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THE MATRIX REPRESENTATION OF CLIFFORD ALGEBRA

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Abstract. In this paper we construct a subalgebra L_8 of $M_8(\mathbb{R})$ which is a generalization of the algebra of quaternions. Moreover we prove that the algebra L_8 is the real Clifford algebra Cl_3, and so L_8 is a matrix representation of Clifford algebra Cl_3.

1. Introduction

In 1843, the algebra of quaternions was discovered by Sir W. R. Hamilton as an extension of complex numbers. In 1844, J. T. Graves found an algebra of octonions with 8 unit elements as an extension of quaternions. The algebra of octonions is a nonassociative extension of the quaternions. In 1878, the Clifford algebra was introduced by uniting the dot product and exterior product into a single geometric product. In particular, the real Clifford algebra Cl_3 is an associative algebra with three generators e_1, e_2, e_3 satisfying the rules of $e_i^2 = -1$ and $e_ie_j = -e_je_i$ for $i \neq j$, and it is an associative extension of the algebra of quaternions.

In this paper we will construct a subalgebra L_8 of $M_8(\mathbb{R})$ which is a generalization of the algebra of quaternions. Moreover we prove that the algebra L_8 is the real Clifford algebra Cl_3, and so L_8 is a matrix representation of Clifford algebra Cl_3.

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2. The matrix representation of Clifford algebra

In this section, we will construct an 8-dimensional associative subalgebra L_8 of $M_8(\mathbb{R})$ in a certain way from the quaternions. It is known that a quaternion can be represented by the following form of 4×4 matrices

$$A = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix},$$

where a_1, a_2, a_3, and a_4 are real numbers.

Define three sets $L_2(\subset M_2(\mathbb{R}))$, $T(\subset M_2(\mathbb{R}))$, $L_4(\subset M_4(\mathbb{R}))$ as follows:

$$L_2 = \left\{ \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \mid s_{11} = s_{22}\,,\ s_{12} = -\, s_{21} \right\},$$

$$T = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mid t_{12} = t_{21}\,,\ t_{11} = -\, t_{22} \right\},$$

$$L_4 = \left\{ \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \mid F_{11} = F_{22} \in L_2\,,\ F_{12} = -\, F_{21} \in T \right\}.$$

Then it is known that L_2 is isomorphic to complex numbers and L_4 is isomorphic to the algebra of quaternions. In this manner, we will extend the rules to 8×8 matrices as follows:
Let G be an 8×8 matrix with sixteen blocks of 2×2 matrices G_{ij} and let H_{rs} be 4×4 matrices satisfying the following:

$$G = \begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where

$$H_{11} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},\ H_{12} = \begin{pmatrix} G_{13} & G_{14} \\ G_{23} & G_{24} \end{pmatrix},$$

$$H_{21} = \begin{pmatrix} G_{31} & G_{32} \\ G_{41} & G_{42} \end{pmatrix},\ H_{22} = \begin{pmatrix} G_{33} & G_{34} \\ G_{43} & G_{44} \end{pmatrix},$$

and

$$G_{11} = G_{22}\,,\ G_{21} = -\, G_{12}\,,\ G_{13} = -G_{24}\,,\ G_{14} = G_{23}\,,$$

$$G_{31} = -G_{42}\,,\ G_{41} = G_{32}\,,\ G_{33} = G_{44}\,,\ G_{34} = -G_{43}\,,$$

$$H_{11} = H_{22}\,,\ H_{12} = -\, H_{21}\,,$$

and $G_{11}, G_{31} \in L_2$ and $G_{21}, G_{41} \in T$. By the above steps, we can obtain an 8×8 matrix $A \in M_8(\mathbb{R})$ of the form

$$A = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 & -a_5 & a_6 & -a_7 & -a_8 \\ a_2 & a_1 & -a_4 & a_3 & -a_6 & -a_5 & -a_8 & a_7 \\ a_3 & a_4 & a_1 & -a_2 & -a_7 & -a_8 & a_5 & -a_6 \\ a_4 & -a_3 & a_2 & a_1 & -a_8 & a_7 & a_6 & a_5 \\ a_5 & -a_6 & a_7 & a_8 & a_1 & -a_2 & -a_3 & -a_4 \\ a_6 & a_5 & a_8 & -a_7 & a_2 & a_1 & -a_4 & a_3 \\ a_7 & a_8 & -a_5 & a_6 & a_3 & a_4 & a_1 & -a_2 \\ a_8 & -a_7 & -a_6 & -a_5 & a_4 & -a_3 & a_2 & a_1 \end{pmatrix}.$$

For the brief notation, we denote the above matrix by

$$A = [a_1; a_2; a_3; a_4; a_5; a_6; a_7; a_8].$$

Let

$$L_8 = \{[a_1; a_2; a_3; a_4; a_5; a_6; a_7; a_8] \mid a_i \in \mathbb{R}\,,\ i = 1, \cdots, 8\}.$$

Then by the simple computations, we can prove the following lemma:

LEMMA 2.1. *L_8 is an 8-dimensional associative algebra.*

Proof. It is enough to show that L_8 is closed under matrix addition and matrix multiplication. Obviously, L_8 is closed under matrix addition. Let $A, B \in L_8$. Then for some $a_i, b_i \in \mathbb{R}, i = 1, \cdots, 8$,

$$A = [a_1; a_2; a_3; a_4; a_5; a_6; a_7; a_8]\ ,\ B = [b_1; b_2; b_3; b_4; b_5; b_6; b_7; b_8].$$

Note that

$$AB = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} & c_{28} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} & c_{37} & c_{38} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} & c_{57} & c_{58} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} & c_{67} & c_{68} \\ c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} & c_{77} & c_{78} \\ c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & c_{88} \end{pmatrix},$$

where

$$\begin{aligned} c_{11} &= a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 + a_6b_6 - a_7b_7 - a_8b_8 \\ &= c_{22} = c_{33} = c_{44} = c_{55} = c_{66} = c_{77} = c_{88}, \end{aligned}$$

$$\begin{aligned} c_{12} &= -a_1b_2 - a_2b_1 - a_3b_4 + a_4b_3 + a_5b_6 + a_6b_5 - a_7b_8 + a_8b_7 \\ &= -c_{21} = c_{34} = -c_{43} = c_{56} = -c_{65} = c_{78} = -c_{87}, \end{aligned}$$

$$\begin{aligned} c_{13} &= -a_1b_3 + a_2b_4 - a_3b_1 - a_4b_2 - a_5b_7 + a_6b_8 + a_7b_5 + a_8b_6 \\ &= -c_{31} = -c_{24} = c_{42} = c_{57} = -c_{68} = -c_{75} = c_{86}, \end{aligned}$$

$$\begin{aligned} c_{14} &= -a_1b_4 - a_2b_3 + a_3b_2 - a_4b_1 - a_5b_8 - a_6b_7 - a_7b_6 + a_8b_5 \\ &= c_{23} = -c_{32} = -c_{41} = c_{58} = c_{67} = -c_{76} = -c_{85}, \end{aligned}$$

$$\begin{aligned} c_{15} &= -a_1b_5 + a_2b_6 + a_3b_7 + a_4b_8 - a_5b_1 + a_6b_2 - a_7b_3 - a_8b_4 \\ &= c_{26} = -c_{37} = -c_{48} = -c_{51} = -c_{62} = c_{73} = c_{84}, \end{aligned}$$

$$\begin{aligned} c_{16} &= a_1b_6 + a_2b_5 + a_3b_8 - a_4b_7 + a_5b_2 + a_6b_1 - a_7b_4 + a_8b_3 \\ &= -c_{25} = -c_{38} = c_{47} = -c_{52} = c_{61} = c_{74} = -c_{83}, \end{aligned}$$

$$\begin{aligned} c_{17} &= -a_1b_7 + a_2b_8 - a_3b_5 - a_4b_6 + a_5b_3 - a_6b_4 - a_7b_1 - a_8b_2 \\ &= -c_{28} = c_{35} = -c_{46} = -c_{53} = c_{64} = -c_{71} = c_{82}, \end{aligned}$$

$$\begin{aligned} c_{18} &= -a_1b_8 - a_2b_7 + a_3b_6 - a_4b_5 + a_5b_4 + a_6b_3 + a_7b_2 - a_8b_1 \\ &= c_{27} = c_{36} = c_{45} = -c_{54} = -c_{63} = -c_{72} = -c_{81}. \end{aligned}$$

Then we can rewrite $A\,B$ as follows:

$$A\,B = \begin{pmatrix} c_{11} & -c_{21} & -c_{31} & -c_{41} & -c_{51} & c_{61} & -c_{71} & -c_{81} \\ c_{21} & c_{11} & -c_{41} & c_{31} & -c_{61} & -c_{51} & -c_{81} & c_{71} \\ c_{31} & c_{41} & c_{11} & -c_{21} & -c_{71} & -c_{81} & c_{51} & -c_{61} \\ c_{41} & -c_{31} & c_{21} & c_{11} & -c_{81} & c_{71} & c_{61} & c_{51} \\ c_{51} & -c_{61} & c_{71} & c_{81} & c_{11} & -c_{21} & -c_{31} & -c_{41} \\ c_{61} & c_{51} & c_{81} & -c_{71} & c_{21} & c_{11} & -c_{41} & c_{31} \\ c_{71} & c_{81} & -c_{51} & c_{61} & c_{31} & c_{41} & c_{11} & -c_{21} \\ c_{81} & -c_{71} & -c_{61} & -c_{51} & c_{41} & -c_{31} & c_{21} & c_{11} \end{pmatrix}$$

$$= [\,c_{11}; c_{21}; c_{31}; c_{41}; c_{51}; c_{61}; c_{71}; c_{81}]$$

and hence we can see that $A\,B$ is contained in L_8. Thus L_8 is closed under the matrix multiplication. Therefore, L_8 is an 8-dimensional associative algebra. □

We may consider $\mathbb{H}$ is a subalgebra of L_8 via the embedding $f : \mathbb{H} \to L_8$ defined by

$$f(a_1 + a_2i + a_3j + a_4k) = [a_1; a_2; a_3; a_4; 0\,; 0\,; 0\,; 0\,] = \begin{pmatrix} R & O \\ O & R \end{pmatrix},$$

where

$$R = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix}.$$

Also, L_8 is not isomorphic to the algebra of octonions since the algebra of octonions is not associative. Thus, we have the following theorem.

THEOREM 2.2. *L_8 is a generalization of the quaternions which is not isomorphic to the algebra of octonions.*

For $i = 1, \cdots, 8$, let $h_i = [0; \cdots; 0; 1; 0; \cdots; 0]$ be an 8×8 matrix having 1 in the i-th position. Then, h_1 is the 8×8 identity matrix I_8. Using the matrix h_i, an element A of L_8 can be denoted by

$$A = [a_1; a_2; a_3; a_4; a_5; a_6; a_7; a_8] = a_1 h_1 + \cdots + a_8 h_8.$$

The multiplication table of h_i's, $i = 1, \cdots, 8$, is as follows:

Table 2.1

$\cdot$	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
h_1	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
h_2	h_2	$-h_1$	h_4	$-h_3$	h_6	$-h_5$	h_8	$-h_7$
h_3	h_3	$-h_4$	$-h_1$	h_2	h_7	$-h_8$	$-h_5$	h_6
h_4	h_4	h_3	$-h_2$	$-h_1$	h_8	h_7	$-h_6$	$-h_5$
h_5	h_5	h_6	$-h_7$	$-h_8$	$-h_1$	$-h_2$	h_3	h_4
h_6	h_6	$-h_5$	$-h_8$	h_7	$-h_2$	h_1	h_4	$-h_3$
h_7	h_7	$-h_8$	h_5	$-h_6$	$-h_3$	h_4	$-h_1$	h_2
h_8	h_8	h_7	h_6	h_5	$-h_4$	$-h_3$	$-h_2$	$-h_1$

From the table 2.1, we can find some interesting relations between h_i's as follows:

$$h_i^2 = -h_1 \,, \; i \neq 1, 6$$

$$h_1^2 = h_6^2 = h_1$$

$$h_i \, h_j = \pm h_j \, h_i$$

$$h_2 \, h_3 \, h_4 \, h_5 \, h_6 \, h_7 \, h_8 = -h_1$$

Furthermore, the table above shows the Corollary 2.3.

COROLLARY 2.3. $\Gamma = \{h_1, \cdots, h_8, -h_1, \cdots, -h_8\}$ *forms a multiplicative group.*

Now we have the main result of this paper.

THEOREM 2.4. *L_8 is the Clifford algebra Cl_3.*

Proof. From the table 2.1, the following relations are satisfied:

$$h_2^2 = h_3^2 = h_7^2 = -h_1 \,,$$

$$h_2h_3 = -h_3h_2 \,,\; h_2h_7 = -h_7h_2 \,,\; h_3h_7 = -h_7h_3 \,.$$

Since h_1 is the multiplicative identity in L_8, L_8 is the Clifford algebra Cl_3. □

REMARK 2.5. From the results in this paper, we know that $L_2 = Cl_1$, $L_{2^2} = Cl_2$ and $L_{2^3} = Cl_3$. These relations give a natural question of $L_{2^n} = Cl_n$ for each natural number n.

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