JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 23, No. 2, June 2010

PROPERTIES OF GENERALIZED BIPRODUCT HOPF ALGEBRAS

Junseok Park* and Wansoon Kim**

Abstract. The biproduct bialgebra has been generalized to generalized biproduct bialgebra $B \times_H^L D$ in [5]. Let (D, B) be an admissible pair and let D be a bialgebra. We show that if generalized biproduct bialgebra $B \times_H^L D$ is a Hopf algebra with antipode s, then D is a Hopf algebra and the identity id_B has an inverse in the convolution algebra $Hom_k(B, B)$. We show that if D is a Hopf algebra with antipode s_D and $s_B \in Hom_k(B, B)$ is an inverse of id_B then $B \times_H^L D$ is a Hopf algebra with antipode s described by $s(b \times_H^L d) = \Sigma(1_B \times_H^L s_D(b_{-1} \cdot d))(s_B(b_0) \times_H^L 1_D)$. We show that the mapping system $B \leftrightarrows^{\Pi_B}_{j_B} B \times_H^L D \rightleftarrows^{\pi_D}_{i_D} D$ (where j_B and i_D are the canonical inclusions, Π_B and π_D are the canonical coalgebra projections) characterizes $B \times_H^L D$. These generalize the corresponding results in [6].

The usual smash product $A\#H$ of an H-module algebra A and a Hopf algebra H has been defined in [7] or [8] and Molnar constructed a smash coproduct $C\natural H$ of an H-comodule coalgebra C and a Hopf algebra H in [4].

Definition 1 [1]. Let H be a bialgebra over a field k and C be a left H-comodule coalgebra. Let E be a left H-module coalgebra. The *generalized smash coproduct* $C\natural_H^L E$ is defined to be $C \otimes_k E$ as a vector space with comultiplication given by

$$\Delta(c\natural_H^L e) = \Sigma(c_1\natural_H^L c_{2,-1} \cdot e_1) \otimes (c_{2,0}\natural_H^L e_2)$$

and counit

$$\varepsilon(c\natural_H^L e) = \varepsilon_C(c)\varepsilon_E(e)$$

Received March 17, 2010; Accepted April 23, 2010.
2000 *Mathematics Subject Classifications*: Primary 16S40.
Key words and phrases: Hopf algebra, biproduct, generalized biproduct.
Correspondence should be addressed to Junseok Park, junspk@hoseo.edu.
* This research was supported by the Academic Research fund of Hoseo University in 2009 (20090038).

for all $c \in C$, $e \in E$.

It is straightforward to show that $\pi_C : C\sharp E \longrightarrow C,\ c\sharp e \longmapsto c\,\varepsilon_E(e)$ and $\pi_E : C\sharp E \longrightarrow E,\ c\sharp e \longmapsto \varepsilon_C(c)e$ are coalgebra surjections since C is a left H-comodule coalgebra and E is a left H-module coalgebra.

Definition 2 [2]. Let H be a bialgebra over a field k and A be a left H-module algebra. Let D be a left H-comodule algebra. The *generalized smash product* $A\#^L_H D$ is defined to be $A \otimes_k D$ as a vector space, with multiplication given by

$$(a\#^L_H d)(b\#^L_H e) = \Sigma a(d_{-1} \cdot b)\#^L_H d_0 e$$

and unit $1_A \otimes 1_D$ for all $a, b \in A$ and $d, e \in D$.

It is straightforward to show that $i_A : A \longrightarrow A\#^L_H D,\ a \longmapsto a\#^L_H 1_D$ and $i_D : D \longrightarrow A\#^L_H D,\ d \longmapsto 1_A\#^L_H d$ are algebra maps since A is a left H-module algebra and D is a left H-comodule algebra.

Definition 3 [5]. Let H be a bialgebra over a field k. Let B be a left H-module algebra and a left H-comodule coalgebra. Let D be a left H-comodule algebra and a left H-module coalgebra. The *generalized biproduct* $B \times^L_H D$ of B and D is defined to be $B\#^L_H D$ as an algebra and $B\sharp^L_H D$ as a coalgebra.

Example 1. A bialgebra H is a left H-comodule algebra via Δ_H because Δ_H is an algebra map. H is a left H-module coalgebra via m_H because m_H is a coalgebra map. The generalized biproduct $B \times^L_H H$ is a biproduct $B \times H$ in [3].

Definition 4. Let H be a bialgebra over k. B is called *a left-left H-crossed module crossed algebra* if B is a left H-module algebra and is a left H-comodule coalgebra such that $\varepsilon_B(h \cdot b) = \varepsilon_H(h)\varepsilon_B(b),\quad b \in B, h \in H$ and $\psi_B(1_B) = 1_H \otimes 1_B$. D is called *a left-left H-crossed comodule crossed algebra* if D is a left H-comodule algebra and a left H-module coalgebra such that $h \cdot 1_D = \varepsilon_H(h)1_D,\ \ h \in H,\ \ \Sigma d_{-1}\varepsilon_D(d_0) = \varepsilon_D(d)1_H,\ \ d \in D$ and $\varepsilon_D(1_D) = 1_k$.

EXAMPLE 2. Let B be a left H-module algebra, a left H-comodule coalgebra, a left H-module coalgebra and a left H-comodule algebra. Then B is a left-left H-crossed module crossed algebra. Let the bialgebra D be a left H-comodule algebra, a left H-module coalgebra, a left H-comodule coalgebra and a left H-module algebra. Then D is a left-left H-crossed comodule crossed algebra.

The followings generalize the corresponding results in [6].

PROPOSITION 1. *Let H be a bialgebra over k. Suppose B is a left-left H-crossed module crossed algebra and D is a left-left H-crossed comodule crossed algebra. Then the followings are equivalent;*

(1) $(\mathrm{B}\times^L_H D,\ m_{B\#^L_H D},\ \eta_{B\#^L_H D},\ \Delta_{B\sharp^L_H D},\ \varepsilon_{B\sharp^L_H D})$ *is a bialgebra.*

(2) ε_B *and* ε_D *are algebra maps,* $\Delta_B(1_B) = 1_B \otimes 1_B$, $\Delta_D(1_D) = 1_D \otimes 1_D$, *and the identities*

(i) $$\Sigma 1_B \times^L_H (b_{-1}\cdot d_1)(b'_{-1}\cdot d'_1) \otimes b_0(d_{2,-1}\cdot b'_0) \times^L_H d_{2,0}d'_2 = \Sigma 1_B \times^L_H [b(d_{-1}\cdot b')]_{-1}\cdot(d_{0,1}d'_1) \otimes [b(d_{-1}\cdot b')]_0 \times^L_H d_{0,2}d'_2.$$

(ii) $$\Sigma [b(d_{-1}\cdot b')]_1 \times^L_H 1_D \otimes [b(d_{-1}\cdot b')]_2 \times^L_H d_0 d' = \Sigma b_1 b'_1 \times^L_H 1_D \otimes b_2(d_{-1}\cdot b'_2) \times^L_H d_0 d'$$

(iii) $$\Sigma b' \times^L_H (b_{-1}\cdot d) \otimes b_0 \times^L_H 1_D = \Sigma (b_{-1}\cdot d)_{-1}\cdot b' \times^L_H (b_{-1}\cdot d)_0 \otimes b_0 \times^L_H 1_D$$

hold for $b, b' \in B$ *and* $d, d' \in D$.

Proof. From Theorem 1 of [5]. □

DEFINITION 5. Let H be a bialgebra and suppose that B is a left-left H-crossed module crossed algebra and D is a left-left H-crossed comodule crossed algebra. In case $(\mathrm{B}\times^L_H D,\ m_{B\#^L_H D},\ \eta_{B\#^L_H D},\ \Delta_{B\sharp^L_H D},\ \varepsilon_{B\sharp^L_H D})$ is a bialgebra, we say the pair (D, B) is *admissible.*

Throughout we let H be a bialgebra over k. Suppose B is a left-left H-crossed module crossed algebra and D is a left-left H-crossed comodule crossed algebra.

THEOREM 1. *Suppose that (D,B) is an admissible pair and that D is a bialgebra.*

(1) If $B\times_H^L D$ is a Hopf algebra with antipode s, then D is a Hopf algebra and the identity id_B has an inverse in the convolution algebra $Hom_k(B,B)$.

(2) If D is a Hopf algebra with antipode s_D and $s_B \in Hom_k(B,B)$ is an inverse of id_B, then $B\times_H^L D$ is a Hopf algebra with antipode s described by

$$s(b\times_H^L d)=\Sigma(1_B\times_H^L s_D(b_{-1}\cdot d))(s_B(b_0)\times_H^L 1_D).$$

Proof. (1): Define $\pi_D : B\times_H^L D\to D$, $b\times_H^L d\mapsto \varepsilon_B(b)d$, $i_D : D\to B\times_H^L D$, $d\mapsto 1_B\times_H^L d$, $j_B : B\to B\times_H^L D$, $b\mapsto b\times_H^L 1_D$, $\Pi_B : B\times_H^L D\to B$, $b\times_H^L d\mapsto \varepsilon_D(d)b$. Let $s_D=\pi_D\circ s\circ i_D$. Then $1_B\times_H^L (\Sigma s_D(d_1)d_2)=1_B\times_H^L \varepsilon_D(d)1_D$. Therefore $\Sigma s_D(d_1)d_2=\varepsilon_D(d)1_D$. Similarly, $\Sigma d_1 s_D(d_2)=\varepsilon_D(d)1_D$. So D is a Hopf algebra with antipode s_D. And $j_B : B\to B\times_H^L D$ is an algebra homomorphism since B is a left H-module algebra. We transfer the coalgebra structure of B to $j_B(B)=B\times_H^L 1_D$ via the algebra isomorphism $j_B : B\to B\times_H^L D$ and identify B with $B\times_H^L 1_D$. Let $\pi=i_D\circ\pi_D$ and $\Pi=j_B\circ\Pi_B$. Let $S\in End_k(B\times_H^L D)$ be defined by $S=\pi * s$. Then $\Delta(b\times_H^L 1_D)=\Sigma(b_1\times_H^L 1_D)(b_2\times_H^L 1_D)$. So $S(b\times_H^L 1_D)=s(b\times_H^L 1_D)$. Therefore

$$\Sigma(b_1\times_H^L 1_D)S(b_2\times_H^L 1_D)=\varepsilon(b\times_H^L 1_D)1_B\times_H^L 1_D. \qquad (*)$$

Thus $S|_{B\times_H^L 1_D}$ is a right inverse of $id_{B\times_H^L 1_D}\in Hom_k(B\times_H^L 1_D, B\times_H^L D)$. Since $(b\times_H^L 1_D)(1_B\times_H^L d)=b\times_H^L d$ and $\Delta(1_B\times_H^L d)=\Sigma(1_B\times_H^L d_1)(1_B\times_H^L d_2)$, we have $S(b\times_H^L d)=\varepsilon_D(d)S(b\times_H^L 1_D)$. So $(S\circ\Pi)(b\times_H^L d)=S(b\times_H^L d)$. Therefore $S\circ\Pi=S$. Since $(\pi*\varepsilon)(b\times_H^L d)=\Sigma\pi(b\times_H^L d)$, $S*id=\pi*s*id=\pi*\varepsilon=\pi$ in $End_k(B\times_H^L D)$. We have $\Sigma[S(b_1\times_H^L 1_D)](b_2\times_H^L 1_D)=\varepsilon(b\times_H^L 1_D)(1_B\times_H^L 1_D)$, and thus $S|_{B\times_H^L 1_D}$ is a left inverse of $id_{B\times_H^L 1_D}$. To complete the proof of (1) we need show that $S(B\times_H^L 1_D)\subseteq B\times_H^L 1_D$, that is, $\Pi\circ S=S$ on $B\times_H^L 1_D$. But since Π is a left $B\times_H^L 1_D$-module homomorphism, applying Π to the equation $(*)$ we see that $\Pi\circ(S\mid_{B\times_H^L 1_D})$ is also a right inverse of $id_{B\times_H^L 1_D}$. This means $\Pi\circ S=S$ on $B\times_H^L 1_D$.

(2): From Theorem 3 of [5]. □

Definition 6. Let (D,B) be an admissible pair and suppose that A is a bialgebra over k. Then

$$B \leftrightarrows_{j}^{\Pi} A \rightleftarrows_{i}^{\pi} D$$

is an *admissible mapping system* if the following conditions hold :

(a) $\Pi \circ j = id_B, \quad \pi \circ i = id_D$,

(b) i and π are algebra maps and coalgebra maps, j is an algebra map, and Π is a coalgebra map,

(c) Π is a D-bicomodule map (A is given the D-bimodule structure via pullback along i and B is given the trivial D-bimodule structure),

(d) $j(B)$ is a sub-D-bimodule of A and $\Pi|_{j(B)}$ is a D-bicomodule map (A is given the D-bicomodule structure via pushout along π, B is given the trivial D-bicomodule structure).

Lemma 1. *Let (D,B) be an admissible pair and suppose that A is a bialgebra over k.*

$$B \leftrightarrows_{j}^{\Pi} A \rightleftarrows_{i}^{\pi} D$$

If i is an algebra map and π is a coalgebra map then

(1) A is a D-bimodule (A is given the D-bimodule structure via pullback along i),

(2) B is a D-bimodule (B is given the trivial D-bimodule structure),

(3) A is a D-bicomodule (A is given the D-bicomodule structure via pushout along π),

(4) B is a D-bicomodule (B is given the trivial D-bicomodule structure).

Proof. (1). Define $A \otimes D \longrightarrow A, a \otimes d \longmapsto a \cdot d = ai(d)$. Then A is a right D-module since i is an algebra map. Define $D \otimes A \longrightarrow A, d \otimes a \longmapsto d \cdot a = i(d)a$. Then A is a left D-module since i is an algebra map. For all $d, d' \in D, a \in A$, $(d \cdot a) \cdot d' = (i(d)a) \cdot d' = i(d)ai(d') = i(d)(a \cdot d') = d \cdot (a \cdot d')$. Therefore A is a D-D-bimodule.

(2). Define $B \otimes D \longrightarrow B, b \otimes d \longmapsto b \cdot d = \varepsilon_D(d)b$. Then B is a right D-module since ε_D is an algebra map. Define $D \otimes B \longrightarrow B, d \otimes b \longmapsto d \cdot b = \varepsilon_D(d)b$. Then B is a left D-module. For all $d, d' \in D, a \in A$,

$(d\cdot b)\cdot d' = (\varepsilon_D(d)b)\cdot d' = \varepsilon_D(d)\varepsilon_D(d')b = \varepsilon_D(d)(b\cdot d') = d\cdot(b\cdot d')$. Therefore B is a D-D-bimodule.

(3). Define $\rho_r : A \longrightarrow A\otimes D, a \longmapsto \Sigma a_1 \otimes \pi(a_2)$. Then $(\rho_r\otimes I)\circ\rho_r = (I\otimes\Delta)\circ\rho_r$. And $((I\otimes\varepsilon_D)\circ\rho_r)(a) = a\otimes 1$ for all $a\in A$. Therefore A is a right D-comodule. Define $\rho_l : A \longrightarrow D\otimes A, a\longmapsto \Sigma\pi(a_1)\otimes a_2$. Then $((I\otimes\rho_l)\circ\rho_l)(a) = ((\Delta\otimes I)\circ\rho_l)(a)$, and $((\varepsilon_D\otimes I)\circ\rho_l)(a) = 1\otimes a$ for all $a\in A$. Therefore A is a left D-comodule. And $((I\otimes\rho_r)\circ\rho_l)(a) = (\rho_l\otimes I)\circ\rho_r(a)$ for all $a\in A$. Therefore A is a D-D-bicomodule.

(4). Define $\rho'_r : B\longrightarrow B\otimes D, b\longmapsto b\otimes 1_D$. For all $b\in B$, $((I\otimes\Delta_D)\circ\rho'_r)(b)((\rho'_r\otimes I)\circ\rho'_r)(b)$ and $((I\otimes\varepsilon_D)\circ\rho'_r)(b) = b\otimes 1_k$. Therefore B is a right D-comodule. Define $\rho'_l : B\longrightarrow D\otimes B, b\longmapsto 1_D\otimes b$. Similarly B is left D-comodule. And $((I\otimes\rho'_r)\circ\rho'_l)(b) = ((\rho'_l\otimes I)\circ\rho'_r)(b)$ for all $b\in B$. Therefore B is a D-D-bicomodule. □

Theorem 2. *Let (D,B) be an admissible pair. Then*

$$B \underset{j_B}{\overset{\Pi_B}{\leftrightarrows}} B\times^L_H D \underset{i_D}{\overset{\pi_D}{\rightleftarrows}} D$$

is an admissible mapping system where $i_D : D\longrightarrow B\times^L_H D, d\longmapsto 1_B\times^L_H d$, $j_B : B\longrightarrow B\times^L_H D, b\longmapsto b\times^L_H 1_D$, $\Pi_B : B\times^L_H D\longrightarrow B, b\times^L_H d\longmapsto \varepsilon_D(d)b$ and $\pi_D : B\times^L_H D\longrightarrow D, b\times^L_H d\longmapsto\varepsilon_B(b)d$.

Proof. (a) By the definitions of mappings, $\Pi_B\circ j_B = I_B$, $\pi_D\circ i_D = id_D$.

(b) The maps $j_B :\longrightarrow B\times^L_H D, b\longmapsto b\times^L_H 1_D$ and $i_D : D\longrightarrow b\times^L_H D, d\longmapsto 1_B\times^L_H d$ are algebra maps since B is a left H-module algebra and D is a left H-comodule algebra. The maps $\Pi_B : B\times^L_H D\longrightarrow B, b\times^L_H d\longmapsto\varepsilon_D(d)b$ and $\pi_D : B\times^L_H D\longrightarrow D, b\times^L_H d\longmapsto\varepsilon_B(b)d$ are coalgebra maps since B is a left H-comodule coalgebra and D is a left H-module coalgebra. For all $d\in D$, $(\Delta_{B\times^L_H D}\circ i_D)(d) = ((id_D\otimes id_D)\circ\Delta_D)(d)$ and $(\varepsilon_{B\times^L_H D}\circ i_D)(d) = \varepsilon_{b\times^L_H D}(1_B\times^L_H d) = \varepsilon_B(1_B)\varepsilon_D(d) = 1_k\varepsilon_D(d) = \varepsilon_D(d)$ by Proposition 1, (2). Therefore i_D is a coalgebra map. $\pi_D((a\times^L_H d)(b\times^L_H e)) = \pi_D(a\times^L_H d)\pi_D(b\times^L_H e)$ and $\pi_D(1_B\times^L_H 1_D) = \varepsilon_B(1_B)1_D = 1_k 1_D = 1_D$ by Proposition 1, (2). Therefore π_D is an algebra map.

(c). $\Pi_B(d'\cdot(b\times^L_H d)) = d'\cdot\Pi_B(b\times^L_H d)$ for all $d,d'\in D, b\in B$. Therefore Π_B is a left D-module map. $\Pi_B((b\times^L_H d)\cdot d') = \Pi_B(b\times^L_H d)\cdot d'$ for all

$b \in B, d, d' \in D$. So Π_B is a right D-module map.

(d). Let $\rho_l : j_B(B) \longrightarrow D \otimes j_B(B)$, $\ b \times^L_H 1_D \longmapsto \Sigma \pi_D((b \times^L_H 1_D)_1) \otimes (b \times^L_H 1_D)_2 = 1_D \otimes (b \times^L_H 1_D)$ be the left sub-D-comodule structure map of $j_B(B) = B \times^L_H 1_D$. Let $\rho_D : B \longrightarrow D \otimes B, b \longmapsto 1_D \otimes b$ be the left D-comodule structure map of B. For all $b \times^L_H 1_D \in j_B(B)$, $(\rho_B \circ \Pi_B)(b \times^L_H 1_D) = \rho_B(\varepsilon_D(1_D)b) = \rho_B(1_k b) = 1_D \otimes b = 1_D \otimes \varepsilon_D(1_D)b = (I \otimes \Pi_B)(1_D \otimes (b \times^L_H 1_D)) = ((I \otimes \Pi_B) \circ \rho_l)(b \times^L_H 1_D)$. Hence $\Pi_B|_{j_B(B)}$ is a left D-comodule map. Let $\rho_r : j_B(B) \longrightarrow j_B(B) \otimes D$, $\ b \times^L_H 1_D \longmapsto \Sigma (b \times^L_H 1_D)_1 \otimes \pi_D((b \times^L_H 1_D)_2) = (b \times^L_H 1_D) \otimes 1_D$ be the right sub-D-comodule structure map of $j_B(B)$. Let $\rho'_B : B \longrightarrow B \otimes D$, $\ b \longmapsto b \otimes 1_D$ be the right D-comodule structure map of b. Similarly, $\Pi_B|_{j_B(B)}$ is a right D-comodule map. Therefore $\Pi_B|_{j_B(B)}$ is a D-bicomodule map.

(e). For all $b \times^L_H d \in B \times^L_H D$, $(j_B \circ \Pi_B) * (i_D \circ \pi_D)(b \times^L_H d) = \Sigma (j_B \circ \Pi_B)((b \times^L_H d)_1)(i_D \circ \pi_D)((b \times^L_H d)_2) = \Sigma(\varepsilon_H(b_{2,-1})\varepsilon_D(d_1) b_1 \times^L_H 1_D)(1_B \times^L_H \varepsilon_B(b_{2,0}) d_2) = \Sigma(\varepsilon_H(\varepsilon_B(b_2)1_H) b_1 \times^L_H 1_D)(1_B \times^L_H d) = b \times^L_H d = id(b \times^L_H d)$. Therefore, $(j_B \circ \Pi_B) * (i_D \circ \pi_D) = id$. □

Lemma 2. *Let (D, B) be an admissible pair and let A be a bialgebra over k. Suppose that $B \leftrightarrows^{\Pi}_{j} A \rightleftarrows^{\pi}_{i} D$ is an admissible mapping system. Then*

$$i(d)j(b) = j(b)i(d)$$

for all $b \in B, d \in D$.

Proof.
$$\begin{aligned} i(d)j(b) &= ((j \circ \Pi) * (i \circ \pi))(i(d)j(b)) \\ &= \Sigma (j \circ \Pi)(i(d_1)j(b)_1)(i \circ \pi)(i(d_2)j(b)_2) \\ &= \Sigma (j \circ \Pi)(d_1 \cdot j(b)_1) i((\pi \circ i)(d_2)\pi(j(b)_2)) \\ &= \Sigma j(d_1 \cdot \Pi(j(b)) i(d_2) i(1_D) \\ &= \Sigma j(d_1 \cdot b) i(d_2) \\ &= \Sigma j(\varepsilon_D(d_1) b) i(d_2) \\ &= j(b) i(d) \end{aligned}$$
for all $b \in B, d \in D$. □

Lemma 3. *Let (D, B) be an admissible pair and let A be a bialgebra over k. Suppose that $B \leftrightarrows^{\Pi}_{j} A \rightleftarrows^{\pi}_{i} D$ is an admissible mapping system. Then*

$$\Sigma j(d_{-1} \cdot b')i(d_0) = j(b')i(d)$$

for all $b' \in B, d \in D$.

Proof. By the definition of admissible pair and Theorem 1, (2), ε_B and ε_D are algebra maps. So $\varepsilon_B(h \cdot b) = \varepsilon_H(h)\varepsilon_B(b)$ and $\Sigma d_{-1}\varepsilon_D(d_0)$

$= \varepsilon_D(d)1_H$. By [5, Corollary 3], $\Sigma(d_{-1} \cdot b' \times^L_H 1_D) \otimes (1_B \times^L_H d_0) = (b' \times^L_H 1_D) \otimes (1_B \times^L_H d)$. If we apply $\Pi_B \otimes \pi_D$ to the two-side of the above, we get $\Sigma d_{-1} \cdot b' \otimes d_0 = b' \otimes d$. So $\Sigma j(d_{-1} \cdot b')i(d_0) = j(b)i(d)$. □

Lemma 4. *Let* (D, B) *be an admissible pair and let* A *be a bialgebra over* k. *Suppose that* $B \leftrightarrows^{\Pi}_{j} A \rightleftarrows^{\pi}_{i} D$ *is an admissible mapping system. Then*

$$\Sigma\Pi(a_1)_{-1} \cdot \pi(a_2) \otimes \Pi(a_1)_0 = \Sigma\pi(a_1) \otimes \Pi(a_2)$$

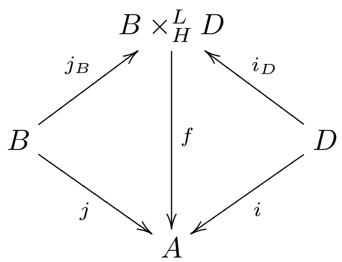
where $B \longrightarrow H \otimes B, b \mapsto \Sigma b_{-1} \otimes b_0$ *is the left* H*-comodule structure map.*

Proof. First let $a \in j(b)$. Then $\Pi|_{j(B)}$ is a right D-comodule map, $\Sigma\Pi(a_1) \otimes \pi(a_2) = \Pi(a) \otimes 1_D$. So, $\Sigma\Pi(a_1)_{-1} \cdot \pi(a_2) \otimes \Pi(a_1)_0 = \Sigma\Pi(a)_{-1} \cdot 1_D \otimes \Pi(a)_0 = \Sigma\varepsilon_B(\Pi(a)_{-1})1_D \otimes \Pi(a)_0 = \Sigma 1_D \otimes \varepsilon_B(\Pi(a)_{-1})$

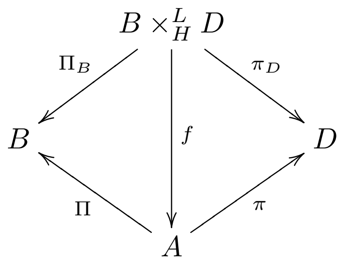
$\Pi(a)_0 = 1_D \otimes \Pi(a) = \Sigma\pi(a_1) \otimes \Pi(a_2)$. From the observation that $\Pi(aa') = \Pi(a)\varepsilon(d')$ for all $a' = i(d') \in i(D)$ and that $A = j(B)i(D)$ for f is surjective, we reduce the general case to the special case. □

Theorem 3. *Let* (D, B) *be an admissible pair and let* A *be a bialgebra over* k. *Suppose that* $B \leftrightarrows^{\Pi}_{j} A \rightleftarrows^{\pi}_{i} D$ *is an admissible mapping system.*

(1) There exists a unique algebra map $f : B \times^L_H D \longrightarrow A$ *such that the diagram*

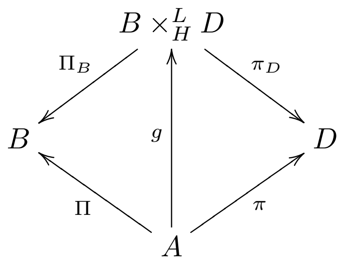


commutes. Furthermore the diagram

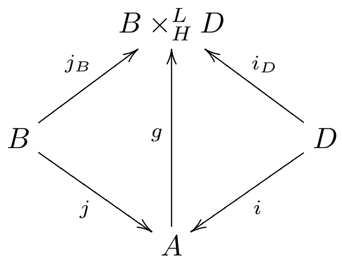


commutes and f is a bialgebra isomorphism.

(2) There exists a unique coalgebra map $g : A \longrightarrow B \times_H^L D$ such that the diagram



commutes. Furthermore the diagram



commutes and g is a bialgebra isomorphism.

Proof. For all $b \in B, d \in D, (b \times_H^L 1_D)(1_B \times_H^L d) = b \times_H^L d$ since D is a left H-comodule algebra and B is a left H-module. If $f : B \times_H^L D \longrightarrow A$ is an algebra map then first diagram commutes if and only if $f(b \times_H^L d) = f(b \times_H^L 1_D) f(1_B \times_H^L d) = j(b)i(d)$ for all $b \in B, d \in D$. If $g : A \longrightarrow B \times_H^L D$ is a coalgebra map then third diagram commutes if and only if $g(a) = I(g(a)) = (j_B \circ \Pi_B) * (i_D \circ \pi_D)(g(a)) = \Sigma \Pi(a_1) \times_H^L \pi(a_2)$ for all $a \in A$ by Theorem 2.

Thus we have the uniqueness of f and g. Let f and g be defined as above. Then $f(g(a)) = f(\Sigma\Pi(a_1) \times_H^L \pi(a_2)) = \Sigma j(\Pi(a_1))i(\pi(a_2)) = ((j \circ \Pi) * (i \circ \pi))(a) = I(a) = a$ and $g(f(b \times_H^L d)) = g(j(b)i(d)) = \Sigma\Pi((j(b)i(d))_1) \times_H^L \pi((j(b)i(d))_2) = \Pi(j(b)) \times_H^L d = b \times_H^L d$. So f and g are inverses. Thus the proof will be complete once we show that f is an algebra map and g is a coalgebra map. $f(1_B \times_H^L 1_D) = j(1_B)i(1_D) = 1_A 1_A = 1_A$ since i and j are algebra maps. We need only show that f is multiplicative. From Lemma 2, and Lemma 3, follow that $f((b \times_H^L d)(b' \times_H^L d')) = \Sigma f(b(d_{-1} \cdot b') \times_H^L d_0 d') = f(b \times_H^L d) f(b' \times_H^L d')$. And $\varepsilon_{B \times_H^L D}(g(a)) = \varepsilon_A(a)$. We need only show that g is comultiplicative. By Lemma 4,

$$
\begin{aligned}
\Sigma(g(a))_1 \otimes (g(a))_2 &= \Delta_{B \times_H^L D}(g(a)) \\
&= \Sigma(\Pi(a)_{1,1} \times_H^L \Pi(a)_{1,2,-1} \cdot \pi(a)_{2,1}) \otimes (\Pi(a)_{1,2,0} \times_H^L \pi(a)_{2,2}) \\
&= \Sigma(\Pi(a_1) \times_H^L \Pi(a_2)_{-1} \cdot \pi(a_3)) \otimes (\Pi(a_2)_0 \times_H^L \pi(a_4)) \\
&= \Sigma(\Pi(a_1) \times_H^L \pi(a_2)) \otimes (\Pi(a_3) \times_H^L \pi(a_4)) \\
&= \Sigma g(a_1) \otimes g(a_2).
\end{aligned}
$$

So g is a coalgebra map. This completes the proof. □

References

[1] S. Caenepeel, G. Militaru and Z. Shenglin, *Crossed Modules and Doi-Hopf Modules,* Israel J. Math. **100** (1997), 221-247.

[2] Zhang Liangyun, *L-R smash products for bimodule algebras,* Progress in Nature Science **16** (6) (2006), 580-587

[3] S. Montgomery, *Hopf Algebras and their actions on Rings,* AMS, Rhode Island, 1992.

[4] R. K. Molnar *Semi-Direct Products of Hopf Algebras,* Journal of Algebra, **47** (1977), 29-51.

[5] J. S. Park, *Generalized Biproduct Hopf Algebras,* J. of the Chungcheong Mathematical Society **21** (2008), no. 3, 301-320

[6] D. E. Radford, *The Structure of Hopf Algebras with a Projection,* Journal of Algebra **92** (1985) 322-347.

[7] M. Takeuchi, $Ext_{ad}(S_pR, \mu^A) \cong \widehat{B_r}(A/k)$, Journal of Algebra, **67** (1980) 436-475

[8] E. Abe, *Hopf Algebras,* Cambridge University Press, Cambridge, 1977.

*

Department of Mathematics
The Research Institute for Basic Sciences,
Hoseo University
Asan 336-795, Republic of Korea

E-mail: junspk@hoseo.edu

**

Department of Mathematics
College of Natural Sciences, Hoseo University
Asan 336-795, Republic of Korea

E-mail: kimws@hoseo.edu