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PERIODIC SOLUTIONS OF VOLTERRA EQUATIONS

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Abstract. We study the existence of periodic solutions of Volterra equations by using the limiting equations and contraction mappings.

1. Introduction

Miller [7] studied forced oscillations in a nonlinear system of Volterra integral equations of the form

$$\begin{aligned} x_1(t) = f_1(t) - & \int_0^t a_1(t-s) g_1(s, x_1(s), x_2(s)) ds \\ - & \int_0^t a_2(t-s) g_2(s, x_1(s), x_2(s)) ds, \\ x_2(t) = f_2(t) - & \int_0^t a_2(t-s) g_1(s, x_1(s), x_2(s)) ds \\ - & \int_0^t a_1(t-s) g_2(s, x_1(s), x_2(s)) ds. \end{aligned} \tag{1.1}$$

where the functions $f_i(t)$ and $g_i(t, x_1, x_2), i = 1, 2$, are asymptotically almost periodic in t. (1.1) arises in a natural way from the initial boundary

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value problem:

$$\begin{aligned} u_t &= u_{xx}, & t > 0,\ 0 < x < \pi, \\ u(0,x) &= F(x), & 0 < x < \pi, \\ u_x(t,0) &= g_1(t, u(t,0), u(t,\pi)), & t > 0, \\ u_x(t,\pi) &= -g_2(t, u(t,0), u(t,\pi), & t > 0. \end{aligned} \tag{1.2}$$

The boundary conditions in this diffusion problem (1.2) are motivated by the theory of superfluidity of liquid helium [7]. Also, see [6].

Burton and Furumochi [1] studied the existence of periodic solutions of

$$x(t) = a(t) - \int_0^t D(t,s,x(s))ds,\ t \in \mathbb{R}^+ = [0,\infty), \tag{1.3}$$

and its limiting equation

$$x(t) = p(t) - \int_{-\infty}^t P(t,s,x(s))ds,\ t \in \mathbb{R} = (-\infty,\infty), \tag{1.4}$$

by using techniques on limiting equations, Liapunov functions, the theory of minimal solutions, and contraction mappings. Also, they investigated the existence of almost periodic solutions of (1.3) and (1.4) in [3].

Furumochi [5] obtained discrete analogues of the results in [1], that is, he obtained the existence of periodic solution of the Volterra difference equations

$$x(n+1) = a(n) - \sum_{k=0}^{n} D(n,k,x(k)),\ n \in \mathbb{Z}^+, \tag{1.5}$$

and

$$x(n+1) = p(n) - \sum_{-\infty}^{n} P(n,k,x(k)),\ n \in \mathbb{Z}. \tag{1.6}$$

For the asymptotic property of linear Volterra difference equations, see [4].

In this paper, we investigate the existence of bounded periodic solutions of (1.3) and (1.4). This study complements [1].

2. Main Results

We are concerned with systems of Volterra equations

$$x(t) = a(t) - \int_0^t D(t,s,x(s))ds,\ t \in \mathbb{R}^+ = [0,\infty), \tag{2.1}$$

$$x(t) = a(t) - \int_{-\infty}^t D(t,s,x(s))ds,\ t \in \mathbb{R} = (-\infty,\infty), \tag{2.2}$$

and

$$x(t) = p(t) - \int_{-\infty}^t P(t,s,x(s))ds,\ t \in \mathbb{R}, \tag{2.3}$$

where $a : \mathbb{R} \to \mathbb{R}^n$, $p : \mathbb{R} \to \mathbb{R}^n$, $D : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $P : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, and

$$p(t+T) = p(t),\ \ q(t) := a(t) - p(t) \to 0 \text{ as } t \to \infty, \tag{2.4}$$

where $T > 0$ is a constant,

$$P(t+T, s+T, x) = P(t,s,x),\ \ Q(t,s,x) := D(t,s,x) - P(t,s,x), \tag{2.5}$$

and for any $J > 0$ there are continuous functions $P_J : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and $Q_J : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ such that

$$\begin{aligned} P_J(t+T, s+T) &= P_J(t,s) \text{ if } t,s \in \mathbb{R}, \\ |P(t,s,x)| &\leq P_J(t,s) \text{ if } t,s \in \mathbb{R} \text{ and } |x| \leq J, \end{aligned}$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$, and $|Q(t,s,x)| \leq Q_J(t,s)$ if $t,s \in \mathbb{R}$ and $|x| \leq J$,

$$\int_{-\infty}^t P_J(t+\tau, s)ds \to 0 \text{ uniformly for } t \in \mathbb{R} \text{ as } \tau \to \infty \tag{2.6}$$

$$\int_0^t P_J(t,s)ds \to 0 \text{ as } t \to \infty \tag{2.7}$$

or

$$\int_{-\infty}^t Q_J(t,s)ds \to 0 \text{ as } t \to \infty \tag{2.8}$$

and

$$\int_{-\infty}^t Q_J(t+\tau, s)ds \to 0 \text{ uniformly for } t \in \mathbb{R} \text{ as } \tau \to \infty.$$

First we obtain a relation between solution of (2.2) and

$$x(t) = p(t+\sigma) - \int_{-\infty}^t P(t+\sigma, s+\sigma, x(s))ds,\ t \in \mathbb{R}, \tag{2.9}$$

where $0 \leq \sigma < T$.

THEOREM 2.1. *Under the assumptions (2.4), (2.5), (2.6) and (2.8), we suppose that (2.2) has an* $\mathbb{R}$*-bounded solution* $x(t)$ *with an initial time in* $\mathbb{R}$*. Let* (s_k) *be a sequence in* $\mathbb{R}$ *with* $s_k \to \infty$ *as* $k \to \infty$*. Then the sequence* $(x_k(t))$ *converges to an* $\mathbb{R}$*-bounded solution* $y(t)$ *of (2.9) uniformly on any compact subset of* $\mathbb{R}$ *as* $k \to \infty$*, where* $x_k(t) := x(t + s_k),\ t \in \mathbb{R}$.

Proof. Since $x(t)$ is $\mathbb{R}$-bounded, the set$\{x_k(t) : t \in \mathbb{R}\}$ is uniformly bounded on $\mathbb{R}$. From (2.4), (2.5), and (2.8) we deduce that $x(t)$ is uniformly continuous on $\mathbb{R}$. Since $x_k(t)$ is obtained by an s_k-translation to the left of $x(t)$, the set $\{x_k(t) : t \in \mathbb{R}\}$ is equicontinuous. By the Ascoli's theorem, the sequence $(x_k(t))$ converges to some $\mathbb{R}$-bounded continuous function $y(t)$ uniformly on any compact subset of $\mathbb{R}$ as $k \to \infty$.

Now, we show that $y(t)$ satisfies (2.9) on $\mathbb{R}$. For any $k \in \mathbb{N}$, let ν_k be an integer with $\nu_k T \leq s_k < \nu_{k+1} T$. Let $\sigma_k = s_k - \nu_k T$. By taking a subsequence if necessary, we may assume that (σ_k) converges to some σ with $0 \leq \sigma < T$. From (2.2), we have

$$\begin{aligned} x_k(t) &= x(t + s_k) \\ &= a(t + s_k) - \int_{-\infty}^{t+s_k} D(t + s_k, s, x(s))ds \\ &= p(t + \sigma_k) + q(t + s_k) - \int_{-\infty}^{t+s_k} P(t + \sigma_k, s + \sigma_k, x(s + s_k))ds \\ &\qquad - \int_{-\infty}^{t+s_k} Q(t + s_k, s, x(s))ds. \end{aligned} \tag{2.10}$$

Note that $p(t + \sigma_k) \to p(t + \sigma)$ and $q(t + s_k) \to 0$ as $k \to \infty$. Let $J > 0$ be a number with $|x| \leq J$. From (2.6), we obtain that for any $\epsilon > 0$ there exists a $\tau > 0$ such that

$$\int_{-\infty}^{t} P_J(t + \tau, s)ds < \epsilon,\ t \in \mathbb{R}. \tag{2.11}$$

In view of (2.8) we have

$$\begin{aligned} \limsup_{k\to\infty} &\left|\int_{-\infty}^{t+s_k} Q(t + s_k, s, x(s))ds\right| \\ &\leq \limsup_{k\to\infty} \int_{-\infty}^{t+s_k} |Q(t + s_k, s, x(s))|ds \\ &= 0. \end{aligned} \tag{2.12}$$

Also,

$$\begin{aligned}
&\limsup_{k\to\infty} \Big| \int_{-\infty}^{t+s_k} P(t+\sigma_k, s+\sigma_k, x(s+s_k))ds \\
&\qquad\qquad - \int_{-\infty}^{t} P(t+\sigma, s+\sigma, y(s))ds \Big| \\
&\le \limsup_{k\to\infty} \Big| \int_{t}^{t+s_k} [P(t+\sigma_k, s+\sigma_k, x_k(s)) \qquad (2.13)\\
&\qquad\qquad - P(t+\sigma, s+\sigma, y(s))]ds \Big| \\
&\quad + \limsup_{k\to\infty} \int_{-\infty}^{t} P_J(t+\sigma_k, s+\sigma_k)ds + \int_{-\infty}^{t} P_J(t+\sigma, s+\sigma)ds \\
&< \epsilon + \epsilon = 2\epsilon,
\end{aligned}$$

by (2.6) and (2.11). Hence it follows from (2.12) and (2.13) that

$$y(t) = p(t+\sigma) - \int_{-\infty}^{t} P(t+\sigma, s+\sigma, y(s))ds, \ t \in \mathbb{R},$$

by letting $k \to \infty$ in (2.10). This completes the proof. □

DEFINITION 2.2. A function $f : \mathbb{R} \to \mathbb{R}^n$ is called *asymptotically T-periodic, $T > 0$ is a constant, if $f = g + h$, where g is T-periodic, i.e, $g(t+T) = g(t)$ for all $t \in \mathbb{R}$, and* $\lim_{t\to\infty} h(t) = 0$.

THEOREM 2.3. *Suppose that (2.4), (2.5), (2.6) and (2.8). If (2.3) has a unique $\mathbb{R}$-bounded solution $x_0(t)$ on $\mathbb{R}$, then the following hold:*

(i) $x_0(t)$ is T-periodic.

(ii) Any $\mathbb{R}$-bounded solution $x(t)$ of (2.2) with an initial time in $\mathbb{R}$ is asymptotically T-periodic and approaches to $x_0(t)$ as $t \to \infty$.

Proof. (i) Let $x_1(t) = x_0(t+T)$, $t \in \mathbb{R}$. We show that $x_1(t) = x_0(t)$ for all $t \in \mathbb{R}$. Since $x_0(t)$ is a unique $\mathbb{R}$-bounded solution of (2.2) on $\mathbb{R}$, $x_1(t)$ is also an $\mathbb{R}$-bounded solution of (2.2) on $\mathbb{R}$. From the uniqueness of solutions, we have $x_1(t) = x_0(t)$ for all $t \in \mathbb{R}$.

(ii) We show that $x(t) \to x_0(t)$ as $t \to \infty$. Let $x_k(t) = x(t+s_k)$ with $s_k = kT$. Then, by Theorem 2.1,

$$x_k(t) \to y(t)$$

uniformly on any compact subset of $\mathbb{R}$ as $k \to \infty$, where $y(t)$ is an $\mathbb{R}$-bounded solution of (2.9) with $0 \le \sigma < T$, and thus is an $\mathbb{R}$-bounded solution of (2.2) when $\sigma = 0$. Also, $y(t) = x_0(t)$ from the uniqueness of solutions. Thus $x(t) = x_0(t) + \varphi(t)$, where $\lim_{t\to\infty} \varphi(t) = 0$. This implies that $x(t)$ is asymptotically T-periodic. This completes the proof. □

THEOREM 2.4. [1] *Suppose that (2.4), (2.5) and (2.6) with* $q(t) \equiv 0$ *and* $Q(t, s, x) \equiv 0$. *Assume that for any* $J > 0$ *there exists a continuous function* $L_J : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ *such that*

$$|P(t,s,x) - P(t,s,y)| \le L_J(t,s)|x-y| \tag{2.14}$$

when $t, s \in \mathbb{R}$ *and* $|x|, |y| \le J$. *Let*

$$\lambda_J := \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} L_J(t,s)ds < 1$$

and

$$\lambda := \sup_{J>0} \lambda_J < 1. \tag{2.15}$$

Then

(i) (2.3) has a unique $\mathbb{R}$*-bounded* T*-periodic solution on* $\mathbb{R}$.

(ii) Any $\mathbb{R}$*-bounded solution of (2.3) with initial time* $t_0 \in \mathbb{R}$ *and bounded continuous initial function* $\varphi : (-\infty, t_0) \to \mathbb{R}^n$ *approaches to the* T*-periodic solution.*

Consider the linear Volterra equation

$$x(t) = p(t) - \int_{-\infty}^{t} P(t,s)x(s)ds, t \in \mathbb{R}, \tag{2.16}$$

where $p : \mathbb{R} \to \mathbb{R}^n$ *and* $P : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous.*

THEOREM 2.5. [1] *If*

$$b(t) := \int_{-\infty}^{t} |P(t,s)|ds < 1, t \in \mathbb{R} \tag{2.17}$$

holds, then for any $t_0 \in \mathbb{R}$ *and any bounded continuous function* $\varphi : (-\infty, t_0) \to \mathbb{R}^n$, *the solution* $x(t) = x(t, t_0, \varphi)$ *of (2.16) satisfies*

$$|x(t)| \le X(t) := \max\left\{ \sup_{t_0\le s\le t} B(s),\ \sup_{s\le t_0} |\varphi(s)|, |x(t_0+)| \right\},\ t \ge t_0,$$

where

$$B(s) := \frac{1}{1-b(s)} \sup_{t_0\le u\le s} |p(u)|,\ s \ge t_0.$$

Now, we obtain the periodicity and attractivity of solution of the linear equation (2.16).

DEFINITION 2.6. The solution $x(t)$ of (2.16) is said to be *globally attractive* if

$$\lim_{t\to\infty}[x(t)-y(t)]=0$$

for any solution $y(t)$ of (2.16).

THEOREM 2.7. *Suppose that $p(t+T)=p(t)$ and $P(t+T,s+T)=P(t,s),t,s\in\mathbb{R}$. In addiction to (2.17), if $b(t)$ is continuous, then (2.16) has a unique $\mathbb{R}$-bounded solution on $\mathbb{R}$ which is T-periodic and globally attractive.*

Proof. In view of Theorem 2.5, the solution $x(t)$ of (2.16) satisfies

$$|x(t)|\le X(t),\ t\ge t_0,t_0\in\mathbb{R},$$

that is, $x(t)$ is $\mathbb{R}$-bounded. From Theorem 2.4, $x(t)$ is T-periodic. Also, $x(t)$ is globally attractive by Theorem 2.4. □

In (2.5), we let $Q(t,s,x)=0$. So we consider

$$x(t)=a(t)-\int_0^t P(t,s,x(s))ds,t\in\mathbb{R}^+, \tag{2.18}$$

where $a:\mathbb{R}^+\to\mathbb{R}^n$ is bounded continuous and $P:\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous.

THEOREM 2.8. *Suppose that (2.4), (2.5) and (2.6) with $Q(t,s,x)\equiv 0$. Under the assumptions (2.14) and (2.15), the following hold:*

(i) *(2.18) has a unique $\mathbb{R}^+$-bounded solution $x(t)$ on $\mathbb{R}^+$.*
(ii) *(2.3) has a unique T-periodic solution $\pi(t)$ on $\mathbb{R}$.*
(iii) *$x(t)\to\pi(t)$ as $t\to\infty$.*

Proof. (i) Let B be the Banach space of all bounded continuous functions $\xi:\mathbb{R}^+\to\mathbb{R}^n$ with

$$\|\xi\|=\sup_{t\ge 0}|\xi(t)|.$$

Define H on B by

$$(H\xi)(t):=a(t)-\int_0^t P(t,s,\xi(s))ds.$$

Then we have

$$|(H\xi)(t)|\le|a(t)|+\int_0^t|P(t,s,\xi(s))|ds.$$

Thus, from (2.6), $H\xi$ is bounded. It follows that $H(B)\subset B$.

We show that H is a contraction. To do this we let $\xi_1, \xi_2 \in H$ with $\|\xi_1\|, \|\xi_2\| \leq J$ for some $J > 0$. Then

$$\begin{aligned} |(H\xi_1)(t) - (H\xi_2)(t)| &\leq \int_0^t |P(t,s,\xi_1(s)) - P(t,s,\xi_2(s))|ds \\ &\leq \int_0^t L_J(t,s)|\xi_1(s) - \xi_2(s)|ds \\ &\leq \lambda_J\|\xi_1 - \xi_2\| \\ &< \lambda\|\xi_1 - \xi_2\|, \end{aligned}$$

by (2.14) and (2.15). This implies that H is a contraction. Hence H has a unique fixed point $x(t)$ of H by the Contraction Mapping Principle.

(ii) Let $x(t)$ denote again $\mathbb{R}$-extension of the given $x(t)$ obtain by defining

$$\begin{cases} x(t) = x(0) = a(0) & \text{for } t < 0, \\ x(t) & \text{for } 0 \leq t < \infty. \end{cases}$$

For any $k \in \mathbb{N}$, set $x_k(t) = x(t + kT)$, $t \in \mathbb{R}$. In view of Theorem 2.4, (2.3) has a unique T-periodic solution, say $\pi(t)$ in $\mathbb{R}$. Therefore $\pi(t)$ is a unique $\mathbb{R}$-bounded solution of (2.3) by Theorem 2.1.

(iii) We can deduce that $x(t) - \pi(t) \to 0$ as $k \to \infty$ since we can show that $x_k \to \pi(t)$ as $k \to \infty$ uniformly on $[0, T]$ as in the proof of Theorem 2.1. This proves the theorem.

□

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