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LINEAR ISOPERIMETRIC INEQUALITY AND GROMOV HYPERBOLICITY ON ALEKSANDROV SURFACES

Byung-Geun Oh*

Abstract. We prove that a simply-connected open Aleksandrov surface that satisfies a linear isoperimetric inequality is hyperbolic in the sense of Gromov.

1. Introduction

The main topic of this paper is *Gromov hyperbolicity*, and we study it on *Aleksandrov surfaces*. First, let us explain what Gromov hyperbolicity means. A metric space (X, d) is called hyperbolic in the sense of Gromov, or *Gromov hyperbolic*, if there exists a constant $\delta > 0$ such that every four points $x, y, z, w \in X$ satisfy the inequality

$$(x, y)_w \geq \min\{(x, z)_w, (y, z)_w\} - \delta, \tag{1.1}$$

where

$$(a, b)_c = \frac{1}{2}\{d(a, c) + d(b, c) - d(a, b)\}.$$

If it is needed to specify the number δ, we will call X δ-hyperbolic. The meaning of the quantity $(a, b)_c$, called the *Gromov product* of a and b with respect to c, is illustrated in Figure 1 in the case when X is Euclidean. Gromov hyperbolic spaces were first introduced in the study of finitely generated free groups, and then it was applied to other branches of mathematics. For further study of Gromov hyperbolic spaces, see [6, 4, 7].

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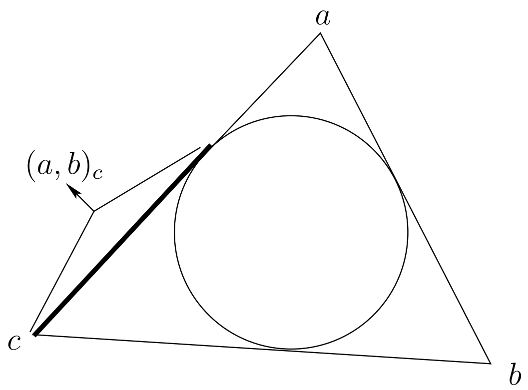

FIGURE 1. Gromov product

Next we explain Aleksandrov surfaces. To do this, let us introduce some terminologies for metric spaces. A metric space (X, d) is called *intrinsic* if we have

$$d(x,y) = \inf\{\text{length}(\gamma) : \gamma \text{ is a curve in } X \text{ joining } x \text{ and } y\} \tag{1.2}$$

for every $x, y \in X$. Here length(γ) denotes the arc length of the curve γ. An intrinsic metric space is called *geodesic* if "infimum" can be replaced by "minimum" in (1.2); i.e., an intrinsic metric space is geodesic if and only if every two points can be joined by a *shortest curve* – a curve whose length is the same as the distance between them. One may check, using the Arzelá-Ascoli Theorem, that a complete locally compact intrinsic space must be geodesic. Also note that if X is a path-connected metric space, by redefining the metric on X if necessary, it is possible to make X intrinsic.

Now we define Aleksandrov surfaces. With the terminology "Aleksandrov surface", we mean a two-dimensional topological manifold with an intrinsic metric whose length element is locally expressed in the form

$$e^u(z)|dz|, \tag{1.3}$$

where z is a local complex coordinate and u is a difference between two subharmonic functions such that $\exp z$ is locally integrable on rectifiable curves in the z-plane. The most typical example of an Aleksandrov surface is, as expected, a two-dimensional Riemannian manifold. By this reason Aleksandrov surfaces are regarded as a generalization of two-dimensional Riemannian manifolds. Another typical example is a surface with a polyhedral metric, which means that each point is locally isometric to a *cone* with the length element

$$|z|^{\alpha-1}|dz|$$

for some $\alpha > 0$ (Figure 2). To study more about Aleksandrov surfaces,

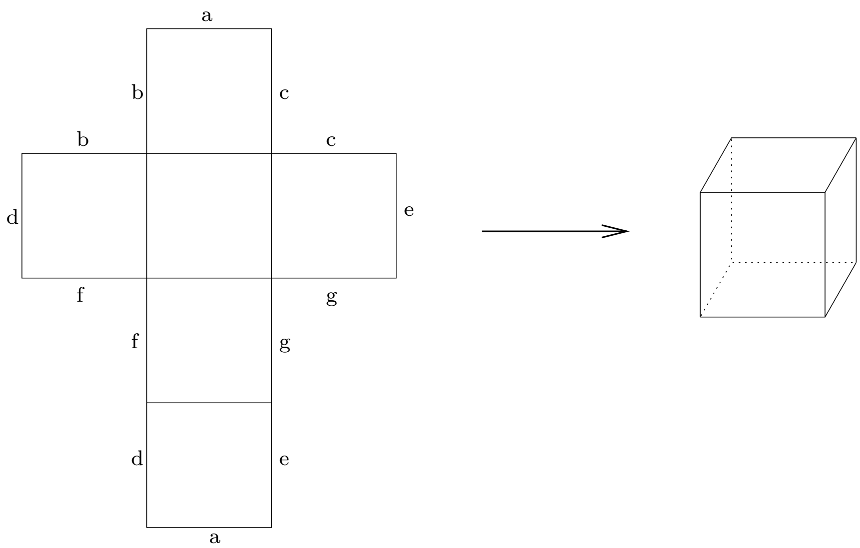

FIGURE 2. An example of a polyhedral surface

see for example [1, 10, 2, 9]. If the reader is not familiar with Aleksandrov surfaces, we recommend to skip the details about Aleksandrov surfaces and just accept them as "nice" surfaces such as Riemannian manifolds or polyhedral surfaces. This should not cause any problem since Aleksandrov surfaces are nice enough to define integrals, curvatures, etc., and more importantly, most applications are found in the category of Riemannian manifolds and polyhedral surfaces.

Now suppose X is an Aleksandrov surface. For a domain $\Omega \subset X$ and a curve $\gamma \subset X$, we denote the area of Ω by Area(Ω), and the length of γ by length(γ). Then we say that X satisfies a *linear isoperimetric inequality* if there exists a constant C such that the inequality

$$\text{Area}(\Omega) \leq C \cdot \text{length}(\partial\Omega) \tag{1.4}$$

holds for every domain $\Omega \subset X$, where $\partial\Omega$ denotes the boundary of Ω. Note that a surface is called *open* if it is not compact and does not have boundary, hence a simply-connected surface is open if and only if it is topologically equivalent to the unit disk. Our main theorem is:

THEOREM 1.1. *Every open simply-connected Aleksandrov surface which satisfies a linear isoperimetric inequality is Gromov hyperbolic.*

The converse of Theroem 1.1 is false. If X is an infinite cylinder, one can show that X is Gromov hyperbolic but does not satisfy linear isoperimetric inequalities (1.4) for any constant C.

Theorem 1.1 should be accepted as a generalization of the result in [4, p. 66], where Theorem 1.1 was proved for open simply-connected two-dimensional Riemannian manifolds that are *complete*. The renovations are that our theorem is about Aleksandrov surfaces instead of Riemannian manifolds, and more importantly that we have proved it without the completeness condition.

2. Incomplete metric space

Suppose X is a geodesic space. A *geodesic triangle* (or for brevity, we just call triangle) λ is a topological circle such that $\lambda = a \cup b \cup c$, where each sides, a, b and c, are shortest curves in X. The end points of a, b, c are called vertices of λ, and we define the *minimal size* of λ by

$$\text{minsize}(\lambda) := \inf \max_{i,j} |y_i - y_j|, \tag{2.1}$$

where the infimum is taken over all $y_1 \in a, y_2 \in b$ and $y_3 \in c$. Note that λ has the minimal size δ if and only if any side of λ is contained in the closed δ-neighborhood of the union of the other two sides.

It is known that when X is a geodesic space, X is Gromov δ-hyperbolic if and only if X satisfies the following condition (cf. [4], p. 10): *there exists a constant* δ' *such that* $\text{minsize}(\lambda) \leq \delta'$ *for every geodesic triangle* $\lambda \subset X$. Moreover, it is known that the constants δ and δ' depends only on each other.

Now suppose X is an Aleksandrov surface with the metric d. For a compact Jordan region $\Omega \subset X$ that is topologically equivalent to the closed unit disk, we define a metric on Ω by

$$d_\Omega(x, y) := \inf\{\text{length}(\gamma) : \gamma \text{ is a curve in } \Omega \text{ joining } x \text{ and } y\}.$$

Note that even though (X, d) is just intrinsic, the metric space (Ω, d_Ω) must be geodesic because of the compactness assumption of Ω (and by the Arzelá-Ascoli Theorem). Moreover if (X, d) satisfies a linear isoperimetric inequality (1.4) for some constant C, then it is also true for (Ω, d_Ω) with the same constant C. Therefore to prove Theorem 1.1, it suffices to show that if X satisfies a linear isoperimetric inequality for some constant C, then every Ω-geodesic triangle has the minimal size at most δ' where δ' depends only on C. Then since Ω is a geodesic space, this will show that Ω is δ-hyperbolic with δ independent of Ω, hence X will be also δ-hyperbolic. In fact, if $\{\Omega_k\}_{k=1}^{\infty}$ is an increasing sequence of compact Jordan regions which are δ-hyperbolic and exhaust X, then since

$$d(x,y) = \lim_{k\to\infty} d_{\Omega_k}(x,y), \text{ for } x,y \in X,$$

one can easily see from (1.1) that X is also δ-hyperbolic.

3. Proof of Theorem 1.1

Suppose $\Omega \subset X$ is a compact Jordan region and λ is an Ω-geodesic triangle. Let Δ be the triangular region[1] enclosed by λ. By the Uniformization Theorem of Huber [8], there is an isometry

$$h : X \to D(R) := \{z \in \mathbb{C} : |z| < R\},\ R \in (0,\infty],$$

where $D(R)$ is equipped with a length element of the form

$$\tilde{\rho}(z)|dz| = e^{u(z)}|dz|.$$

Here u is a difference of two subharmonic functions such that $\exp u$ is locally integrable on reticfiable curves in $D(R)$. Note that Huber's Uniformization Theorem says that the function u in the definition of Aleksandrov surfaces (1.3) is in fact defined globally. Then since Δ is a closed Jordan region, the Riemann mapping Theroem and the Carathéodory Theorem [3] (cf. [5], p.41) imply that there is a homeomorphism $\varphi : h(\Delta) \to T$, where T is a triangular region in $\mathbb{C}$ whose boundary is the Euclidean equilateral triangle of side-length 2, such that φ is conformal in the interior of $h(\Delta)$, continuous on $h(\Delta)$, and sends the vertices of $h(\Delta)$ to the vertices of T. We define

$$\mu(z) := \begin{cases} \tilde{\rho}(\varphi^{-1}(z))|(\varphi^{-1})'(z)|, & \text{if } z \in T; \\ 0, & \text{otherwise,} \end{cases}$$

[1] With a triangular region, we mean a closed set such that it is homeomorphic to the closed unit disc and its boundary is a topological triangle.

for all $z \in \mathbb{C}$. Also we define for $x \in \mathbb{C}$ and $r > 0$,

$$\begin{aligned} D(x,r) &:= \{z \in \mathbb{C} : |z-x| \le r\}, \\ L(x,r) &:= \int_{\partial D(x,r)} \mu(z)|dz| = \int_{|z-x|=r} \mu(z)|dz|, \\ A(x,r) &:= \int_{D(x,r)} \mu(z)^2 dm = \int_{|z-x|\le r} \mu(z)^2 dm, \end{aligned}$$

where dm denotes the Lebesque measure in $\mathbb{C}$. Finally let $\psi := h^{-1} \circ \varphi^{-1}$.

Note that even though μ is not smooth, the function $r \to A(x,r)$ is still differentiable. In fact, by the definition of Aleksandrov surfaces and our construction, μ is integrable over rectifiable curves in $\mathbb{C}$. In particular, $\int_{\partial T} \mu(z)|dz| < \infty$. Therefore if the Aleksandrov surface X satisfies a linear isoperimetric inequality for some constant C, then

$$\int_{\mathbb{C}} \mu(z)^2 dm = \int_T \mu(z)^2 dm \le C \int_{\partial T} \mu(z)|dz| < \infty,$$

hence μ^2 is integrable in $\mathbb{C}$. Now because

$$A(x, r_0) = \int_0^{r_0} \int_0^{2\pi} \mu(r,\theta)^2\, r\, d\theta\, dr,$$

the Fubini's Theorem implies that the function $r \to \int_0^{2\pi} \mu(r,\theta)^2 r d\theta$ is finite almost every r and integrable over $[0,\infty)$. Therefore $A(x,r)$ is an absolutely continuous function and it is differentiable at almost every r with the derivative

$$\frac{d}{dr} A(x,r) = \int_0^{2\pi} \mu(r,\theta)^2 r d\theta.$$

For the rest of the proof, we follow the proof given in [4] (where the proof is written in French). Suppose that every Jordan region $D \subset X$ satisfies the inequality (1.4) for some constant C which does not depend on D. Then it is easy to see that there are three constants $A_0 > 0, c > 1$, and $a > 2$ such that

(i) $2a^{1/c}(a-2)^{-1} < 1$, and
(ii) any Jordan region $D \subset X$ with $A_0 \le \mathrm{Area}(D) \le a^2 A_0$ satisfies the inequality

$$16\pi c \cdot \mathrm{Area}(D) \le \{\mathrm{length}(\partial D)\}^2. \tag{3.1}$$

In fact, one may take $A_0 = 16\pi C^2 + 1$, $c = A_0/(A_0 - 1)$, and a sufficiently large so that (i) holds.

Lemma 3.1. *Let $x \in \mathbb{C}$ and $r > 0$ be given. We suppose that $D := D(x,r)$ meets ∂T at most one side of T and $A_0 \le A(x,r) \le a^2 A_0$. Then the following inequality holds:*

$$4\pi c \cdot A(x,r) \le L(x,r)^2.$$

Proof. The statement is clear when $\partial T \cap \partial D = \emptyset$. So we assume that D meets ∂T at one side of T. (Figure 3)

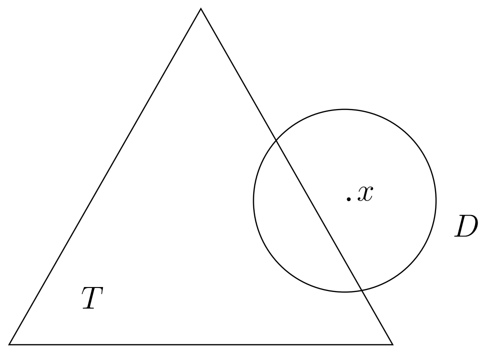

Figure 3. The case $\partial T \cap \partial D \neq \emptyset$

Let $\gamma = \partial D \cap T$ and $\tau = \partial T \cap D$. Then since $\psi(T \cap D)$ is a Jordan region in $\Omega \subset X$ with boundary $\psi(\gamma) \cup \psi(\tau)$, the inequality (3.1) implies that

$$16\pi c \int_{T \cap D} \mu(z)^2 dm \le \left(\int_{\tau \cup \gamma} \mu(z)|dz| \right)^2.$$

But note that $\psi(\tau)$ is an Ω-shortest curve because it is contained in a side of Δ and all sides of Δ are Ω-shortest curves. Now since $\psi(\gamma)$ is a curve in Ω with the same end points as $\psi(\tau)$, the μ-length of τ is less than or equal to that of γ. Hence

$$\begin{aligned} 4\pi c A(x,r) &= 4\pi c \int_D \mu(z)^2 dm = 4\pi c \int_{T \cap D} \mu(z)^2 dm \\ &\le \frac{1}{4} \left(\int_{\tau \cup \gamma} \mu(z)|dz| \right)^2 \le \left(\int_{\gamma} \mu(z)|dz| \right)^2 \\ &= \left(\int_{\partial D} \mu(z)|dz| \right)^2 = L(x,r)^2, \end{aligned}$$

as desired. □

Let $\mathcal{R}$ be a rhombus which is the union of T and a reflection of T with respect to a side of T. For each $x \in \mathcal{R}$, we denote by $d(x)$ the (Euclidean) distance from x to the boundary of $\mathcal{R}$, and for a fixed constant $A > 0$ we define

$$r(x, A) := \begin{cases} \text{the smallest } r \in (0, d(x)) \text{ with } A(x,r) = A, & \text{if such } r \text{ exists;} \\ d(x), & \text{otherwise.} \end{cases}$$

LEMMA 3.2. *For all* $x \in \mathcal{R}$,

$$\left(\frac{r(x, a^2 A_0)}{r(x, A_0)}\right)^{2c} \leq a^2.$$

Proof. If $r(x, A_0) = d(x)$, there is nothing to prove. Next, we consider the case $r(x, a^2 A_0) < d(x)$.

Let $r_0 := r(x, A_0)$ and $r_a := r(x, a^2 A_0)$. By Lemma 3.1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{4\pi c}{2\pi r} A(x,r) \leq \frac{1}{2\pi r} L(x,r)^2 &= \frac{1}{2\pi r}\left(\int_{\partial D(x,r)} \mu(z)|dz|\right)^2 \\ &\leq \frac{1}{2\pi r}\left(\int_{\partial D(x,r)} \mu(z)^2 |dz|\right)\left(\int_{\partial D(x,r)} |dz|\right) \\ &= \left(\int_{\partial D(x,r)} \mu(z)^2 |dz|\right) = \frac{d}{dr} A(x,r), \end{aligned}$$

for all $r \in [r_0, r_a]$. Then by integrating this inequality from r_0 to r_a, we have

$$\left(\frac{r_a}{r_0}\right)^{2c} \leq \frac{A(x, r_a)}{A(x, r_0)} = a^2,$$

as desired.

Finally if $r(x, a^2 A_0) = d(x)$ and $r(x, A_0) < d(x)$, we take the smallest $b \leq a$ such that $r(x, b^2 A_0) = d(x)$. Then by the same argument as above,

$$\left(\frac{r(x, b^2 A_0)}{r(x, A_0)}\right)^{2c} \leq b^2,$$

hence

$$\left(\frac{r(x, a^2 A_0)}{r(x, A_0)}\right)^{2c} = \left(\frac{r(x, b^2 A_0)}{r(x, A_0)}\right)^{2c} \leq b^2 \leq a^2.$$

This completes the proof of Lemma 3.2. □

LEMMA 3.3. *For all $x \in \mathcal{R}$, we have*

$$r(x, A_0) \geq (1-\alpha)a^{-1/c}d(x),$$

where $\alpha := 2a^{1/c}(a-2)^{-1} < 1$.

Proof. Suppose there is $x_0 \in \mathcal{R}$ such that

$$r_0 := r(x_0, A_0) < (1-\alpha)a^{-1/c}d(x_0). \tag{3.2}$$

Then by Lemma 3.2,

$$R_0 := r(x_0, a^2A_0) \leq a^{1/c}r(x_0, A_0) < (1-\alpha)d(x_0) < d(x_0), \tag{3.3}$$

hence $D(x_0, R_0) \subset \mathcal{R}$.

Now we claim that there is a point $x_1 \in D(x_0, R_0)$ such that

(a) $r_1 := r(x_1, A_0) \leq \alpha r_0$ and
(b) $r_1 < (1-\alpha)a^{-1/c}d(x_1)$.

Suppose we have shown the existence of $x_1 \in D(x_0, R_0)$ satisfying the condition (a) above. Then

$$\begin{aligned}
d(x_1) &\geq d(x_0) - |x_1 - x_0| \\
&\geq d(x_0) - R_0 = d(x_0) - r(x_0, a^2A_0) && (\text{since } x_1 \in D(x_0, R_0)) \\
&\geq d(x_0) - a^{1/c}r(x_0, A_0) = d(x_0) - a^{1/c}r_0 && (\text{by } (3.3)) \\
&> d(x_0) - (1-\alpha)d(x_0) = \alpha d(x_0) && (\text{by } (3.2)) \\
&> \alpha a^{1/c}(1-\alpha)^{-1}r_0 && (\text{by } (3.2)) \\
&\geq a^{1/c}(1-\alpha)^{-1}r_1 && (\text{by the condition (a)}),
\end{aligned}$$

which shows the inequality (b).

To show the existence of $x_1 \in D(x_0, R_0)$ satisfying the condition (a), let $\mathcal{M}_0$ be the average of the function $A(\,\cdot\,, \alpha r_0)$ over the disc $D(x_0, R_0)$. Then by the Fubini's Theorem,

$$\begin{aligned}
\mathcal{M}_0 &= \frac{1}{\pi R_0^2}\int_{D(x_0,R_0)} A(x, \alpha r_0)\, dm(x) \\
&= \frac{1}{\pi R_0^2}\int_{D(x_0,R_0)}\int_{D(x,\alpha r_0)} \mu(y)^2\, dm(y)\, dm(x) \\
&= \frac{1}{\pi R_0^2}\int_{D(x_0,R_0+\alpha r_0)}\int_{D(x_0,R_0)\cap D(y,\alpha r_0)} \mu(y)^2\, dm(x)\, dm(y) \\
&\geq \frac{1}{\pi R_0^2}\int_{D(x_0,R_0)} \mu(y)^2 |D(x_0, R_0) \cap D(y, \alpha r_0)|_{\mathbb{C}}\, dm(y).
\end{aligned}$$

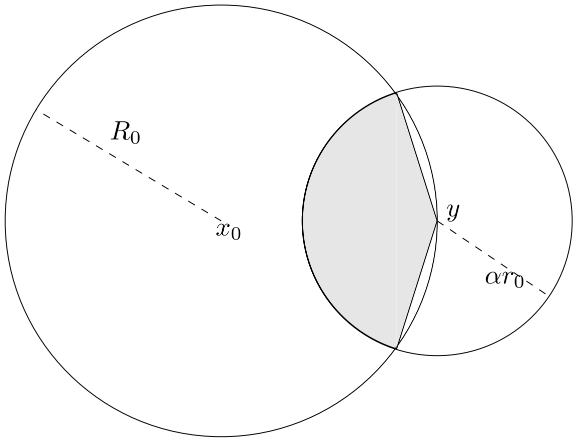

FIGURE 4. The sector of $D(y,\alpha r_0)$ contained in $D(x_0,R_0)\cap D(y,\alpha r_0)$

Here the quantity $|D(x_0,R_0)\cap D(y,\alpha r_0)|_{\mathbb{C}}$ denotes the Euclidean area of the region, and its value takes the minimum when y is on the boundary of $D(x_0,R_0)$. Hence it is greater than the area of the largest sector of $D(y,\alpha r_0)$ contained in $D(x_0,R_0)\cap D(y,\alpha r_0)$ (Figure 4), or

$$(\alpha r_0)^2 \arccos\left(\frac{\alpha r_0}{2R_0}\right).$$

Then since

$$\frac{1}{\pi}\arccos\left(\frac{x}{2}\right) > \frac{1}{4}\arccos\left(\frac{x}{2}\right) > \frac{1}{4(1+x^2)}$$

for $0<x<1$, we have

$$\begin{aligned}\mathcal{M}_0 &\geq \frac{1}{\pi R_0^2}(\alpha r_0)^2 \arccos\left(\frac{\alpha r_0}{2R_0}\right)\int_{D(x_0,R_0)}\mu(y)^2 dm(y)\\ &\geq \frac{1}{4}\left(\frac{\alpha r_0}{R_0}\right)^2 \frac{1}{(1+(\alpha r_0/R_0)^2)}A(x_0,R_0)\\ &= \frac{1}{4}\frac{(\alpha r_0)^2}{R_0^2+(\alpha r_0)^2}a^2A_0 \geq \frac{1}{4}\left(\frac{\alpha r_0}{R_0+\alpha r_0}\right)^2 a^2 A_0.\end{aligned}$$

Now note that $R_0 \leq a^{1/c}r_0$ by (3.3). Then since $\alpha = 2a^{1/c}(a-2)^{-1}$,

$$\mathcal{M}_0 \geq \frac{1}{4}\left(\frac{\alpha r_0}{a^{1/c}r_0+\alpha r_0}\right)^2 a^2A_0 = \frac{1}{4}\left(\frac{\alpha}{a^{1/c}+\alpha}\right)^2 a^2 A_0 = A_0.$$

Hence there is $x_1 \in D(x_0, R_0)$ such that $A(x_1, \alpha r_0) \geq A_0$. In particular, this implies $r(x_1, A_0) \leq \alpha r_0$, which proves the claim.

Now we apply the above argument repeatedly to x_n in place of x_0, and get a sequence $\{x_n\}$ such that $r_n := r(x_n, A_0) \leq \alpha^n r_0$, $r_n < (1 - \alpha)a^{-1/c}d(x_n)$, and $|x_{n+1} - x_n| \leq r(x_n, a^2 A_0)$. Then we have

$$\begin{aligned}|x_n - x_0| &\leq \sum_{k=0}^{n-1} |x_{k+1} - x_k| \leq \sum_{k=0}^{n-1} r(x_k, a^2 A_0) \leq \sum_{k=0}^{n-1} a^{1/c} r_k \\ &\leq \sum_{k=0}^{n-1} a^{1/c}\alpha^k r_0 \leq a^{1/c} r_0 (1-\alpha)^{-1} < d(x_0).\end{aligned}$$

This means that there is a limit point $x \in \mathcal{R}$ of $\{x_n\}$ such that every neighborhood of x has μ-area greater than or equal to A_0, since $r_n \leq \alpha^n r_0 \to 0$. But this is impossible because μ is a square integrable function, and this contradiction proves the lemma. □

We next consider the tripod $\mathbb{T}$ obtained by joining the center of T to the midpoint of each sides (Figure 5). Note that each foot of $\mathbb{T}$ has length $1/\sqrt{3}$.

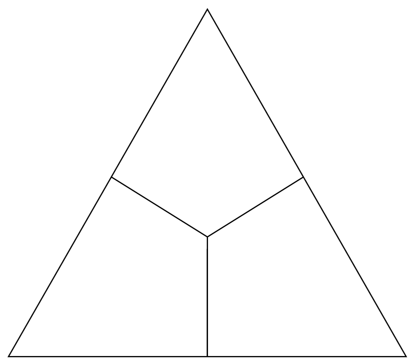

FIGURE 5. Tripod $\mathbb{T}$

For simplicity, let $\beta := (1-\alpha)a^{-1/c}/\sqrt{3}$.

LEMMA 3.4. *For all $x \in \mathbb{T}$, there exists a number r_x such that*

$$\frac{1}{2}\beta \leq r_x \leq \beta \quad \text{and} \quad L(x, r_x) \leq 2\sqrt{\pi A_0}.$$

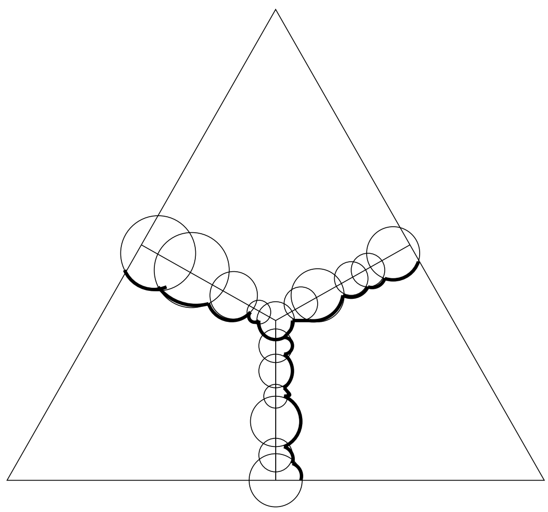

FIGURE 6. Three curves joining each pairs of sides of $\mathbb{T}$

Proof. By symmetry, we may assume that x lies on the diagonal of $\mathcal{R}$ which has length $2\sqrt{3}$, where $\mathcal{R}$ is the rhombus in Lemma 3.3. Then by Lemma 3.3 and the fact $d(x) \geq 1/\sqrt{3}$,

$$r(x, A_0) \geq (1-\alpha)a^{-1/c}d(x) \geq \beta,$$

or

$$A(x, \beta) \leq A_0.$$

If $L(x,r)^2 \geq 4\pi A_0$ for all $r \in [\beta/2, \beta]$, then the Cauchy-Schwarz inequality implies

$$4\pi A_0 \leq \left(\int_{\partial D(x,r)} \mu(z)|dz|\right)^2 \leq 2\pi r \int_{\partial D(x,r)} \mu(z)^2|dz|.$$

Integrating this inequality from $\beta/2$ to β, we have

$$2A_0 \log 2 \leq A(x,\beta) \leq A_0,$$

which is a contradiction since $A_0 > 0$. This completes the proof of Lemma 3.4. □

Now we are ready to proof the main theorem.

Proof of Theorem 1.1. Let P be a set of points in $\mathbb{T}$ such that for every $x \in P$ the Euclidean distance from x to $P \setminus \{x\}$ is equal to $\beta = (1-\alpha)a^{-1/c}/\sqrt{3}$ and that $\mathbb{T}$ is contained in the (closed) $(\beta/2)$-neighborhood of P. Then trivially the number of points in P is at most $3(1+3^{-1/2}/\beta) = 3(1+(1-\alpha)^{-1}a^{1/c})$. We also consider the circles $\partial D(x, r_x)$ for all $x \in P$, where r_x is the number in Lemma 3.4. The sum of their μ-length is at most $3(1 + (1-\alpha)^{-1}a^{1/c})2(\pi A_0)^{1/2}$.

Now it is easy to find three points on ∂T, one in each side of T, and three curves joining each pair of these points and consisting of arcs of $\partial D(x, r_x), x \in P$ (Figure 6). Each of these curves has μ-length at most $\delta' := 3(1 + (1-\alpha)^{-1}a^{1/c})2(\pi A_0)^{1/2}$, hence the minimal size of the Ω-geodesic triangle $\lambda = \psi(\partial T)$ is at most δ'. This, and the argument given in Section 2, completes the proof of Theorem 1.1. □

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Department of Mathematics Education
Hanyang University
Seoul 133-791, Republic of Korea
E-mail: bgoh@hanyang.ac.kr