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h-STABILITY OF THE NONLINEAR DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

Yoon Hoe Goo*

Abstract. In this paper, we investigate h-stability of the nonlinear differential systems using the notion of t_∞-similarity.

1. Introduction and basic facts

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ and $f(t, 0) = 0$. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x|\leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \ v(t_0) = v_0 \tag{1.2}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \ z(t_0) = z_0. \tag{1.3}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.2).

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We recall some notions of h-stability [11].

DEFINITION 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called *h-stable (hS)* if there exist $c \geq 1$, $\delta > 0$, and a positive bounded continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$, and is called *h-stable in variation (hSV)* if (1.3) (or $z = 0$ of (1.3)) is h-stable.

The notion of h-stability (hS) was introduced by Pinto [11, 12] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems.

Choi et al. studied the important properties about hS for the various differential systems [3] and for hS of nonlinear differential systems via t_∞-similarity [4].

Goo et al. investigated hS for the nonlinear Volterra integro-differential system [8] and for the linear perturbed Volterra integro-differential systems [7].

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [5].

DEFINITION 1.2. A matrix $A(t) \in \mathcal{M}$ is *t_∞-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.4}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [5, 9].

In this paper, we investigate h-stability of the nonlinear differential systems using the notion of t_∞-similarity.

We give some related properties that we need in the sequal.

LEMMA 1.3. [12] *The linear system*

$$x' = A(t)x, \; x(t_0) = x_0, \tag{1.5}$$

where $A(t)$ is an $n \times n$ continuous matrix, is hS if and only if there exist $c \geq 1$ and a positive bounded continuous function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{1.6}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.5).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \; y(t_0) = y_0, \tag{1.7}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.7) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 1.4. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

THEOREM 1.5. [3] *If the zero solution of (1.1) is hS, then the zero solution of (1.2) is hS.*

THEOREM 1.6. [4] *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.2) is hS, then the solution $z = 0$ of (1.3) is hS.*

THEOREM 1.7. [10] *Let $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, and $f_x = \partial f / \partial x$ exist and be continuous on $\mathbb{R}^n \times \mathbb{R}^n$. Assume that $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ are any two solutions of (1.1) through (t_0, x_0) and (t_0, y_0), respectively, existing for $t \geq t_0$, such that x_0, y_0 belong to a convex subset of $\mathbb{R}^n$. Then*

$$x(t, t_0, x_0) - x(t, t_0, y_0) = \left[\int_0^1 \Phi(t, t_0, sx_0 + (1-s)y_0) ds \right] (x_0 - y_0).$$

hods for $t \geq t_0$.

We need to modify Theorem 3.6 in [3] into the following:

THEOREM 1.8. *Suppose that the solution $x = 0$ of (1.1) is hS with a nondecreasing function h and the perturbed term g in (1.7) satisfies*

$$|\Phi(t, s, z)g(t, z)| \leq \gamma(s)|z|, \ t \geq t_0 \geq 0,$$

where $\gamma \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\int_{t_0}^{\infty} \gamma(s)ds < \infty$. Then $y = 0$ of (1.7) is hS.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.7), respectively. By Lemma 1.4, we obtain

$$\begin{aligned} |y(t) &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))g(s, y(s))|ds \\ &\leq c|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t} \gamma(s)|y(s)|ds. \end{aligned}$$

Thus, it follows from the Gronwall inequality that we have

$$\begin{aligned} |y(t)| &\leq c|y_0|h(t)h(t_0)^{-1} \exp(\int_{t_0}^{t} \gamma(s)ds) \\ &\leq c_1|y_0|h(t)h(t_0)^{-1}, \ t \geq t_0, \end{aligned}$$

where $c_1 = c \exp(\int_{t_0}^{\infty} \gamma(s)ds)$. This implies that $y = 0$ of (1.7) is hS. □

2. Main Results

In this section, we investigate hS for the nonlinear differential systems via t_∞-similarity.

THEOREM 2.1. *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. Then the solution $v = 0$ of (1.2) is hS if and only if the solution $z = 0$ of (1.3) is hS.*

Proof. First, suppose $v = 0$ of (1.2) is hS. Then by Theorem 1.6, the solution $z = 0$ of (1.3) is hS.

Conversely, suppose the solution $z = 0$ of (1.3) is hS. Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1). Then by Theorem 1.7, we have

$$x(t, t_0, x_0) = \left[\int_0^1 \Phi(t, t_0, sx_0)ds \right] x_0.$$

By Lemma 1.3, since the solution $z = 0$ of (1.3) is hS, there exist $c \geq 1$ and a positive bounded continuous function h on $\mathbb{R}^+$ such that

$$|\Phi(t, t_0, x_0)| \leq c\, h(t)\, h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$, where $\Phi(t, t_0, x_0)$ is a fundamental matrix of (1.3). From (1.6), we have

$$|x(t, t_0, x_0)| \leq \int_0^1 |\Phi(t, t_0, sx_0)|\, ds\, |x_0| \leq c\, |x_0|\, h(t)\, h(t_0)^{-1}.$$

This implies that the zero solution of (1.1) is hS. Therefore, by Theorem 1.5, the solution $v = 0$ of (1.2) is hS and so the proof is complete. □

Corollary 2.2. *Under the same conditions of Theorem 2.1, the zero solution of (1.1) is hSV.*

Corollary 2.3. *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$ and the solution $z = 0$ of (1.3) is hS with a nondecreasing function h. Also, suppose that for all $t \geq t_0 \geq 0$,*

$$|\Phi(t, s, z)\, g(t, z)| \leq \gamma(s)|z|,$$

where $\gamma \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\int_{t_0}^{\infty} \gamma(s)ds < \infty$. Then $y = 0$ of (1.7) is hS.

Proof. It follows from Theorem 2.1 that the solution $v = 0$ of (1.2) is hS. In the proof of Theorem 2.1, the solution $x = 0$ of (1.1) is hS. Hence, by Theorem 1.8, the solution $y = 0$ of (1.7) is hS. This completes the proof. □

Also, we examine the property of hS for the perturbed system

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds,\ y(t_0) = y_0, \tag{2.1}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$.

Lemma 2.4. *Let $u, p, q \in C(\mathbb{R}^+, \mathbb{R}^+)$ and suppose that, for some $c \geq 0$, we have*

$$u(t) \leq c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} q(\tau)u(\tau)d\tau ds,\ t \geq t_0. \tag{2.2}$$

Then

$$u(t) \leq c \exp(\int_{t_0}^{t} p(s) \int_{t_0}^{s} q(\tau)d\tau ds),\ t \geq t_0. \tag{2.3}$$

Proof. Setting $v(t) = c + \int_{t_0}^{t} p(s) \int_{t_0}^{s} q(\tau)u(\tau)d\tau ds$, we have $v(t_0) = c$ and

$$
\begin{aligned}
v'(t) &= p(t)\int_{t_0}^{t} q(s)u(s)ds \le p(t)\int_{t_0}^{t} q(s)v(s)ds \\
&\le [p(t)\int_{t_0}^{t} q(s)ds]v(t),\ t \ge t_0,
\end{aligned}
\tag{2.4}
$$

since $v(t)$ is nondecreasing and $u(t) \le v(t)$. It follows from the Gronwall inequality that (2.4) yields the estimate (2.3). □

THEOREM 2.5. *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$. If the solution $x = 0$ of (1.1) is hS with a positive bounded continuous function h and g in (2.1) satisfies*

$$|g(t, y)| \le \lambda(t)|y|,\ t \ge t_0,\ y \in \mathbb{R}^n,$$

where $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with

$$\int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} h(\tau)\lambda(\tau)d\tau ds < \infty, \tag{2.5}$$

for all $t_0 \ge 0$, then the solution $y = 0$ of (2.1) is hS.

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$. By Theorem 1.5, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.2) is hS. Therefore, by Theorem 2.1, the solution $z = 0$ of (1.3) is hS. By Lemma 1.4, we have

$$
\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} |g(\tau, y(\tau))|d\tau ds \\
&\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2\frac{h(t)}{h(s)} \int_{t_0}^{s} h(\tau)\lambda(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau ds.
\end{aligned}
$$

Setting $u(t) = |y(t)|h(t)^{-1}$ and using Lemma 2.4, we obtain

$$
\begin{aligned}
|y(t)| &\le c_1|y_0|h(t)\, h(t_0)^{-1} e^{c_2 \int_{t_0}^{t} \frac{1}{h(s)} \int_{t_0}^{s} h(\tau)\lambda(\tau)d\tau ds} \\
&\le c|y_0|h(t)\, h(t_0)^{-1},\ t \ge t_0,
\end{aligned}
$$

where $c = c_1 e^{c_2 \int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} h(\tau)\lambda(\tau)d\tau ds}$. It follows that $y = 0$ of (2.1) is hS. Hence, the proof is complete. □

REMARK 2.1. We further suppose that h is nondecreasing in Theorem 2.5, then the condition (2.5) can be replaced by

$$\int_{t_0}^{\infty} \int_{t_0}^{s} \lambda(\tau)d\tau ds < \infty, \tag{2.6}$$

for all $t_0 \geq 0$.

Corollary 2.6. *Under the assumptions of Theorem 2.5, we suppose furthermore that the condition (2.5) is replace by (2.6) and* $\frac{h(s)}{h(t)}$ *is bounded for for each* $t \geq s \geq 0$. *Then the solution* $y = 0$ *of (2.1) is hS.*

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Department of Mathematics
Hanseo University
Seosan 356-706, Republic of Korea
E-mail: yhgoo@hanseo.ac.kr