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ON THE HYERS-ULAM-RASSIAS STABILITY OF A BI-PEXIDER FUNCTIONAL EQUATION

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Abstract. In this paper, we study the Hyers-Ulam-Rassias stability of a bi-Pexider functional equation

$$f(x+y,z)-f_1(x,z)-f_2(y,z)=0,$$
$$f(x,y+z)-f_3(x,y)-f_4(x,z)=0.$$

Moreover, we establish stability results on the punctured domain.

1. Introduction

The stability problem of functional equations originated from a question of S. M. Ulam [17] concerning the stability of group homomorphisms: Given a group G_1, a metric group (G_2, d) and $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $h : G_1 \to G_2$ satisfies

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say the equation of homomorphism $H(xy) = H(x)H(y)$ stable.
In 1941, D. H. Hyers [4] gave first affirmative answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [1] for additive mappings and by Th. M. Rassias [16] for linear mappings by considering an unbounded Cauchy difference(See the recent Maligranda's paper [13]). Since then, further generalizations of the Hyers-Ulam theorem have been extensively investigated by a number

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of mathematicians [3, 5, 9, 11, 12, 14].

Throughout this paper, let X be a vector space and Y a Banach space. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively, a Jensen mapping) if g satisfies the functional equation $g(x + y) = g(x) + g(y)$ (respectively, $2g(\frac{x+y}{2}) = g(x) + g(y)$).

For given mappings $f, f_1, f_2, f_3, f_4 : X \times X \to Y$, we define

$$\begin{aligned}
&C_1 f(x, y, z) := f(x + y, z) - f(x, z) - f(y, z),\\
&C_2 f(x, y, z) := f(x, y + z) - f(x, y) - f(x, z),\\
&J_1 f(x, y, z) := 2f(\frac{x + y}{2}, z) - f(x, z) - f(y, z),\\
&J_2 f(x, y, z) := 2f(x, \frac{y + z}{2}) - f(x, y) - f(x, z),\\
&P_1(f, f_1, f_2)(x, y, z) := f(x + y, z) - f_1(x, z) - f_2(y, z),\\
&P_2(f, f_3, f_4)(x, y, z) := f(x, y + z) - f_3(x, y) - f_4(x, z),\\
&P(f, f_1, f_2, f_3, f_4)(x, y, z, w)\\
&\qquad := f(x + y, z + w) - f_1(x, z) - f_2(x, w) - f_3(y, z) - f_4(y, w)
\end{aligned}$$

for all $x, y, z, w \in X$. If a mapping f satisfies the functional equations $C_1 f = 0$ and $C_2 f = 0$ ($C_1 f = 0$ and $J_2 f = 0$, $C_2 f = 0$ and $J_1 f = 0$,$J_1 f = 0$ and $J_2 f = 0$, $P_1(f, f_1, f_2) = 0$ and $P_2(f, f_3, f_4) = 0$, respectively), we say that $f : X \times X \to Y$ satisfies a biadditive (Cauchy-Jensen, Jensen-Cauchy, bi-Jensen, bi-Pexider, respectively) functional equation. It is easy to see that f satisfies a biadditive (Cauchy-Jensen and Jensen-Cauchy respectively) functional equation, then f satisfies a bi-Jensen functional equation.

In 2006, Bae and Park [2, 15] obtained the generalized Hyers-Ulam stability of a Cauchy-Jensen functional equation and a bi-Jensen functional equation. In 2007, Lee et al. [6, 7, 8, 10] improved the Bae and Park's results.

In this paper, we investigate the stability of a bi-Pexider functional equation.

2. Stability of a bi-Jensen functional equation

Throughout in this paper, denote $P_1(f, f_1, f_2)$ and $P_2(f, f_3, f_4)$ by P_1 and P_2 briefly, respectively. One can easily prove the basic properties of a bi-Jensen mapping in the following lemmas.

Lemma 2.1. [7] *Let $f : X \times X \to Y$ be a bi-Jensen mapping. Then*

$$f(x,y) = \frac{f(2^n x, 2^n y)}{4^n} + \big(\frac{1}{2^n} - \frac{1}{4^n}\big)\big(f(2^n x, 0) + f(0, 2^n y)\big) + \big(1 - \frac{1}{2^n}\big)^2 f(0,0),$$

$$f(x,y) = \frac{f(2^n x, 2^n y)}{4^n} + (2^n - 1)\big(f\big(\frac{x}{2^n}, 0\big) + f\big(0, \frac{y}{2^n}\big)\big) - (2^{n+1} - 3 + \frac{1}{4^n})f(0,0)),$$

$$f(x,y) = 4^n f\big(\frac{x}{2^n}, \frac{y}{2^n}\big) + (2^n - 4^n)\big(f\big(\frac{x}{2^n}, 0\big) + f\big(0, \frac{y}{2^n}\big)\big) + (2^n - 1)^2 f(0,0)),$$

$$f(x,y) = \frac{1}{2^n}(f(2^n x, y) - f(0,y)) + \frac{1}{2^n}(f(0, 2^n y) - f(0,0)) + f(0,0)$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Lemma 2.2. *For given mappings $f, f_1, f_2, f_3, f_4 : X \times X \to Y$, let $f', f'', f''', A_3 : X \times X \to Y$ and $A_1, A_2 : X \times X \times X \to Y$ be defined by*

$$f'(x,y) = f(x,y) - f(0,y),$$
$$f''(x,y) = f(x,y) - f(x,0),$$
$$f'''(x,y) = f(x,y) - f(x,0) - f(0,y) + f(0,0),$$
$$A_1(x,y,z) = P_1(x,y,z) + P_1(y,x,z) - P_1(x,x,z) - P_1(y,y,z),$$
$$A_2(x,y,z) = P_2(x,y,z) + P_2(x,z,y) - P_2(x,y,y) - P_2(x,z,z),$$
$$A_3(x,y) = \frac{1}{8}(A_1(x,0,2y) + 2A_1(x,0,y) - 3A_1(x,0,0) + A_2(2x,0,y) + 2A_2(x,0,y) - 3A_2(0,0,y))$$

for all $x, y \in X$. Then

$$f'(x,y) - \frac{f'(2x,y)}{2} = \frac{1}{2}A_1(x,0,y),$$
$$f''(x,y) - \frac{f''(x,2y)}{2} = \frac{1}{2}A_2(x,0,y),$$
$$f'''(x,y) - \frac{f'''(2x,2y)}{4} = A_3(x,y),$$
$$J_1 f'''(x,y,z) = A_1(\frac{x}{2}, \frac{y}{2}, z) - A_1(\frac{x}{2}, \frac{y}{2}, 0),$$
$$J_2 f'''(x,y,z) = A_2(x, \frac{y}{2}, \frac{z}{2}) - A_2(0, \frac{y}{2}, \frac{z}{2})$$

for all $x, y \in X$.

THEOREM 2.3. *Let* $0 \le p < 1, 0 < \varepsilon$ *and let* $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ *be the mappings such that*

$$\|P_1(f, f_1, f_2)(x, y, z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p), \tag{2.1}$$

$$\|P_2(f, f_3, f_4)(x, y, z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.2}$$

for all $x, y, z \in X$. *Then there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\|f(x, y) - F(x, y)\| \le \frac{4\varepsilon}{2 - 2^p}\|x\|^p + (\frac{4}{2 - 2^p} + 4)\varepsilon\|y\|^p \tag{2.3}$$

for all $x, y \in X$ *with* $F(0, 0) = f(0, 0)$. *The mapping* $F : X \times X \to Y$ *is given by*

$$F(x, y) := \lim_{j\to\infty} \frac{f(2^j x, y) + f(0, 2^j y)}{2^j} + f(0, 0)$$

for all $x, y \in X$.

Proof. By Lemma 2.2, (2.1) and (2.2), we get

$$\begin{aligned} \|\frac{f(2^j x, y) - f(0, y)}{2^j} - \frac{f(2^{j+1}x, y) - f(0, y)}{2^{j+1}}\| &= \|\frac{A_1 f(2^j x, 0, y)}{2^{j+1}}\| \\ &\le \frac{(\|2^j x\|^p + \|y\|^p)\varepsilon}{2^{j-1}}, \\ \|\frac{f(0, 2^j y) - f(0, 0)}{2^j} - \frac{f(0, 2^{j+1}y) - f(0, 0)}{2^{j+1}}\| &= \|\frac{A_2 f(0, 0, 2^j y)}{2^{j+1}}\| \\ &\le \frac{\|2^j y\|^p \varepsilon}{2^{j-1}} \end{aligned}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. For given integers l, m $(0 \le l < m)$,

$$\begin{aligned} \|\frac{f'(2^l x, y)}{2^l} - \frac{f'(2^m x, y)}{2^m}\| &= \|\sum_{j=l}^{m-1} \frac{A_1 f(2^j x, 0, y)}{2^{j+1}}\| \\ &\le \sum_{j=l}^{m-1} \frac{2^{jp}\|x\|^p + \|y\|^p}{2^{j-1}}\varepsilon, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \|\frac{f'(0, 2^l y)}{2^l} - \frac{f'(0, 2^m y)}{2^m}\| &= \|\sum_{j=l}^{m-1} \frac{A_2 f(0, 0, 2^j y)}{2^{j+1}}\| \\ &\le \sum_{j=l}^{m-1} \frac{2^{jp}\|y\|^p}{2^{j-1}}\varepsilon \end{aligned} \tag{2.5}$$

for all $x, y \in X$. By $p < 1$, both the sequences $\{\frac{1}{2^j}(f(2^j x, y) - f(0, y))\}$ and $\{\frac{1}{2^j}(f(0, 2^j y) - f(0, 0))\}$ are Cauchy sequences for all $x, y \in X$. Since Y is complete, the sequences $\{\frac{1}{2^j}(f(2^j x, y) - f(0, y))\}$ and $\{\frac{1}{2^j}(f(0, 2^j y) - f(0, 0))\}$ converge for all $x, y \in X$. Define $F_1, F_2 : X \times X \to Y$ by

$$F_1(x, y) := \lim_{j\to\infty} \frac{f(2^j x, y)}{2^j},$$
$$F_2(x, y) := \lim_{j\to\infty} \frac{f(0, 2^j y)}{2^j}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.4) and (2.5), then one can obtain the inequalities

$$\|f(x, y) - f(0, y) - F_1(x, y)\| \le \frac{4\varepsilon}{2 - 2^p}\|x\|^p + 4\varepsilon\|y\|^p,$$
$$\|f(0, y) - f(0, 0) - F_2(x, y)\| \le \frac{4\varepsilon}{2 - 2^p}\|y\|^p$$

for all $x, y \in X$. By (2.1), (2.2) and the definitions of F_1 and F_2, we get

$$J_1F_1(x, y, z) = \lim_{j\to\infty} \frac{A_1(2^{j-1}x, 2^{j-1}y, z)}{2^j} = 0,$$
$$J_2F_1(x, y, z) = \lim_{j\to\infty} \frac{A_2(2^j x, y, z) - A_2(2^j x, \frac{y+z}{2}, \frac{y+z}{2})}{2^j} = 0,$$
$$J_1F_2(x, y, z) = 0,$$
$$J_2F_2(x, y, z) = \lim_{j\to\infty} \frac{A_2(0, 2^{j-1}y, 2^{j-1}z)}{2^j} = 0$$

for all $x, y, z \in X$ and so F is a bi-Jensen mapping satisfying (2.3), where F is given by

$$F(x, y) = F_1(x, y) + F_2(x, y) + f(0, 0).$$

Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.3) with $F'(0, 0) = f(0, 0)$. By Lemma 2.1, we have

$$\begin{aligned}
&\|F(x, y) - F'(x, y)\| \\
&\qquad = \frac{1}{2^n}\|(F - F')(2^n x, y) + (1 - \frac{1}{2^n})(F - F')(0, 2^n y)\| \\
&\qquad \le \frac{1}{2^n}\big(\|(F - f)(2^n x, y)\| + \|(F - f)(0, 2^n y)\|
\end{aligned}$$

$$+ \|(f - F')(2^n x, y)\| + \|(f - F')(0, 2^n y)\|\big)$$
$$\leq \big(\frac{2^p}{2}\big)^n \frac{8\varepsilon}{2 - 2^p}\|x\|^p + \big(\frac{2^{np+1}}{2^n}\big)\big(\frac{8}{2 - 2^p} + 8\big)\varepsilon\|y\|^p$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x, y) = F'(x, y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Let $f, f_1, f_2, f_3, f_4, F, F' : X \times X \to Y$ be the bi-Jensen maps defined by

$$f(x, y) = f_1(x, y) = f_2(x, y) = f_3(x, y) = f_4(x, y) := 0,$$

$$F(x, y) := \varepsilon, F'(x, y) := -\varepsilon$$

for all $x, y \in X$. Then $f, f_1, f_2, f_3, f_4, F, F'$ satisfy the conditions in Theorem 2.3 but $F' \neq F$. Hence the condition $F(0, 0) = f(0, 0)$ is necessary to show that the map F is unique.

THEOREM 2.4. *Let $2 < p$ and let $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ be the mappings satisfying (2.1) and (2.2) for all $x, y, z \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x, y) - F(x, y)\| \leq \big(\frac{16}{2^p - 4} + \frac{2 \cdot 2^p}{2^p - 4} + \frac{4}{2^p - 2}\big)\varepsilon(\|x\|^p + \|y\|^p) \tag{2.6}$$

for all $x, y \in X$. The mapping F is given by

$$F(x, y) := \lim_{j\to\infty} \big(4^j f(\frac{x}{2^j}, \frac{y}{2^j}) - (4^j - 2^j)(f(\frac{x}{2^j}, 0) + f(0, \frac{y}{2^j})) + (2^j - 1)^2 f(0, 0)\big)$$

for all $x, y \in X$.

Proof. By Lemma 2.2, (2.1) and (2.2), we get

$$\|2^l(f(\frac{x}{2^l}, 0) - f(0, 0)) - 2^m(f(\frac{x}{2^m}, 0) - f(0, 0))\|$$
$$= \|\sum_{j=l}^{m-1} 2^{j+1} A_1(\frac{x}{2^{j+1}}, 0, 0)\| \leq \sum_{j=l}^{m-1} \frac{2^{j+3}}{2^{(j+1)p}}\varepsilon\|x\|^p, \tag{2.7}$$

$$\|2^l(f(0,\frac{y}{2^l})-f(0,0))-2^m(f(0,\frac{y}{2^m})-f(0,0))\|$$

$$=\|\sum_{j=l}^{m-1}2^{j+1}A_2(0,0,\frac{y}{2^{j+1}})\|\le\sum_{j=l}^{m-1}\frac{2^{j+3}}{2^{(j+1)p}}\varepsilon\|y\|^p, \tag{2.8}$$

$$\|4^l f'''(\frac{x}{2^l},\frac{y}{2^l})-4^m f'''(\frac{x}{2^m},\frac{y}{2^m})\|=\|\sum_{j=l}^{m-1}4^{j+1}A_3(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}})\|$$

$$\le\sum_{j=l}^{m-1}(\frac{4^{j+2}}{2^{(j+1)p}}+\frac{4^{j+1}}{2\cdot 2^{jp}})\varepsilon(\|x\|^p+\|y\|^p) \tag{2.9}$$

for all $x,y\in X$ and given integers l,m $(0\le l<m)$. By the similar method in Theorem 2.3, we can define $F_1,F_2,F_3 : X\times X\to Y$ by

$$F_1(x,y):=\lim_{j\to\infty}2^j(f(\frac{x}{2^j},0)-f(0,0)),$$
$$F_2(x,y):=\lim_{j\to\infty}2^j(f(0,\frac{y}{2^j})-f(0,0)),$$
$$F_3(x,y):=\lim_{j\to\infty}4^j f'''(\frac{x}{2^j},\frac{y}{2^j})$$

for all $x,y\in X$. Putting $l=0$ and taking $m\to\infty$ in (2.7), (2.8) and (2.9), one can obtain the inequalities

$$\|f(0,y)-f(0,0)-F_1(x,y)\|\le\frac{4\varepsilon}{2^p-2}\|x\|^p,$$
$$\|f(0,y)-f(0,0)-F_2(x,y)\|\le\frac{4\varepsilon}{2^p-2}\|y\|^p,$$
$$\|f'''(x,y)-F_3(x,y)\|\le(\frac{16}{2^p-4}+\frac{2\cdot 2^p}{2^p-4})\varepsilon(\|x\|^p+\|y\|^p)$$

By Lemma 2.2, (2.1), (2.2) and the definitions of F_1 and F_2, we get

$$J_1F_1(x,y,z)=\lim_{j\to\infty}2^jA_1(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},0)=0,$$
$$J_2F_1(x,y,z)=0,$$
$$J_1F_2(x,y,z)=0,$$
$$J_2F_2(x,y,z)=\lim_{j\to\infty}2^jA_2(0,\frac{y}{2^{j+1}},\frac{z}{2^{j+1}})=0,$$
$$J_1F_3(x,y,z)=\lim_{j\to\infty}4^j(A_1(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},\frac{z}{2^j})-A_1(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}},0))=0,$$
$$J_2F_3(x,y,z)=\lim_{j\to\infty}4^j(A_2(\frac{x}{2^j},\frac{y}{2^{j+1}},\frac{z}{2^{j+1}})-A_2(0,\frac{y}{2^{j+1}},\frac{z}{2^{j+1}}))=0$$

for all $x, y, z \in X$ and so F is a bi-Jensen mapping satisfying (2.6) where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + F_3(x,y) + f(0,0).$$

Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.6) with $F'(0,0) = f(0,0)$. By Lemma 2.1 and $F'(0,0) = f(0,0) = F(0,0)$, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&\quad = \|4^n(F-F')(\frac{x}{2^n}, \frac{y}{2^n}) + (2^n - 4^n)\big((F-F')(\frac{x}{2^n}, 0) \\
&\qquad\qquad + (F-F')(0, \frac{y}{2^n})\big)\| \\
&\quad \le 4^n\big(\|(F-f)(\frac{x}{2^n}, \frac{y}{2^n})\| + \|(f-F')(\frac{x}{2^n}, \frac{y}{2^n})\| + \|(F-f)(\frac{x}{2^n}, 0)\| \\
&\quad + \|(f-F')(\frac{x}{2^n}, 0)\| + \|(F-f)(0, \frac{y}{2^n})\| + \|(f-F')(0, \frac{y}{2^n})\|\big) \\
&\quad \le \frac{4^{n+1}}{2^{np}}(\frac{16}{2^p-4} + \frac{2\cdot 2^p}{2^p-4} + \frac{4}{2^p-2})\varepsilon(\|x\|^p + \|y\|^p)
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

THEOREM 2.5. *Let $1 < p < 2$ and let $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ be the mappings satisfying (2.1) and (2.2) for all $x, y, z \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \le \big(\frac{16}{4-2^p} + \frac{2\cdot 2^p}{4-2^p} + \frac{4}{2^p-2}\big)\varepsilon(\|x\|^p + \|y\|^p) \tag{2.10}$$

for all $x, y \in X$. The mapping F is given by

$$\begin{aligned}
F(x,y) &:= \lim_{j\to\infty} \frac{1}{4^j}[f(2^jx, 2^jy) - f(2^jx, 0) - f(0, 2^jy)] \\
&\quad + \lim_{j\to\infty}[2^j(f(\frac{x}{2^j}, 0) + f(0, \frac{y}{2^j})) - (2^{j+1}-1)f(0,0)]
\end{aligned}$$

for all $x, y \in X$.

Proof. Let F_1, F_2 be as in the proof of Theorem 2.4. By Lemma 2.2, (2.1) and (2.2), we get

$$\begin{aligned}
\|\frac{1}{4^j}f'''(2^jx, 2^jy) - \frac{1}{4^{j+1}}f'''(2^{j+1}x, 2^{j+1}y)\| &= \frac{1}{4^j}\|A_3(2^jx, 2^jy)\| \\
&\le (\frac{2^{jp}}{4^{j-1}} + \frac{2^{(j+1)p}}{2\cdot 4^j})\varepsilon(\|x\|^p + \|y\|^p)
\end{aligned}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. By the similar method in Theorem 2.3, we define $F_3 : X \times X \to Y$ by

$$F_3(x, y) := \lim_{j\to\infty} \frac{1}{4^j} f'''(2^j x, 2^j y)$$

for all $x, y \in X$ and obtain the inequality

$$\|f'''(x, y) - F_3(x, y)\| \leq (\frac{16}{4 - 2^p} + \frac{2 \cdot 2^p}{4 - 2^p})(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. By (2.1), (2.2) and the definition of F_3, we get

$$J_1 F_3(x, y, z) = \lim_{j\to\infty} \frac{A_1(2^{j-1}x, 2^{j-1}y, 2^j z) - A_1(2^{j-1}x, 2^{j-1}y, 0)}{4^j} = 0,$$
$$J_2 F_3(x, y, z) = \lim_{j\to\infty} \frac{A_2(2^j x, 2^{j-1}y, 2^{j-1}z) - A_2(0, 2^{j-1}y, 2^{j-1}z)}{4^j} = 0$$

for all $x, y, z \in X$ and so F is a bi-Jensen mapping satisfying (2.10) where F is given by

$$F(x, y) = F_1(x, y) + F_2(x, y) + F_3(x, y) + f(0, 0).$$

Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2.10) with $F'(0, 0) = f(0, 0)$. By Lemma 2.1 and $F'(0, 0) = f(0, 0) = F(0, 0)$, we have

$$\begin{aligned}
&\|F(x, y) - F'(x, y)\| \\
&\quad = \|\frac{(F - F')(2^n x, 2^n y)}{4^n} + (2^n - 1)\big((F - F')(\frac{x}{2^n}, 0) \\
&\qquad\qquad + (F - F')(0, \frac{y}{2^n})\big)\| \\
&\quad \leq \|\frac{(F - f)(2^n x, 2^n y)}{4^n}\| + \|\frac{(f - F')(2^n x, 2^n y)}{4^n}\| + 2^n\big(\|(F - f)(\frac{x}{2^n}, 0)\| \\
&\quad + 2^n\|(F - f)(0, \frac{y}{2^n})\| + \|(f - F')(\frac{x}{2^n}, 0)\| + \|(f - F')(0, \frac{y}{2^n})\|\big) \\
&\quad \leq \big(\frac{2^{np}}{4^n} + \frac{2^n}{2^{np}}\big)\big(\frac{32}{4 - 2^p} + \frac{4 \cdot 2^p}{4 - 2^p} + \frac{8}{2^p - 2}\big)\varepsilon(\|x\|^p + \|y\|^p)
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x, y) = F'(x, y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

THEOREM 2.6. *Let $0 \le p(\neq 1), 0 < \varepsilon$ and let $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ be the mappings such that*

$$\|P_1(f, f_1, f_2)(x, y, z)\| \le \varepsilon(\|x\|^p + \|y\|^p)\|z\|^p, \tag{2.11}$$

$$\|P_2(f, f_3, f_4)(x, y, z)\| \le \varepsilon\|x\|^p(\|y\|^p + \|z\|^p) \tag{2.12}$$

for all $x, y, z \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \le \frac{4\varepsilon}{|2 - 2^p|}\|x\|^p\|y\|^p$$

for all $x, y \in X$ with $F(0, 0) = f(0, 0)$. The mapping $F : X \times X \to Y$ is given by

$$F(x, y) = \lim_{j\to\infty} \frac{f(2^j x, y)}{2^j} + f(0, y) \qquad \text{if} \quad 0 \le p < 1,$$

$$F(x, y) = \lim_{j\to\infty} 2^j(f(\frac{x}{2^j}, y) - f(0, y)) + f(0, y) \qquad \text{if} \quad p > 1.$$

Proof. By Lemma 2.2, (2.11) and (2.12), we get

$$\begin{aligned}\|\frac{f(2^j x, y) - f(0, y)}{2^{j+1}} - \frac{f(2^{j+1}x, y) - f(0, y)}{2^{j+1}}\| &= \|\frac{A_1(2^j x, 0, y)}{2^{j+1}}\| \\ &\le \frac{2^{jp}\|x\|^p\|y\|^p}{2^{j-1}}\varepsilon,\end{aligned}$$

$$\|f(0, y) - f(0, 0) - \frac{f(0, 2^n y) - f(0, 0)}{2^n}\| = \|\sum_{j=0}^{n-1} \frac{A_2(0, 0, 2^j y)}{2^{j+1}}\| = 0$$

for all $x, y \in X, j \in \mathbb{N}$ if $0 \le p < 1$ and

$$\begin{aligned}&\|2^j(f(\frac{x}{2^j}, y) - f(0, y)) - 2^{j+1}f(\frac{x}{2^{j+1}}, y) - f(0, y))\| \\ &\qquad = \|2^j A_1(\frac{x}{2^{j+1}}, 0, y)\| \le \frac{2^{j+2}\|x\|^p\|y\|^p}{2^{(j+1)p}}\varepsilon,\end{aligned}$$

$$\begin{aligned}&\|f(0, y) - f(0, 0) - 2^n(f(0, \frac{y}{2^n}) - f(0, 0))\| \\ &\qquad = \|\sum_{j=0}^{n-1} 2^{j+1}A_2(0, 0, \frac{y}{2^{j+1}})\| = 0\end{aligned}$$

for all $x, y \in X, j \in \mathbb{N}$ if $p > 1$. The remainder of proof is same to the proof of Theorem 2.3. □

THEOREM 2.7. *Let $0 \le p(\neq 1), 0 < \varepsilon$ and let $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ be the mappings satisfying (2.11) and (2.12) for all $x, y, z \in X$.*

Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ *such that*

$$\|f(x,y) - F(x,y)\| \le \frac{\varepsilon}{|4-4^p|}\|x\|^p\|y\|^p$$

for all $x, y \in X$. *The mapping* $F : X \times X \to Y$ *is given by*

$$F(x,y) = \lim_{j\to\infty} \frac{f(2^j x, 2^j y)}{4^j} + f(x,0) + f(y,0) - f(0,0) \qquad \text{if } 0 \le p < 1,$$

$$F(x,y) = \lim_{j\to\infty} 4^j(f(\frac{x}{2^j},\frac{y}{2^j}) - f(\frac{x}{2^j},0) - f(0,\frac{y}{2^j}) + f(0,0)) \\ + f(x,0) + f(y,0) - f(0,0) \qquad \text{if } p > 1.$$

Proof. By Lemma 2.2, (2.11) and (2.12), we get

$$\|\frac{f'''(2^j x, 2^j y)}{4^j} - \frac{f'''(2^{j+1}x, 2^{j+1}y)}{4^{j+1}}\| = \|\frac{A_3(2^j x, 2^j y)}{4^j}\| \\ \le \frac{4^{jp}(2+2^p)}{4^j}\varepsilon\|x\|^p\|y\|^p$$

for all $x, y \in X, j \in \mathbb{N}$ if $0 \le p < 1$ and

$$\|4^j f'''(\frac{x}{2^j},\frac{y}{2^j}) - 4^{j+1}f'''(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}})\| = 4^{j+1}\|A_3(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}})\| \\ \le \frac{4^{j+1}(2+2^p)}{4^{(j+1)p}}\varepsilon\|x\|^p\|y\|^p$$

for all $x, y \in X, j \in \mathbb{N}$ if $p > 1$. As in the proof of Theorem 2.6, we get

$$f(x,0) - f(0,0) = \frac{f(2^n x, 0) - f(0,0)}{2^n} = 2^n(f(\frac{x}{2^n},0) - f(0,0)),$$
$$f(0,y) - f(0,0) = \frac{f(0,2^n y) - f(0,0)}{2^n} = 2^n(f(0,\frac{y}{2^n}) - f(0,0))$$

for all $x, y \in X, n \in \mathbb{N}$. The remainder of proof is same to the proof of Theorem 4. □

3. Stability of a bi-Pexider functional equation on the punctured domain

The following theorem can be found in [8].

THEOREM 3.1. *Let* $p < 0$ *and* $\varepsilon > 0$. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$\|J_1 f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$
$$\|J_2 f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X\backslash\{0\}$. *Then there exists a (unique for the bi-Jensen mapping* F' *with* $F(0,0) = F'(0,0)$*) bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\|f(x,y) - F(x,y)\| \le \frac{3 - 2^p + 3^p}{2 - 2^p}\varepsilon\|x\|^p + \frac{12 - 3\cdot 2^p + 2\cdot 3^p}{2(2-2^p)}\varepsilon\|y\|^p$$

for all $x, y \in X\backslash\{0\}$.

THEOREM 3.2. *Let* $p < 0$ *and* $\varepsilon > 0$ *and let* $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ *be the mappings satisfying (2.1) and (2.2) for all* $x, y, z \in X\backslash\{0\}$. *Then there exists a (unique for the bi-Jensen mapping* F' *with* $F(0,0) = F'(0,0)$*) bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\|f(x,y) - F(x,y)\| \le \frac{12 - 4\cdot 2^p + 4\cdot 3^p}{2^p(2-2^p)}\varepsilon\|x\|^p + \frac{24 - 6\cdot 2^p + 4\cdot 3^p}{2^p(2-2^p)}\varepsilon\|y\|^p$$

for all $x, y \in X\backslash\{0\}$.

Proof. Since

$$\begin{aligned} \|J_1 f(x,y,z)\| &= \|A_1(\frac{x}{2}, \frac{y}{2}, z)\| \le \frac{4\varepsilon}{2^p}(\|x\|^p + \|y\|^p + \|2z\|^p), \\ \|J_2 f(x,y,z)\| &= \|A_2(x, \frac{y}{2}, \frac{z}{2})\| \le \frac{4\varepsilon}{2^p}(\|x\|^p + \|y\|^p + \|2z\|^p) \end{aligned}$$

for all $x, y, 2z \in X\backslash\{0\}$, we can apply Theorem 3.1 and obtain the desired result. □

The following theorem can be found in [8].

THEOREM 3.3. *Let* $p < 0$ *and* $\varepsilon > 0$. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$\begin{aligned} \|J_1 f(x,y,z)\| &\le \varepsilon(\|x\|^p + \|y\|^p)\|z\|^p, \\ \|J_2 f(x,y,z)\| &\le \varepsilon\|x\|^p(\|y\|^p + \|z\|^p) \end{aligned}$$

for all $x, y, z \in X\backslash\{0\}$. *Then there exists a bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$f(x,y) = F(x,y)$$

for all $(x,y) \ne (0,0)$.

THEOREM 3.4. *Let* $p < 0$ *and* $\varepsilon > 0$. *Let* $f, f_1, f_2, f_3, f_4 : X \times X \to Y$ *be the mappings satisfying (2.11) and (2.12) for all* $x, y, z \in X\backslash\{0\}$. *Then there exists a bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$f(x,y) = F(x,y)$$

for all $(x,y) \ne (0,0)$.

Proof. Since

$$\|J_1 f(x,y,z)\| = \|A_1(\frac{x}{2},\frac{y}{2},z)\| \le \frac{4\varepsilon}{2^p}(\|x\|^p + \|y\|^p)\|z\|^p,$$
$$\|J_2 f(x,y,z)\| = \|A_2(x,\frac{y}{2},\frac{z}{2})\| \le \frac{4\varepsilon}{2^p}\|x\|^p(\|y\|^p + \|z\|^p)$$

for all $x,y,z \in X\backslash A$. Hence we can apply Theorem 3.3 and obtain the desired result. □

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