(1) = 0. \tag{4}$$

Thus we define the singular component $w \in C^0(\Omega)$ as the solution of

$$\begin{cases} L^- w(x) = 0, & x \in \Omega^- \\ L^+ w(x) = 0, & x \in \Omega^+, \end{cases} \tag{5}$$

$$w(0) = 0, \quad [w'](d) = -[v'](d), \quad w(1) = u(1) - v(1), \tag{6}$$

and further we decompose w as $w = w_1 + w_2$, where w_1 is the solution of

$$\begin{cases} w_1(x) = 0, & x \in \Omega^-, \quad w_1(d) = -[v](d), \\ L^+ w_1(x) = 0, & x \in \Omega^+, \quad w_1(1) = u(1) - v(1) \end{cases}$$

and $w_2 \in C^0(\Omega)$ is the solution of

$$\begin{cases} L^- w_2(x) = 0, & x \in \Omega^-, \quad w_2(0) = 0, [w_2'](d) = -[v'](d) - [w_1'](d), \\ L^+ w_2(x) = 0, & x \in \Omega^+, \quad w_2(1) = 0. \end{cases}$$

Theorem 3. *For each integer k, satisfying $0 \leq k \leq 4$, the solutions v and w of (3)-(4) and (5)-(6) respectively satisfy the following bounds:*

$$\| v^{(k)} \|_{\bar{\Omega}} \quad \leq \quad C(1 + \varepsilon^{3-k}), \tag{7}$$

$$|w^{(k)}(x)| \quad \leq \quad C\varepsilon^{-k+1/2} e^{-(d-x)\alpha/\varepsilon}, \quad x \in \Omega^-, \tag{8}$$

$$|w^{(k)}(x)| \quad \leq \quad C\varepsilon^{-k/2} (e^{-(x-d)\sqrt{\gamma/\varepsilon}} + e^{-(1-x)\sqrt{\gamma/\varepsilon}}), \quad x \in \Omega^+. \tag{9}$$

2.1. Cubic Spline Difference Scheme. In this section, first we derive the cubic spline scheme on variable meshes.

Let $x_0 = 0, x_N = 1, \quad x_i = x_0 + \sum_{k=1}^{i} h_k, \quad h_k = x_i - x_{i-1}, i = 1, ..., N$ be the mesh. For given values $U(x_0), U(x_1), ..., U(x_N)$ of a function $u(x)$, at the nodal points $x_0, x_1, ..., x_N$ there exists an interpolating cubic spline function $S(x)$ with the following properties:

(i) $S(x)$ coincides with a polynomial of degree three on each subintervals $[x_{i-1}, x_i], i = 1, ..., N$

(ii) $S(x) \in C^2(\bar{\Omega})$; (iii) $S(x_i) = U(x_i), \quad i = 0, 1, ..., N.$

Then the cubic spline function can be written as

$$\begin{aligned} S(x) \quad = \quad & \frac{(x_i - x)^3}{6h_i} M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i} M_i + (U(x_{i-1}) - \frac{h_i^2}{6} M_{i-1})(\frac{x_i - x}{h_i}) \\ + \quad & (U(x_i) - \frac{h_i^2}{6} M_i)(\frac{x - x_{i-1}}{h_i}), \quad x \in [x_{i-1}, x_i], \quad i = 1, ..., N, \end{aligned}$$

where $M_i = S''(x_i), i = 0, ..., N$. From the basic properties of spline, it should satisfy the following condition of continuity for $i = 1, ..., N - 1$

$$\frac{h_i}{6} M_{i-1} + \frac{h_i + h_{i+1}}{3} M_i + \frac{h_{i+1}}{6} M_{i+1} = \frac{U(x_{i+1}) - U(x_i)}{h_{i+1}} - \frac{U(x_i) - U(x_{i-1})}{h_i}. \tag{10}$$

For obtaining second order approximation of the first order derivative of $u(x)$, we use the Taylor series expansion for U around x_i, to get the following approximations for $U(x_{i+1})$ and $U(x_{i-1})$

$$U(x_{i+1}) \quad \simeq \quad U(x_i) + h_{i+1} U'(x_i) + \frac{h_{i+1}^2}{2} U''(x_i) \tag{11}$$

$$U(x_{i-1}) \quad \simeq \quad U(x_i) - h_i U'(x_i) + \frac{h_i^2}{2} U''(x_i). \tag{12}$$

Multiplying (12) by h_{i+1}^2/h_i^2, then subtracting from (11) and multiplying (12) by h_{i+1}/h_i, then adding it to (11), we get the following approximation for $U'(x_i)$ and $U''(x_i)$, respectively

$$U'(x_i) \simeq \frac{1}{h_i h_{i+1}(h_i + h_{i+1})}(-h_{i+1}^2 U(x_{i-1}) + (h_{i+1}^2 - h_i^2)U(x_i) + h_i^2 U(x_{i+1})),$$

$$U''(x_i) \simeq \frac{2}{h_i h_{i+1}(h_i + h_{i+1})}(h_{i+1}U(x_{i-1}) - (h_i + h_{i+1})U(x_i) + h_i U(x_{i+1})).$$

Using these approximations in $U'(x_{i+1}) \simeq U'(x_i) + h_{i+1}U''(x_i)$ and $U'(x_{i-1}) \simeq U'(x_i) - h_i U''(x_i)$, we get the following approximation

$$U'_j(x_{i+1}) \simeq \frac{1}{h_i h_{i+1}(h_i + h_{i+1})}[h_{i+1}^2 U(x_{i-1}) - (h_i + h_{i+1})^2 U(x_i) + (h_i^2 + 2h_i h_{i+1})U(x_{i+1})]$$

and

$$U'_j(x_{i-1}) \simeq \frac{1}{h_i h_{i+1}(h_i + h_{i+1})}[-(h_{i+1}^2 + 2h_i h_{i+1})U(x_{i-1}) + (h_i + h_{i+1})^2 U(x_i) - h_i^2 U(x_{i+1})].$$

We now derive linear system of equations for convection-diffusion and reaction-diffusion equations. For the convection-diffusion equation, consider the expression

$$-\varepsilon M_j + a(x_j)U'(x_j) + b(x_j)U(x_j) = f(x_j), \; j = i, i \pm 1.$$

Substituting this in (10), we get the following linear system of equations, for $i = 1, ..., N-1$

$$r_{1,i}^- U(x_{i-1}) + r_{1,i}^c U(x_i) + r_{1,i}^+ U(x_{i+1}) = F_1(x_i), \tag{13}$$

where

$$r_{1,i}^- = -\frac{(h_{i+1} + 2h_i)}{2(h_i + h_{i+1})}a(x_{i-1}) - \frac{h_{i+1}}{h_i}a(x_i) + \frac{h_{i+1}^2}{2h_i(h_i + h_{i+1})}a(x_{i+1}) + \frac{h_i}{2}b(x_{i-1}) - \frac{3\varepsilon}{h_i},$$

$$r_{1,i}^c = \frac{(h_i + h_{i+1})}{2h_{i+1}}a(x_{i-1}) + \frac{(h_{i+1}^2 - h_i^2)}{h_i h_{i+1}}a(x_i) - \frac{(h_i + h_{i+1})}{2h_i}a(x_{i+1}) + (h_i + h_{i+1})b(x_i)$$
$$+ \frac{3\varepsilon(h_i + h_{i+1})}{h_i h_{i+1}},$$

$$r_{1,i}^+ = -\frac{h_i^2}{2h_{i+1}(h_i + h_{i+1})}a(x_{i-1}) + \frac{h_i}{h_{i+1}}a(x_i) + \frac{(2h_{i+1} + h_i)}{2(h_i + h_{i+1})}a(x_{i+1}) + \frac{h_{i+1}}{2}b(x_{i+1}) - \frac{3\varepsilon}{h_{i+1}},$$

$$F_1(x_i) = F_{1,i}^- f(x_{i-1}) + F_{1,i}^c f(x_i) + F_{1,i}^+ f(x_{i+1}),$$

$$F_{1,i}^- = \frac{h_i}{2}, \quad F_{1,i}^c = (h_i + h_{i+1}), \quad F_{1,i}^+ = \frac{h_{i+1}}{2}.$$

Similarly for the reaction-diffusion equation, substituting the expression

$$-\varepsilon M_j + c(x_j)U(x_j) = f(x_j), \; j = i, i \pm 1$$

in (10), we get the following linear system of equations, for $i = 1, ..., N-1$

$$r_{2,i}^- U(x_{i-1}) + r_{2,i}^c U(x_i) + r_{2,i}^+ U(x_{i+1}) = F_2(x_i), \tag{14}$$

where

$$r_{2,i}^- = \frac{h_i c(x_{i-1})}{2} - \frac{3\varepsilon}{h_i}, \quad r_{2,i}^c = (h_i + h_{i+1})c(x_i) + \frac{3\varepsilon(h_i + h_{i+1})}{h_i h_{i+1}}, \quad r_{2,i}^+ = \frac{h_{i+1}c(x_{i+1})}{2} - \frac{3\varepsilon}{h_{i+1}},$$

$$F_2(x_i) = F_{2,i}^- f(x_{i-1}) + F_{2,i}^c f(x_i) + F_{2,i}^+ f(x_{i+1}), \quad F_{2,i}^- = \frac{h_i}{2}, \quad F_{2,i}^c = (h_i + h_{i+1}), \quad F_{2,i}^+ = \frac{h_{i+1}}{2}.$$

3. Hybrid Difference Schemes

To approximate the solution of the problem (1)-(2), two hybrid difference schemes are introduced. On Ω a piecewise uniform mesh of N mesh interval is constructed as follows. The domain $\bar{\Omega}^-$ is subdivided into the two subintervals $[0, d - \sigma_1] \cup [d - \sigma_1, d]$ for some σ_1, that satisfy $0 < \sigma_1 \leq \frac{d}{2}$, and the domain $\bar{\Omega}^+$ is subdivided into the three subintervals $[d, d + \sigma_2] \cup [d + \sigma_2, 1 - \sigma_2] \cup [1 - \sigma_2, 1]$, for some σ_2, that satisfy $0 < \sigma_2 \leq \frac{1-d}{4}$. We shall construct a piecewise uniform mesh $\bar{\Omega}_\varepsilon^N$ condensed near to boundary $x = 1$ and around interface point $x = d$,

$$\bar{\Omega}_\varepsilon^N = \{x_i, \ x_i = x_{i-1} + h_i, \ 1 \leq i \leq m + n = N, \ x_0 = 0, x_m = d, x_N = 1\},$$

where

$$h_i = \begin{cases} h_1 = 2(d - \sigma_1)/m, & i = 1, ..., m/2, \\ h_2 = 2\sigma_1/m, & i = m/2 + 1, ..., m, \\ h_3 = 4\sigma_2/n, & i = m + 1, ..., m + n/4, \\ h_4 = 2(1 - d - 2\sigma_2)/n, & i = m + n/4 + 1, ..., m + 3n/4 \\ h_5 = 4\sigma_2/n, & i = m + 3n/4 + 1, ..., N. \end{cases}$$

In general one takes the transition parameters as $\sigma_1 = \min\{\frac{d}{2}, \frac{2\varepsilon}{\alpha} \ln m\}$ and $\sigma_2 = \min\{\frac{1-d}{4}, 2\sqrt{\frac{\varepsilon}{\gamma}} \ln n\}$. But for our analysis we assume that $\sigma_1 = \dfrac{2\varepsilon}{\alpha} \ln m$ and $\sigma_2 = 2\sqrt{\frac{\varepsilon}{\gamma}} \ln n$, since otherwise N^{-1} is exponentially small compared with ε.

Hybrid Difference Scheme-I (HDS - I): In this scheme, we discretize (1) and (2) using the central difference scheme on the fine mesh region and the midpoint scheme in the coarse region on $\bar{\Omega}_\varepsilon^{-N}$ and a central difference scheme on $\bar{\Omega}_\varepsilon^{+N}$, that is,

$$L_u^{-N} U_i \equiv \frac{-2\varepsilon}{h_i + h_{i+1}} \left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i} \right) + a_{i-1/2} \frac{U_i - U_{i-1}}{h_i} + b_{i-1/2} \bar{U}_i = f_{i-1/2}, \ 0 < i \leq m/2$$

$$\tag{15}$$

$$L_c^{-N} U_i \equiv \frac{-2\varepsilon}{h_i + h_{i+1}} \left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i} \right) + a_i \frac{U_{i+1} - U_{i-1}}{h_i + h_{i+1}} + b_i U_i = f_i, \ m/2 < i < m \tag{16}$$

$$L_c^{+N} U_i \equiv \frac{-2\varepsilon}{h_i + h_{i+1}} \left(\frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i} \right) + c_i U_i = f_i, \quad m < i < N, \tag{17}$$

where $U_i = U(x_i)$, $\bar{U}_i = \frac{U_{i-1} + U_i}{2}$, $a_{i-1/2} \equiv a((x_{i-1} + x_i)/2)$; similarly for $b_{i-1/2}, f_{i-1/2}$ and $a_i = a(x_i)$; similarly for b_i, c_i, f_i. At the interface point $x_m = d$, we shall use the difference operator

$$L_t^N U_m \equiv \frac{-U_{m+2} + 4U_{m+1} - 3U_m}{2h_{m+1}} - \frac{U_{m-2} - 4U_{m-1} + 3U_m}{2h_{m-1}} = 0. \tag{18}$$

From equation (16) and (17) we get

$$U_{m-2} = \frac{2h_2}{-2\varepsilon - h_2 a_{m-1}}(h_2 f_{m-1} - (\frac{2\varepsilon + h_2^2 b_{m-1}}{h_2})U_{m-1} - (\frac{-2\varepsilon + h_2 a_{m-1}}{2h_2})U_m),$$

$$U_{m+2} = -\frac{h_3^2}{\varepsilon}f_{m+1} + (2 + \frac{h_3^2}{\varepsilon}c_{m+1})U_{m+1} - U_m.$$

Inserting the expression for U_{m-2} and U_{m+2} in (18) gives

$$L_T^N U_m = (\frac{2}{h_2} + \frac{(2\varepsilon + h_2^2 b_{m-1})}{h_2(-2\varepsilon - h_2 a_{m-1})})U_{m-1} - (\frac{3(h_2 + h_3)}{2h_2 h_3} - \frac{1}{2h_3} - \frac{(-2\varepsilon + h_2 a_{m-1})}{2h_2(-2\varepsilon - h_2 a_{m-1})})U_m$$

$$+ (\frac{2}{h_3} - (\frac{1}{h_3} + \frac{h_3 c_{m+1}}{2\varepsilon}))U_{m+1} = \frac{h_2}{(-2\varepsilon - h_2 a_{m-1})}f_{m-1} - \frac{h_3}{2\varepsilon}f_{m+1}.$$

Thus, we have

$$L_H^N U_i = f_i, \quad \text{for} \quad i = 1, ..., N - 1, \tag{19}$$

where, $L_H^N U_i = \begin{cases} L_u^{-N} U_i, & \text{for} \quad i = 1, ..., m/2, \\ L_c^{-N} U_i, & \text{for} \quad i = m/2 + 1, ..., m - 1, \\ L_T^N U_i, & \text{for} \quad i = m, \\ L_c^{+N} U_i, & \text{for} \quad i = m + 1, ..., N - 1 \end{cases}$

and $f_i = \begin{cases} f_{i-1/2}, & \text{for} \quad i = 1, ..., m/2, \\ f_i, & \text{for} \quad i = m/2 + 1, ..., m - 1, \\ \frac{h_2}{(-2\varepsilon - h_2 a_{m-1})}f_{m-1} - \frac{h_3}{2\varepsilon}f_{m+1}, & \text{for} \quad i = m, \\ f_i, & \text{for} \quad i = m + 1, ..., N - 1. \end{cases}$

Hybrid Difference Scheme-II (HDS - II): In this scheme, we use the cubic spline difference scheme in the fine mesh region and we use the schemes as in the HDS-I on the coarse region. Thus we have

$$L_H^N U_i = f_i, \quad \text{for} \quad i = 1, ..., N - 1, \tag{20}$$

where

$$L_H^N U_i = \begin{cases} L_u^{-N} U_i, & \text{for} \quad i = 1, ..., m/2, \\ L_{cu}^{-N} U_i, & \text{for} \quad i = m/2 + 1, ..., m - 1, \\ L_T^N U_i, & \text{for} \quad i = m, \\ L_{cu}^{+N} U_i, & \text{for} \quad i = m + 1, ..., m + n/4 - 1, m + 3n/4 + 1, ..., N - 1 \\ L_c^{+N} U_i, & \text{for} \quad i = m + n/4, .., m + 3n/4 \end{cases}$$

$$f_i = \begin{cases} f_{i-1/2}, & \text{for} \quad i = 1, ..., m/2, \\ \frac{h_i}{2}f_{i-1} + (h_i + h_{i+1})f_i + \frac{h_{i+1}}{2}f_{i+1}, & \text{for} \quad i = m/2 + 1, ..., m - 1, \\ & \hspace{2.5cm} m + 1, ..., m + n/4 - 1, m + 3n/4 + 1, ..., N - 1 \\ \frac{h_{m+1}}{r_{1,m-1}^-}(\frac{h_{m-1}}{2}f_{m-2} + (h_{m-1} + h_m)f_{m-1} + \frac{h_m}{2}f_m) \\ + \frac{h_{m-1}}{r_{2,m+1}^+}(\frac{h_{m+1}}{2}f_m + (h_{m+1} + h_{m+2})f_{m+1} + \frac{h_{m+2}}{2}f_{m+2}), & \text{for} \quad i = m, \\ f_i, & \text{for} \quad i = m + n/4, ..., m + 3n/4 \end{cases}$$

$$L_{cu}^{-N}U_i \equiv r_{1,i}^- U_{i-1} + r_{1,i}^c U_i + r_{1,i}^+ U_{i+1} = F_1(x_i), \quad m/2 < i < m$$

$$L_{cu}^{+N}U_i \equiv r_{2,i}^- U_{i-1} + r_{2,i}^c U_i + r_{2,i}^+ U_{i+1} = F_2(x_i), \quad m+1, ..., m+n/4-1, m+3n/4+1, ..., N-1,$$

$$L_T^N U_m = (4h_{m+1} + \frac{h_{m+1} r_{1,m-1}^c}{r_{1,m-1}^-})U_{m-1} - (3(h_{m-1}+h_{m+1}) - \frac{h_{m-1} r_{2,m+1}^-}{r_{2,m+1}^+} - \frac{h_{m+1} r_{1,m-1}^+}{r_{1,m-1}^-})U_m$$

$$+ (4h_{m-1} + \frac{h_{m-1} r_{2,m+1}^c}{r_{2,m+1}^+})U_{m+1} = \frac{h_{m+1}}{r_{1,m-1}^-}(\frac{h_{m-1}}{2}f_{m-2} + (h_{m-1}+h_m)f_{m-1} + \frac{h_m}{2}f_m)$$

$$+ \frac{h_{m-1}}{r_{2,m+1}^+}(\frac{h_{m+1}}{2}f_m + (h_{m+1}+h_{m+2})f_{m+1} + \frac{h_{m+2}}{2}f_{m+2}),$$

$r_{1,i}^-, r_{1,i}^c, r_{1,i}^+, r_{2,i}^-, r_{2,i}^c, r_{2,i}^+, F_1(x_i), F_2(x_i)$, are defined in Section 2.1 and L_u^{-N}, L_c^{+N} are defined in (15), (17) and $f(x_m) = \dfrac{h_2 f(x_m - h_2) + h_3 f(x_m + h_3)}{h_2 + h_3}$.

Note: It may be noted that the same operator symbol L_H^N is used for both the schemes. In the following whatever discussion is carried out, it is true for both the schemes.

4. Numerical Solution Estimates

To guarantee the monotonicity property of the difference operator L_H^N, we impose the following mild assumption on the minimum number of mesh points [4],

$$\frac{N}{\ln N} \geq 4\frac{\| a \|}{\alpha}. \tag{21}$$

Lemma 1. *Assume that the inequality (21) holds true. Then the operators L_H^N defined by (19) and (20) satisfy a discrete minimum principle, that is, if $Z(x_i)$, $i = 0, 1, ..., N$ is a mesh function that satisfy $Z(x_0) \geq 0$, $Z(x_N) \geq 0$ and $L_H^N Z(x_i) \geq 0$, for $1, ..., N-1$, then $Z(x_i) \geq 0$ for all $i = 0, ..., N$.*

Proof. See [10]. $\square$

Define the mesh function V to be the solution of the following discrete problem

$$L_H^N V(x_i) = f(x_i), \quad \text{for} \quad x_i \in \Omega_\varepsilon^N, \tag{22}$$

$$V(x_0) = v(0), V(x_m) = v(d), V(x_N) = v(1). \tag{23}$$

We define the mesh function W to be the solution of

$$L_H^N W(x_i) = 0, \quad \text{for} \quad x_i \in \Omega_\varepsilon^N \setminus \{d\}, \tag{24}$$

$$W(0) = w(0), L_T^N W(x_m) = -L_T^N V(x_m), W(x_N) = w(1). \tag{25}$$

Analogous to the continuous case we can further decompose W as $W = W_1 + W_2$, where W_1, the discrete analogous of the boundary layer function w_1 is defined as the solution of

$$\begin{cases} W_1(x_i) = 0, & \text{for} \quad x_i \in \Omega_\varepsilon^N \cap (0, d), \\ L_H^N W_1(x_i) = 0, & \text{for} \quad x_i \in \Omega_\varepsilon^N \cap (d, 1), \end{cases} \tag{26}$$

$$W_1(x_m) = w_1(x_m), \quad W_1(x_N) = w_1(1) \tag{27}$$

and W_2, the discrete analogous of the weak interior layer function w_2 is defined as the solution of

$$L_H^N W_2(x_i) = 0, \quad \text{for} \quad x_i \in \Omega_\varepsilon^N \tag{28}$$

$$W_2(x_0) = 0, \quad W_2(x_N) = 0, \tag{29}$$

$$L_T^N W_2(x_m) = -L_T^N W_1(x_m) - L_T^N V(x_m). \tag{30}$$

Now, we can define $U(x_i)$ to be

$$U(x_i) = V(x_i) + W(x_i) = \begin{cases} V(x_i) + W_2(x_i), & \text{for} \quad i = 1, ..., m-1, \\ V(x_i) + W_1(x_i) + W_2(x_i), & \text{for} \quad i = m, ..., m+n-1. \end{cases} \tag{31}$$

Using the procedure adopted in [4], [15, §4], we can deduce the truncation error for the Hybrid Difference Scheme - I as

$$|L_H^N(U-u)(x_i)| \leq \begin{cases} \varepsilon h_i \parallel u^{(3)} \parallel + Ch_i^2(\parallel u^{(3)} \parallel + \parallel u^{(2)} \parallel), \, i = 1, ..., m/2, \\ \varepsilon h_i^2 \parallel u^{(4)} \parallel + \parallel a \parallel h_i^2 \parallel u^{(3)} \parallel, \, i = m/2+1, ..., m-1 \\ \varepsilon h_i^2 \parallel u^{(4)} \parallel, \, i = m+1, ..., N-1. \end{cases}$$

Using the procedure adopted in [5], [14, §3.1], we can deduce the truncation error for the Hybrid Difference Scheme - II as

$$|L_H^N(U-u)(x_i)| \leq \begin{cases} \varepsilon h_i \parallel u^{(3)} \parallel + Ch_i^2(\parallel u^{(3)} \parallel + \parallel u^{(2)} \parallel), \quad i = 1, ..., m/2, \\ \varepsilon h_i^2 \parallel u^{(4)} \parallel, \quad i = m/2+1, ..., m-1, m+1, ..., N-1. \end{cases}$$

Using these mesh functions the nodal error $|(U-u)(x_i)| = |(V-v)(x_i) + (W-w)(x_i)|$ is then bounded separately outside and inside the layer.

Lemma 2. *For both the schemes, at each mesh point $x_i \in \Omega_\varepsilon^N$, the regular component of the error satisfies the estimate*

$$|(V-v)(x_i)| \leq \begin{cases} Cm^{-2}x_i, & \text{for} \quad i = 1, ..., m-1 \\ Cn^{-2}x_i, & \text{for} \quad i = m+1, ..., m+n-1. \end{cases} \tag{32}$$

Proof. For the HDS - I, let us now consider the truncation error at the mesh points. Using standard truncation error bounds and the bounds on the derivatives of v, we have

$$|L_H^N(V-v)(x_i)| \leq \begin{cases} \varepsilon h_i \parallel v^{(3)} \parallel + Ch_i^2(\parallel v^{(3)} \parallel + \parallel v^{(2)} \parallel), & i = 1, ..., m/2, \\ \varepsilon h_i^2 \parallel v^{(4)} \parallel + \parallel a \parallel h_i^2 \parallel v^{(3)} \parallel, & i = m/2+1, ..., m-1 \\ \varepsilon h_i^2 \parallel v^{(4)} \parallel, & i = m+1, ..., N-1 \end{cases}$$

$$\leq \begin{cases} Cm^{-2}, & \text{for} \quad i = 1, ..., m-1 \\ Cn^{-2}, & \text{for} \quad i = m+1, ..., m+n-1. \end{cases}$$

For the HDS - II,

$$|L_H^N(V-v)(x_i)| \leq \begin{cases} \varepsilon h_i \parallel v^{(3)} \parallel + Ch_i^2(\parallel v^{(3)} \parallel + \parallel v^{(2)} \parallel), & i = 1, ..., m/2, \\ \varepsilon h_i^2 \parallel v^{(4)} \parallel, & i = m/2+1, ..., N-1 \end{cases}$$

$$\leq \begin{cases} Cm^{-2}, & \text{for} \quad i = 1, ..., m-1 \\ Cn^{-2}, & \text{for} \quad i = m+1, ..., m+n-1. \end{cases}$$

Consider the two mesh functions

$$\Psi^{\pm}(x_i) = \begin{cases} \frac{C}{\alpha} m^{-2} x_i \pm (V-v)(x_i), & i = 0, ..., m \\ \frac{C}{\gamma} n^{-2} x_i \pm (V-v)(x_i), & i = m+1, ..., N. \end{cases}$$

Then, we have $\Psi^{\pm}(x_0) = 0$ and $\Psi^{\pm}(x_N) \geq 0$. For $i = 1, ..., m/2$, we have

$$L_H^N \bar{\Psi}^{\pm}(x_i) = \frac{C}{\alpha} m^{-2} a_{i-1/2} + \frac{C}{\alpha} m^{-2} b_{i-1/2} \frac{x_i + x_{i-1}}{2} \pm Cm^{-2} \geq 0,$$

for both the schemes. For the HDS - I, we have

$$L_H^N \bar{\Psi}^{\pm}(x_i) = \frac{C}{\alpha} m^{-2} a_i + \frac{C}{\alpha} m^{-2} b_i x_i \pm Cm^{-2} \geq 0, \, i = m/2+1, ...m-1$$

$$L_H^N \bar{\Psi}^{\pm}(x_i) = \frac{C}{\gamma} n^{-2} c_i x_i \pm Cn^{-2} \geq 0, \, i = m+1, ..., N-1.$$

For the HDS - II, we have

$$L_H^N \bar{\Psi}^{\pm}(x_i) = \frac{C}{\alpha} m^{-2} (r_{1,i}^- + r_{1,i}^c + r_{1,i}^+)(x_i) \pm Cm^{-2} > 0, \, i = m/2+1, ..., m-1$$

$$L_H^N \bar{\Psi}^{\pm}(x_i) = \frac{C}{\gamma} n^{-2} (r_{2,i}^- + r_{2,i}^c + r_{2,i}^+)(x_i) \pm Cn^{-2} > 0, \, i = m+1, ..., N-1.$$

Applying Theorem 1 to $\bar{\Psi}^{\pm}(x_i)$, $x_i \in \bar{\Omega}_\varepsilon^N$, we get the required result.   □

Lemma 3. *For both the schemes, at each mesh point $x_i \in \Omega_\varepsilon^N$, the layer component of the error satisfies the estimate*

$$|(W-w)(x_i)| \leq \begin{cases} Cm^{-2}(\ln m)^3, & \text{for} \quad x_i \in \Omega_\varepsilon^N \cap (0, d) \\ Cn^{-2}(\ln n)^2, & \text{for} \quad x_i \in \Omega_\varepsilon^N \cap (d, 1). \end{cases}$$

Proof. For the HDS - I, by [1, p47], for all $x_i \in \Omega_\varepsilon^N \cap [d+\sigma_2, 1-\sigma_2]$, we have

$$|L_H^N(W_1 - w_1)(x_i)| \leq 2\varepsilon \max_{x \in [x_{i-1}, x_{i+1}]} |w''(x)| \leq Cn^{-2}.$$

Now by [1, p46], for all $x_i \in \Omega_\varepsilon^N \cap (d, d+\sigma_2)$ and $x_i \in \Omega_\varepsilon^N \cap (1-\sigma_2, 1)$

$$|L_H^N(W_1 - w_1)(x_i)| \leq \varepsilon(x_{i+1} - x_{i-1})^2 |w_1^{(4)}| \leq Cn^{-2}(\ln n)^2.$$

Since $|U(x_m)| \leq C$ and with (32), we have $|W_2(x_m)| \leq C\sqrt{\varepsilon}$. Using the arguments in [3, Chap. 3], for $x_i \leq d - \sigma_1$, we have

$$|W_2(x_i)| \leq |W_2(x_m)| m^{-2} \leq C\sqrt{\varepsilon} m^{-2}$$

and

$$|(W_2 - w_2)(x_i)| \leq |W_2(x_i)| + |w_2(x_i)| \leq C\sqrt{\varepsilon} N^{-2}. \tag{33}$$

Therefore, $|(W_2 - w_2)(x_i)| \le CN^{-2}$.

Now, for all $x_i \in \Omega_\varepsilon^N \cap (d - \sigma_1, d)$, we get

$$|L_H^N(W_2 - w_2)(x_i)| \quad \le \quad \varepsilon h_i^2 \parallel w_2^{(4)} \parallel + \parallel a \parallel h_i^2 \parallel w_2^{(3)} \parallel \le Cm^{-2}\sigma_1^2\varepsilon^{-5/2}.$$

At the interface point $x_m = d$,

$$|L_T^N W_2(x_m) - \frac{h_2}{(-2\varepsilon - h_2 a_{m-1})}f_{m-1} + \frac{h_3}{2\varepsilon}f_{m+1}| \le Cm^{-2}\sigma_1^2\varepsilon^{-5/2} + Cn^{-2}\sigma_2^2\varepsilon^{-1}.$$

For $x_i \in \Omega_\varepsilon^N \cap (d, d + \sigma_2)$ and $\Omega_\varepsilon^N \cap (1 - \sigma_2, 1)$

$$|L_H^N(W_2 - w_2)(x_i)| \quad \le \quad \varepsilon h_i^2 \parallel w_2^{(4)} \parallel \le Cn^{-2}\sigma_2^2\varepsilon^{-1}.$$

For the HDS - II, for all $x_i \in \Omega_\varepsilon^N \cap (d, 1)$, we have

$$|L_H^N(W_1 - w_1)(x_i)| \le \varepsilon(x_{i+1} - x_{i-1})^2|w_1^{(4)}| \le Cn^{-2}(\ln n)^2,$$

for all $x_i \in \Omega_\varepsilon^N \cap (d - \sigma_1, d)$,

$$|L_H^N(W_2 - w_2)(x_i)| \quad \le \quad \varepsilon h_i^2 \parallel w_2^{(4)} \parallel \le Cm^{-2}\sigma_1^2\varepsilon^{-5/2}$$

and for $x_i \in \Omega_\varepsilon^N \cap (d, d + \sigma_2)$ and $\Omega_\varepsilon^N \cap (1 - \sigma_2, 1)$

$$|L_H^N(W_2 - w_2)(x_i)| \quad \le \quad \varepsilon h_i^2 \parallel w_2^{(4)} \parallel \le Cn^{-2}\sigma_2^2\varepsilon^{-1}.$$

At the point of interface $x_m = d$,

$$|L_T^N(W_2 - w_2)(x_m)| \le Cm^{-2}\sigma_1^2\varepsilon^{-5/2} + Cn^{-2}\sigma_2^2\varepsilon^{-1}.$$

Consider the barrier function

$$\Phi^\pm(x_i) = \Psi(x_i) \pm |(W_2 - w_2)(x_i)|, \quad x_i \in \Omega_\varepsilon^N \cap (d - \sigma_1, 1),$$

where,

$$\Psi(x_i) = \begin{cases} Cm^{-2} + Cm^{-2}\sigma_1^2\varepsilon^{-5/2}(x_i - d + \sigma_1), & \text{for} \quad x_i \in \Omega_\varepsilon^{-N} \cap (d - \sigma_1, d) \\ Cn^{-2} + Cn^{-2}\sigma_2^2\varepsilon^{-1}(1 - x_i), & \text{for} \quad x_i \in \Omega_\varepsilon^{+N} \cap (d, 1) \end{cases}$$

We observe that $\Phi^\pm(d - \sigma_1) = Cm^{-2} \pm Cm^{-2} \ge 0$ and $\Phi^\pm(1) = 0$. Also, for the HSD-I,

$$L_H^N\Phi^\pm(x_i) \quad \ge \quad \begin{cases} a(x_i)Cm^{-2}\sigma_1^2\varepsilon^{-5/2} + b(x_i)\Psi(x_i) \pm Cm^{-2}\sigma_1^2\varepsilon^{-5/2}, \text{for } x_i \in \Omega_\varepsilon^{-N} \cap (d - \sigma_1, d) \\ c(x_i)\Psi(x_i) \pm Cn^{-2}\sigma_2^2\varepsilon^{-1}, \quad \text{for} \quad x_i \in \Omega_\varepsilon^{+N} \cap (d, 1) \end{cases}$$

$$> \quad 0, \quad \text{for all} \quad x_i \in \Omega_\varepsilon^N \cap (d - \sigma_1, 1)$$

and at the point of interface we have $\Phi^\pm(x_m) > 0$. Applying Theorem 1, we get $\bar{\Psi}^\pm(x_i) \ge 0$, $x_i \in \bar{\Omega}_\varepsilon^N$. Using the above barrier function for the HSD-II, one can easily establish that $\bar{\Psi}^\pm(x_i) \ge 0$, $x_i \in \bar{\Omega}_\varepsilon^N$. Thus, we have

$$|(W_1 - w_1)(x_i)| \quad \le \quad Cn^{-2}(\ln n)^2, \text{for} \quad x_i \in \Omega_\varepsilon^N \cap (d, 1);$$

$$|(W_2 - w_2)(x_i)| \quad \le \quad \begin{cases} Cm^{-2}\sigma_1^3\varepsilon^{-5/2}, & \text{for} \quad x_i \in \Omega_\varepsilon^N \cap (0, d), \\ Cn^{-2}\sigma_2^2\varepsilon^{-1}, & \text{for} \quad x_i \in \Omega_\varepsilon^N \cap (d, 1), \end{cases}$$

$$\le \quad \begin{cases} C\sqrt{\varepsilon}m^{-2}(\ln m)^3, & \text{for} \quad x_i \in \Omega_\varepsilon^N \cap (0, d), \\ Cn^{-2}(\ln n)^2, & \text{for} \quad x_i \in \Omega_\varepsilon^N \cap (d, 1). \end{cases}$$

Therefore, $|(W-w)(x_i)| \leq |(W_1-w_1)(x_i)|+|(W_2-w_2)(x_i)| \leq CN^{-2}(\ln N)^3.$ $\square$

Remark 1. *The requirement $n \approx m \approx N/2$ are a technicality. In general case it is clear from the analysis above that the order of convergence will be $O(m^{-2}(\ln m)^3 + n^{-2}(\ln n)^2).$*

Theorem 4. *Let $u(x)$ be the solution of (1)-(2) and $U(x_i)$ be the corresponding numerical solution generated by difference scheme HDS - I or HDS - II. Then for each i, $0 \leq i \leq N$, we have*

$$|(U - u)(x_i)| \leq CN^{-2}(\ln N)^3.$$

Proof. Proof follows immediately, if one applies the above Lemmas 2 and 3 to $U - u = (V - v) + (W - w)$. $\square$

5. Numerical Derivative Estimates

Let us consider the higher order discrete approximation to the derivative defined by

$$D^0U(x_i) = \frac{h_iD^+U(x_i) + h_{i+1}D^-U(x_i)}{h_{i+1} + h_i}, \; h_i = x_i - x_{i-1}. \text{ In this section,}$$

we approximate the scaled derivative $\sqrt{\varepsilon}u'$ of the solution of the problem (1)-(2) by the scaled centred discrete derivative $\sqrt{\varepsilon}D^0U(x_i)$ at all internal points x_i, $i = 1, ..., N - 1$ for the HSD-I. We note that for $i = 1, ..., m$, the error $e(x_i) \equiv U(x_i)-u(x_i)$ satisfies the equations $L_H^N e(x_i)-b(x_i)e(x_i) = -b(x_i)e(x_i)+$ truncation error, where, by Theorem 4, $b(x_i)e(x_i) = O(N^{-2}(\ln N)^3)$. These equations will be used in the proofs of the following lemmas and theorems. Hence the analysis carried out in [3, §3.5] and [12, §2] can be applied immediately with a slight modification where ever necessary. Therefore, for some theorems short proves are given.

Lemma 4. *At each mesh point $x_i \in \Omega_\varepsilon^N$ and for all $x \in \bar{\Omega}_i = [x_i, x_{i+1}]$, we have*

$$|\sqrt{\varepsilon}(D^0u(x_i) - u'(x))| \leq CN^{-2}(\ln N)^2,$$

where $u(x)$ is the solution of (1)-(2).

Proof. Consider $\sigma_1 = \frac{2\varepsilon}{\alpha} \ln m$ and $\sigma_2 = 2\sqrt{\frac{\varepsilon}{\gamma}} \ln n$. Then, for $x_i \in \Omega_\varepsilon^N$

$$|\sqrt{\varepsilon}(D^0u(x_i) - u'(x))| \leq |\sqrt{\varepsilon}(D^0v(x_i) - v'(x))| + |\sqrt{\varepsilon}(D^0w(x_i) - w'(x))|.$$

Using standard truncation error bounds and the bounds on the derivatives of v, we have

$$|D^0v(x_i) - v'(x)| \leq CN^{-2}$$

which gives the required bound in the first term. For the second term we have for $x_i \leq d - \sigma_1$,

$$|\sqrt{\varepsilon}(D^0w(x_i) - w'(x))| \leq C\sqrt{\varepsilon}||w'||_{[0,d-\sigma_1]} \leq CN^{-2},$$

since $\sqrt{\varepsilon}|w'(x)| \leq Ce^{-\alpha\sigma_1/\varepsilon} \leq CN^{-2}$. Also, for $x_i \in \bar{\Omega}_\varepsilon^N \cap [d+\sigma_2, 1-\sigma_2]$, we have

$$|\sqrt{\varepsilon}(D^0 w(x_i) - w'(x))| \leq C\sqrt{\varepsilon}\|w'\|_{[d+\sigma_2,1-\sigma_2]} \leq CN^{-2}.$$

Finally for $x_i \in \bar{\Omega}_\varepsilon^N \cap ((d-\sigma_1, d) \cup (d, d+\sigma_2) \cup (1-\sigma_2, 1))$, we have

$$|\sqrt{\varepsilon}(D^0 w(x_i) - w'(x))| \leq C\sqrt{\varepsilon}(x_{i+1} - x_i)^2|w'''(x_i)| \leq CN^{-2}(\ln N)^2,$$

which completes the proof. $\qquad\square$

Lemma 5. *Let $v(x)$ and $V(x_i)$ be the exact and discrete regular components of the solutions of (1)-(2) respectively. Then for all $x_i \in \Omega_\varepsilon^N$, we have*

$$|\sqrt{\varepsilon}D^0(V-v)(x_i)| \leq CN^{-2}.$$

Proof. We denote the error and the local truncation error, respectively at each mesh point by $e(x_i) = V(x_i) - v(x_i)$ and $\tau(x_i) = L_H^N e(x_i)$. We have

$$|\sqrt{\varepsilon}D^- e(x_{m/2})| = |\frac{\sqrt{\varepsilon}(e(x_{m/2}) - e(x_{m/2-1}))}{x_{m/2} - x_{m/2-1}}| \leq C\sqrt{\varepsilon}m^{-2}. \qquad (34)$$

Now we write $\tau(x_j) = L_H^N e(x_j)$ in the form

$$\varepsilon D^- e(x_j) - \varepsilon D^- e(x_{j+1}) + \frac{1}{2}(x_{j+1} - x_{j-1})a(x_j)D^- e(x_j) = \frac{1}{2}(x_{j+1} - x_{j-1})(\tau(x_j) - b(x_j)e(x_j)),$$

for $x_j \in \Omega_\varepsilon^N \cap (0, d)$.
Multiplying throughout by $(1/\sqrt{\varepsilon})$ and summing, rearranging for each i, $0 < i < m/2$, we get

$$
\begin{aligned}
|\sqrt{\varepsilon}D^- e(x_i)| &\leq |\sqrt{\varepsilon}D^- e(x_{m/2})| + \frac{1}{2}\sum_{j=i}^{m/2-1} \frac{(x_{j+1} - x_{j-1})}{\sqrt{\varepsilon}}(|\tau(x_j)| + |b(x_j)e(x_j)|) \\
&\quad + |\frac{1}{2}\sum_{j=i}^{m/2-1} \frac{(x_{j+1} - x_{j-1})}{\sqrt{\varepsilon}}a(x_j)D^- e(x_j)|.
\end{aligned}
$$

Using the telescoping effect for the last term, (34), $|e(x_i)| \leq Cm^{-2}x_i$ and $|a(x_j) - a(x_{j-1})| \leq \| a' \| (x_j - x_{j-1})$, we get for all i, $0 < i \leq m/2$,

$$|\sqrt{\varepsilon}D^- e(x_i)| \leq Cm^{-2}.$$

Similarly, we can prove that $|\sqrt{\varepsilon}D^+ e(x_i)| \leq Cm^{-2}$ for all i, $0 < i \leq m/2$.
To prove the result for $m/2 + 1 \leq i \leq m$, we rewrite the relation $\tau(x_i) = L_H^N e(x_i)$, in the form,

$$\varepsilon D^+ e(x_j) - \varepsilon D^+ e(x_{j-1}) - \frac{1}{4}(x_{j+1} - x_{j-1})a(x_j)(D^+ e(x_j) + D^+ e(x_{j-1}))$$

$$= \frac{1}{2}(x_{j+1} - x_{j-1})(b(x_j)e(x_j) - \tau(x_j)).$$

Multiplying the above equation throughout by $(1/\sqrt{\varepsilon})$, summing, rearranging and using the telescoping effect for the last term, $|e(x_i)| \leq Cm^{-2}x_i$ and $|a(x_j) - a(x_{j-1})| \leq \| a' \| (x_j - x_{j-1})$, we get for all i, $m/2 < i \leq m$,

$$|\sqrt{\varepsilon}D^+ e(x_i)| \leq Cm^{-2}.$$

Similarly, we can prove that $|\sqrt{\varepsilon}D^-e(x_i)| \le Cm^{-2}$ for all i, $m/2 < i \le m$.

For $x_i \in \Omega_\varepsilon^N \cap (d,1)$, adopting the above procedure, we get

$$|\sqrt{\varepsilon}D^+e(x_i)| \le |\sqrt{\varepsilon}D^+e(x_m)| + \frac{1}{2}\sum_{j=m+1}^{i}(x_{j+1}-x_{j-1})(|b(x_j)e(x_j)| - |\tau(x_j)|) \le Cn^{-2}.$$

Similarly, we can prove that $|\sqrt{\varepsilon}D^-e(x_i)| \le Cn^{-2}$. This implies that

$$|\sqrt{\varepsilon}D^0e(x_i)| \equiv |\frac{\sqrt{\varepsilon}(D^+ + D^-)e(x_i)}{2}| \le CN^{-2}, \quad x_i \in \Omega_\varepsilon^N,$$

which completes the proof. $\square$

Lemma 6. *Let $w(x)$ be the singular component of the solution of (1)-(2) and $W(x_i)$ be the corresponding discrete singular component. Then we have*

$$|W(x_i)| \le \begin{cases} C\sqrt{\varepsilon}m^{-2}x_i, & x_i \in \Omega_\varepsilon^N \cap (0, d-\sigma_1] \\ Cn^{-2}(1-x_i), & x_i \in \Omega_\varepsilon^N \cap [d+\sigma_2, 1-\sigma_2] \end{cases} \tag{35}$$

and

$$|\sqrt{\varepsilon}D^0W(x_i)| \le \begin{cases} Cm^{-2}, & x_i \in \Omega_\varepsilon^N \cap (0, d-\sigma_1] \\ Cn^{-2}, & x_i \in \Omega_\varepsilon^N \cap [d+\sigma_2, 1-\sigma_2]. \end{cases} \tag{36}$$

Proof. To prove (35), we use the barrier functions

$$\Psi^\pm(x_i) = \begin{cases} |W(d-\sigma_1)|\dfrac{x_i}{d-\sigma_1} & x_i \in \Omega_\varepsilon^N \cap (0, d-\sigma_1) \\ |W(1-\sigma_2)|\dfrac{1-x_i}{1-d} & x_i \in \Omega_\varepsilon^N \cap (d+\sigma_2, 1-\sigma_2) \end{cases} \pm |W(x_i)|$$

and Theorem 1, to get the required result.

Finally, to prove (36), we use (35) and the procedure followed in [3, Lemma 3.15] to get $|\sqrt{\varepsilon}D^+W(x_i)| \le CN^{-2}$. Similarly it can be proved that $|\sqrt{\varepsilon}D^-W(x_i)| \le CN^{-2}$. This implies $|\sqrt{\varepsilon}D^0W(x_i)| \le \sqrt{\varepsilon}(|D^+W(x_i)| + |D^-W(x_i)|)/2 \le CN^{-2}$. $\square$

Lemma 7. *Let $w(x)$ and $W(x_i)$ be the exact and discrete singular components of the solutions of (1) and (2) respectively. Then for all $x_i \in \Omega_\varepsilon^N$, we have*

$$|\sqrt{\varepsilon}D^0(W-w)(x_i)| \le CN^{-2}(\ln N)^2.$$

Proof. For all $x_i \in \Omega_\varepsilon^N \cap (0, d-\sigma_1]$ and $x_i \in \Omega^N \cap [d+\sigma_1, 1-\sigma_2]$, using triangle inequality we have

$$|\sqrt{\varepsilon}D^0(W-w)(x_i)| \le |\sqrt{\varepsilon}(D^0W-w')(x_i)| + |\sqrt{\varepsilon}(D^0w-w')(x_i)|.$$

By Lemma 4, it is obvious to see that the second term is bounded. To bound the first term, using triangle inequality, we write it as $|\sqrt{\varepsilon}(D^0W-w')(x_i)| \le |\sqrt{\varepsilon}D^0W(x_i)| + |\sqrt{\varepsilon}w'(x_i)| \le CN^{-2}$. Now consider $x_i \in \Omega_\varepsilon^N \cap (d-\sigma_1, d)$. For convenience we introduce the notation $\hat{e}(x_i) = (W-w)(x_i)$ and $\hat{\tau}(x_i) = L_H^N\hat{e}(x_i)$. We have already established that

$$|\hat{e}(x_i)| \le C\sqrt{\varepsilon}m^{-2}(\ln m)^3 \quad \text{and} \quad |\hat{\tau}(x_i)| \le C\sigma_1^2\varepsilon^{-5/2}m^{-2}e^{-\alpha(d-x_i)/\varepsilon}. \tag{37}$$

We write the equation $\hat{\tau}(x_i) = L_H^N \hat{e}(x_i)$ in the form

$$\varepsilon D^+(\hat{e}(x_{j-1}) - \hat{e}(x_j)) + \frac{1}{2}a(x_j)(x_{j+1} - x_{j-1})D^0\hat{e}(x_j) = \frac{1}{2}(x_{j+1} - x_{j-1})[\hat{\tau}(x_j) - b(x_j)\hat{e}(x_j)].$$

Multiplying throughout by $(1/\sqrt{\varepsilon})$, summing and rearranging gives

$$\sqrt{\varepsilon}D^+\hat{e}(x_i) = \sqrt{\varepsilon}D^+\hat{e}(x_{m-1}) + \frac{1}{2\sqrt{\varepsilon}}\sum_{j=i+1}^{m-1}[a(x_j)(\hat{e}(x_{j+1}) - \hat{e}(x_{j-1})) - h_i[\hat{\tau}(x_j) - b(x_j)\hat{e}(x_j)]]$$

$$\leq \sqrt{\varepsilon}D^+\hat{e}(x_{m-1}) + a(x_{m-1})\hat{e}(x_m) - a(x_i)\hat{e}(x_{i+1}) + a(x_{m-1})\hat{e}(x_{m-1}) - a(x_i)\hat{e}(x_i)$$

$$-\frac{1}{2\sqrt{\varepsilon}}\sum_{j=i+1}^{m-1}[(a(x_j) - a(x_{j-1}))\hat{e}(x_j) + (a(x_j) - a(x_{j-1}))\hat{e}(x_{j-1}) - h_i\hat{\tau}(x_j) + b(x_j)\hat{e}(x_j)].$$

Hence using the result at the point x_{m-1} and (37) we have

$$\sqrt{\varepsilon}D^+\hat{e}(x_i) \leq Cm^{-2}(\ln^3 m + \frac{\sigma_1^2}{\varepsilon^2}\frac{\alpha h_i/\varepsilon}{1 - e^{-\alpha h_2/\varepsilon}}).$$

But $y = \alpha h_2/\varepsilon = 4m^{-1}\ln m$ and $B(y) = \frac{y}{1-e^{-y}}$ are bounded and it follows that $|\sqrt{\varepsilon}D^+\hat{e}(x_i)| \leq Cm^{-2}\ln^2 m$ as required.

For $x_i \in \Omega_\varepsilon^N \cap (d, d + \sigma_2)$ and $x_i \in \Omega_\varepsilon^N \cap (1 - \sigma_2, 1)$, adopting the above procedure, we get respectively

$$\sqrt{\varepsilon}D^+\hat{e}(x_i) = \sqrt{\varepsilon}D^+\hat{e}(x_{m+n/4-1}) + \frac{1}{2\sqrt{\varepsilon}}\sum_{j=i+1}^{m+n/4-1}[h_i[\hat{\tau}(x_j) - b(x_j)\hat{e}(x_j)]]$$

$$\text{and} \quad \sqrt{\varepsilon}D^+\hat{e}(x_i) = \sqrt{\varepsilon}D^+\hat{e}(x_{m+n-1}) + \frac{1}{2\sqrt{\varepsilon}}\sum_{j=i+1}^{m+n-1}[h_i[\hat{\tau}(x_j) - b(x_j)\hat{e}(x_j)]].$$

Hence using the results $|\hat{e}(x_i)| \leq Cn^{-2}(\ln n)^2$ and

$$|\hat{\tau}(x_i)| \leq C\sigma_2^2\varepsilon^{-1}n^{-2}\big(e^{-(x_i-d)\sqrt{\gamma/\varepsilon}} + e^{-(1-x_i)\sqrt{\gamma/\varepsilon}}\big),$$

we have

$$\sqrt{\varepsilon}D^+\hat{e}(x_i) \leq Cn^{-2}(\ln^2 n + \frac{\sigma_2^2}{\varepsilon}\frac{h_i\sqrt{\gamma/\varepsilon}}{1 - e^{-h_i\sqrt{\gamma/\varepsilon}}}) \leq Cn^{-2}(\ln n)^2.$$

Similarly, we get $|\sqrt{\varepsilon}D^-\hat{e}(x_i)| \leq Cn^{-2}(\ln n)^2$. Thus, we have

$$|\sqrt{\varepsilon}D^0 e(x_i)| \equiv |\frac{\sqrt{\varepsilon}(D^+ + D^-)e(x_i)}{2}| \leq CN^{-2}\ln^2 N.$$

$\square$

Theorem 5. *Let $u(x)$ be the solution of (1), (2) and $U(x_i)$ the corresponding numerical solution generated by the difference scheme HSD-I. Then for each i, $1 \leq i \leq N - 1$ we have*

$$|\sqrt{\varepsilon}(D^0 U - u')(x_i)| \leq CN^{-2}(\ln N)^2.$$

Proof. From triangular inequality we have $|\sqrt{\varepsilon}(D^0 U(x_i) - u'(x))| \le |\sqrt{\varepsilon}D^0(U - u)(x_i)| + |\sqrt{\varepsilon}(D^0 u(x_i) - u'(x))|$. From Lemma 4 we get $|\sqrt{\varepsilon}(D^0 u(x_i) - u'(x))| \le CN^{-2}(\ln N)^2$. To bound $|\sqrt{\varepsilon}D^0(U - u)(x_i)|$, it can be written as

$$|\sqrt{\varepsilon}D^0(U - u)(x_i)| \le |\sqrt{\varepsilon}D^0(V - v)(x_i)| + |\sqrt{\varepsilon}D^0(W - w)(x_i)| \le CN^{-2}(\ln N)^2,$$

where each term is bounded by Lemmas 5 and 7. $\qquad\square$

Remark 2. *Let $\bar{U}$ denote the piecewise linear interpolant of the finite difference solution $\{U(x_i)\}_{i=0}^N$. As done in [3, p.66], we get*

$$\sup_{0 < \varepsilon \le 1} \| \sqrt{\varepsilon}(\bar{D}^0 U - u') \|_{\overline{\Omega}_i} \quad \le \quad CN^{-2}(\ln N)^2, \, i = 1, ..., N - 1$$

where, $\bar{D}^0 U(x) = D^0 U(x_i)$, for $x \in (x_{i-1}, x_i]$, $i = 1, ..., N$.

We can also obtain the $\varepsilon-$uniform error estimate between the scaled derivative of the continuous solution and the corresponding numerical solution in the fine mesh region. Further, in the coarse mesh, an estimate can be obtained without scaling the derivative. As done in [9], we get

$$\sup_{0 < \varepsilon \le 1} \| \bar{D}^- U - u' \|_{\overline{\Omega}_i} \quad \le \quad CN^{-1}, \, i = 1, ..., m/2,$$

$$\sup_{0 < \varepsilon \le 1} \| \varepsilon(\bar{D}^0 U - u') \|_{\overline{\Omega}_i} \quad \le \quad CN^{-2}(\ln N)^2, \, i = m/2 + 1, ..., m,$$

$$\sup_{0 < \varepsilon \le 1} \| \varepsilon(\bar{D}^0 U - u') \|_{\overline{\Omega}_i} \quad \le \quad CN^{-2}(\ln N)^2, \, i = m + 1, ..., m + n/4 - 1, m + 3n/4 + 1, ..., N - 1$$

$$\sup_{0 < \varepsilon \le 1} \| \bar{D}^+ U - u' \|_{\overline{\Omega}_i} \quad \le \quad CN^{-1}, \, i = m + n/4, ..., m + 3n/4 + 1$$

where, $\bar{D}^0 U(x) = D^0 U(x_i)$, for $x \in [x_{i-1}, x_i]$, $i = m/2+1, ..., m+n/4$, $\bar{D}^- U(x) = D^- U(x_i)$, for $x \in (x_{i-1}, x_i]$, $i = 1, ..., m/2$ and $\bar{D}^+ U(x) = D^+ U(x_i)$, for $x \in (x_{i-1}, x_i]$, $i = m + 1, ..., m + n/4 - 1, m + 3n/4 + 1, ..., N - 1$.

Remark 3. *For the HSD-II, the numerical derivative estimate will be given in the future article. However, numerical results are given for the scaled discrete derivative generated by the HSD-II.*

6. Numerical Experiments

In this section, we consider the following examples to illustrate the results obtained in the paper.

Example 1. [6]*:*

$$-\varepsilon u'' + (1 + \cos(\pi x))u' + (1 + \sin(\pi x/2))u = 1 + \sin(\pi x)\cos(\pi x), \, x \in \Omega^-,$$
$$-\varepsilon u'' + (4 + \cos(\pi x/2))u = 3 + 2\sin(\pi x/2)\cos(\pi x/2), \, x \in \Omega^+,$$
$$u(0) = 0, \quad u(1) = 0.$$

Example 2.

$$-\varepsilon u'' + (1 + x^2)u' = 2, \, x \in \Omega^-, \, -\varepsilon u'' + (4 + x^3)u = 1.8x, \, x \in \Omega^+,$$
$$u(0) = 0, \quad u(1) = 0.$$

The nodal errors and their corresponding orders of convergence are estimated using the double mesh principle [3]. Define the parameter uniform double mesh nodal difference D_ε^N to be

$$D^N = \max_{\varepsilon \in R_\varepsilon} D_\varepsilon^N \quad \text{and} \quad D_\varepsilon^N = \max_{x_i \in \bar{\Omega}_\varepsilon^N} |(Y^N - \bar{Y}^{4096})(x_i)|,$$

$$S^N = \max_{\varepsilon \in R_\varepsilon} S_\varepsilon^N \quad \text{and} \quad S_\varepsilon^N = \max_{x_i \in \bar{\Omega}_\varepsilon^N} |\sqrt{\varepsilon}(D^0 Y^N - \bar{D}^0 Y^{4096})(x_i)|,$$

where $\bar{Y}^{4096}$ is the piecewise linear interpolant of the mesh function Y^{4096} onto $[0, 1]$. Here R_ε is the range of singular perturbation parameters $\varepsilon \in R_\varepsilon = \{2^{-10}, ..., 2^{-35}\}$, over which numerical performance of the schemes will be tested. From these quantities the parameter-robust orders of convergence are computed from
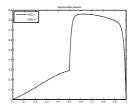
$$p^N = \log_2\left(\frac{D^N}{D^{2N}}\right), \quad r^N = \log_2\left(\frac{S^N}{S^{2N}}\right).$$

TABLE 1.   *Values of D^N, p^N and S^N, r^N for the Example 1*

	Number of mesh points N					
	32	64	128	256	512	1024
HDS-I						
D^N	1.8000e-2	8.0004e-3	3.0108e-3	1.1106e-3	3.9050e-4	1.3419e-4
p^N	1.1699	1.4099	1.4385	1.5089	1.5410	-
S^N	1.2549e-1	8.3974e-2	4.8036e-2	2.1621e-2	8.1242e-3	2.7595e-3
r^N	0.5795	0.8058	1.1517	1.4121	1.5578	-
HDS-II						
D^N	2.4857e-2	9.2016e-3	3.1824e-3	1.0849e-3	3.4942e-4	1.3259e-4
p^N	1.4337	1.5318	1.5526	1.6345	1.3259	-
S^N	1.4054e-1	9.0605e-2	4.6597e-2	1.9953e-2	7.3446e-3	2.3112e-3
r^N	0.6363	0.9593	1.2236	1.4418	1.6680	-

TABLE 2.   *Values of D^N, p^N and S^N, r^N for the Example 2*

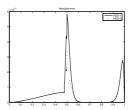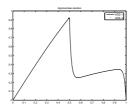|	Number of mesh points N					
	32	64	128	256	512	1024
HDS-I						
D^N	1.8730e-2	8.4711e-3	3.1172e-3	1.0470e-3	3.3521e-4	1.0136e-4
p^N	1.1447	1.4423	1.5740	1.6481	1.7256	-
S^N	1.2489e-1	7.3523e-2	3.4575e-2	1.5510e-2	6.1005e-3	2.0632e-3
r^N	0.7643	1.0885	1.1565	1.3462	1.5640	-
HDS-II						
D^N	2.4340e-2	9.0718e-3	3.1161e-3	1.0627e-3	3.4299e-4	1.2923e-4
p^N	1.4239	1.5416	1.5520	1.6315	1.4082	-
S^N	1.3596e-1	8.8353e-2	4.5690e-2	1.9637e-2	7.2455e-3	2.2515e-3
r^N	0.6218	0.9514	1.2183	1.4384	1.6862	-

FIGURE 1. *Approximate solution and error for the methods HDS-I and HDS-II of the Example 1 for $\varepsilon = 2^{-10}$ with $N = 128$.*
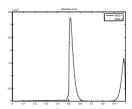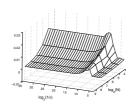

FIGURE 2. *Approximate solution and error for the methods HDS-I and HDS-II of the Example 2 for $\varepsilon = 2^{-10}$ with $N = 128$.*
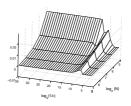

FIGURE 3. *Surface plots of the maximum pointwise errors as a function of N and ε for the solution generated by methods HDS-I and HDS-II of the Example 1.*

From the tables, the performance of the two schemes appears to be almost the same but these two schemes are derived from different methods. It is expected that they may significantly differ for certain problems as the truncation error derived for HDS - II is smaller than HDS - I.

In Fig 3, the maximum pointwise errors D_ε^N at the mesh points for the Example 1 are plotted as function of N and ε. Note that for all values of $\varepsilon \in \{2^{-10}, ..., 2^{-35}\}$, that is, the case $\varepsilon \leq N^{-1}$, the error decreases steadily with increasing N whereas for all values of $\varepsilon \in \{2^{-1}, ..., 2^{-9}\}$, that is, the case $\varepsilon \geq N^{-1}$, the error increases and the ridge of persistent error is immediately apparent.

7. Conclusion

A mixed type singularly perturbed one dimensional elliptic problem with discontinuous source term was examined. The domain under consideration was partitioned into two subdomains. A convection-diffusion and a reaction-diffusion type equations are posed on the first and second subdomains respectively. Two hybrid difference schemes on the Shishkin mesh were constructed for solving this problem which generates almost second order $\varepsilon-$uniform convergent numerical approximation to the solution as well as to the scaled first derivative of the solution. Numerical results were presented which are in agreement with the theoretical results.

References

[1] J.J.H. Miller, E. O'Riordan, G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions*, World Scientific Publishing Co. Pte. Ltd, 1996.

[2] H.G. Roos, M. Stynes, L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*, Springer-Verlag, Berlin Heidelberg, 1996.

[3] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, G.I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman and Hall/CRC Press, Boca Raton, U.S.A., 2000.

[4] M. Stynes, H.G. Roos, *The Mid-point Upwind Scheme*, Applied Numerical Mathametics, **23** (1997), 361-374.

[5] S. Natesan, B. Sundar Deb, *A Robust Computational Method for Singularly Perturbed Coupled System of Reaction-Diffusion Boundary-Value Problems*, Journal of Applied Mathematics and Computation, **188** (2007), 353-364.

[6] Iliya A. Brayanov, *Uniformly Convergent Difference Scheme for Singularly Perturbed Problem of Mixed Type*, Electronic Transaction on Numerical Analysis, **23** (2006), 288-303.

[7] P.A. Farrel, J.J.H. Miller, E. O'Riordan, G.I. Shishkin, *Singularly Perturbed Differential Equations with Discontinuous Source Terms*, Proceedings of "Analytical and Numerical Methods for Convection-Dominated and Singularly Perturbed Problems", Lozenetz, Bulgaria, 1998, J.J.H.Miller, G.I.Shishkin and L.Vulkov eds., Nova Science Publishers. New York, 23-32, USA (2000).

[8] P.A. Farrel, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, G.I. Shishkin, *Global Maximum-Norm Parameter-Uniform Numerical Method for a Singularly Perturbed Convection-Diffusion Problem with Discontinuous Convection Coefficient*, Mathematical and Computer Modelling,**40** (2004), 1375-1392.

[9] R. K. Dunne, E. O'Riordan, *Interior layers arising in linear singularly perturbed differential equations with discontinuous coefficients*, Proceedings of the fourth International Conference on finite difference methods: Theory and Applications, Lozenetz, Bulgaria, August 26–29, 2006 (I. Farago, p. Vabishchevich and L. Vulkov eds.), Rousse University, Bulgaria, 29–38, 2007.

[10] Z. Cen, *A Hybrid Difference Scheme for a Singularly Perturbed Convection-Diffusion Problem with Discontinuous Convection Coefficient*, Applied Mathematics and Computation, **169** (2005), 689-699.

[11] T. Valanarasu, N. Ramanujam, *An Asymptotic Numerical Method for Singularly Perturbed Third Order Ordinary Differential Equations With Weak Interior Layer*, International Journal of Computer Mathematics, **84** (2007), 333-346.

[12] R. Mythili Priyadharshini, N. Ramanujam, *Approximation of Derivative to a Singularly Perturbed Second-Order Ordinary Differential Equation with Discontinuous Convection Coefficient Using Hybrid Difference Scheme*, International Journal of Computer Mathematics, **86 (8)** (2009), 1355-1369.

[13] R. Mythili Priyadharshini and N. Ramanujam, *Approximation of derivative for a singularly perturbed second-order ordinary differential equation of Robin type with discontinuous convection coefficient and source term*, Numer. Math. Theor. Meth. Appl. **2(1)** (2009), 100-118.

[14] R. Mythili Priyadharshini and N. Ramanujam, *Hybrid Difference Schemes for a System of Singularly Perturbed Convection-Diffusion Equations*, J. Appl. Math. & Informatics, **27(5-6)**(2009), 1001-1015.

[15] J.L. Gracia, E. O'Riordan, M. L. Pickett, *A Parameter Robust Highedr Order Numerical Method for a Singularly Perturbed Two-Parameter Problem*, Applied Numerical Mathematics, **56** (2006), 962-980.

N. Ramanujam is working as a Professor and Head, Department of Mathematics; Chair, School of Mathematical Sciences and Dean, Faculty of Engineering & Technology, Bharathidasn University. His area of interests are Differential Equations, Differential Inequalities and Numerical Analysis.

Department of Mathematics, School of Mathematics and Computer Science, Bharathidasn University, Tiruchirappalli - 620 024, Tamilnadu, India.
e-mail: matram@bdu.ac.in, Website: http://www.bdu.ac.in

R. Mythili Priyadharshini is doing her Ph. D at Bharathidasan University under the direction of Dr. N. Ramanujam. Her area of interests are Differential Equations and Numerical Analysis.

Department of Mathematics, School of Mathematics and Computer Science, Bharathidasn University, Tiruchirappalli - 620 024, Tamilnadu, India.
e-mail: mythiliroy777@yahoo.co.in

T. Valanarasu is working as a Lecturer, Department of Mathematics, Bharathidasan University College. His area of interests are Differential Equations and Numerical Analysis.

Department of Mathematics, Bharathidasan University College, Perambalur-621 212, Tamilnadu, India.
e-mail: valan_tmj@yahoo.co.in