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ON CONVERGENCE FOR THE GR_k-INTEGRAL

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Abstract. In this paper, we study some convergence results for the GR_k-integral.

1. Introduction and preliminaries

In [1], S. Pal, D. K. Ganguly and Lee Peng Yee introduced the GR_k-integral. It is a Stieltjes type integral which for $k = 1$ includes classical Henstock Stieltjes integral in particular case. In [1], some elementary results for the GR_k-integral and also analogue of the Saks-Henstone lemma are studied.

In this paper, we obtain some convergence results for the GR_k-integral.

Let k be a fixed positive integer and δ be a positive function defined on $[a, b]$. We shall have a division D of $[a, b]$ given by $a = x_0 < x_1 < \cdots < x_n = b$ with associated points $\{\xi_0, \xi_1, \cdots, \xi_{n-k}\}$ satisfying

$$\xi_i \in [x_i, x_{i+k}] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \text{ for } i = 0, 1, \cdots, n-k$$

a δ^k-fine division of $[a, b]$. For a given positive δ, we denote a δ^k-fine division D by $\{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\cdots,n-k}$. When $k = 1$, it coincides with the usual definition of δ-fine division.

In [1], the GR_k-integral is defined as follows:

Definition 1.1. Let g be real-valued function defined on a closed interval $[a, b]^{k+1}$ in the $(k + 1)$-dimensional space, and f a real-valued function defined on $[a, b]$. We say that f is GR_k-integrable with respect to g to I on $[a, b]$ if for every $\epsilon > 0$ there is a function $\delta(\xi) > 0$ for

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$\xi \in [a,b]$ such that for any δ^k-fine division $D = \{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\cdots,n-k}$ we have

$$|\sum_{i=0}^{n-k} f(\xi_i) g(x_i, \cdots, x_{i+k}) - I| < \epsilon.$$

We shall denote the above Riemann sum by $s(f,g;D)$. If f is integrable with respect to g in the above sense, we write $(f,g) \in GR_k[a,b]$ and denote the integral by $\int_a^b f dg$.

Let $x \in [x_i, x_{i+k}]$ where $x_i < x_{i+1} < \cdots < x_{i+k}$. The jump of g at x, denoted by $J(g;x)$, is defined by

$$J(g;x) = \lim_{x_i \to x,\, x_{i+k} \to x} g(x_i, \cdots, x_{i+k}),$$

if the limit exists finitely.

Let $[a_i, b_i],\ i = 1, 2, \cdots, p$ be pairwise non-overlapping, and $\cup_{i=1}^p [a_i, b_i] \subset [a,b]$. Then $\{D_i\}_{i=1,2,\cdots,p}$ is said to be a δ^k-fine partial division of $[a,b]$ if each D_i is a δ^k-fine division of $[a_i, b_i]$. Its corresponding partial Riemann sum is given by $\sum_{i=1}^p s(f,g;D_i)$.

With this notion of partial division we have proved in [1] the following theorem.

THEOREM 1.2 (Saks-Henstone lemma analogue for GR_k-integral). *If $(f,g) \in GR_k[a,b]$ and $J(g;c)$ exists for all $c \in (a,b)$, then for every $\epsilon > 0$ there exists a positive function δ on $[a,b]$ such that for any δ^k-fine partial division $\{D_i\}_{i=1,2,\cdots,p}$ of $[a,b]$*

$$|s(f,g;D) - F(a,b)| < \epsilon \text{ and } |\sum_{i=1}^p \{s(f,g;D_i) - F(a_i,b_i)\}| < (k+1)\epsilon$$

where D_i is a δ^k-fine division of $[a_i, b_i]$ and $F(u,v)$ denotes the GR_k-integral on $[u,v] \subset [a,b]$.

2. Some results of convergence theorems for GR_k-integral

DEFINITION 2.1. For $X \subset [a,b]$, we define

$$V_g^k(X) = \inf_\delta \sup_D \sum_{\xi_i \in X} |g(x_i, \cdots, x_{i+k})|,$$

where supremum is taken over all δ^k-fine partial division $D = \{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\cdots,n-k}$ with $\xi_i \in X$.

$X \subset [a,b]$ is said to be of g^k-variation zero if $V_g^k(X) = 0$. Let g be a function from $[a,b]^{k+1}$ to $\mathbb{R}$. Then g is said to be of $BV^k[a,b]$ if $V_g^k[a,b]$ is finite. Also g is said to be $BV^kG(X)$ if $X = \cup_{j=1}^{\infty} X_j$ such that g is $BV^k(X_j)$ for each j. Clearly, g is of $BV^k[a,b]$ if there exists a positive function δ on $[a,b]$ and $M \in \mathbb{R}$ such that $\sum_{i=0}^{n-k} |g(x_i, \cdots, x_{i+k})| < M$ for any δ^k-fine partial division $D = \{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\cdots,n-k}$ of $[a,b]$.

A property is said to hold g^k *a.e.* if it holds everywhere in $[a,b]$ except on set of g^k-variation zero.

THEOREM 2.2. *If $f = 0$ g^k a.e., then $(f,g) \in GR_k[a,b]$ and $\int_a^b f dg =$ 0.*

Proof. Let $f(x) = 0$ for all $x \in [a,b]$ except for a set X of g^k-variation zero and let $X_i = \{x \in X : i-1 < |f(x)| \leq i\}$, $i = 1, 2, \cdots$. So $X_i \subset X$ and $X = \cup_{i=1} X_i$, and $V_g^k(X_i) = 0$ for $i = 1, 2, \cdots$. Hence given $\epsilon > 0$, for each $i \in \mathbb{N}$, there exists $\delta_i(x) > 0$ defined on $[a,b]$ such that $\sum_{\xi_j \in X_i} |g(x_j, \cdots, x_{j+k})| < \frac{\epsilon}{i2^i}$ for $i = 1, 2, \cdots$, $D = \{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\cdots,n-k}$ of $[a,b]$. We define $\delta(x) = \delta_i(x)$ for $x \in X_i$ and 1 otherwise. Let $D = \{[y_j, y_{j+k}], \eta_j\}_{j=0,1,\cdots,n-k}$ be a δ^k-fine partial division of $[a,b]$. Then

$$|s(f,g;D)| = |\sum_{\eta_j \in X} f(\eta_j) g(y_j, \cdots, y_{j+k})| < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$

Thus $(f,\, g) \in GR_k[a,b]$ and $\int_a^b f dg = 0$. □

COROLLARY 2.3. *If f is GR_k-integrable with respect to g to I on $[a,b]$ and $f = h$ g^ka.e. in $[a,b]$, then h is GR_k-integrable with respect to g to I on $[a,b]$ and $\int_a^b f dg = \int_a^b h dg$.*

We now give some convergence theorems for the GR_k-integral.

DEFINITION 2.4. Let $(f_n,\, g) \in GR_k[a,b]$. $\{(f_n,\, g)\}$ is said be equi-GR_k-integrable on $[a,b]$ if for all $\epsilon > 0$ there exists $\delta(x) > 0$, $x \in [a,b]$ such that

$$|s(f_n,\, g; D) - \int_a^b f_n dg| < \epsilon,$$

for all n, whenever $D = \{[x_i, x_{i+k}], \xi_i\}_{i=0,1.\cdots,n-k}$ is a δ^k-fine division of $[a,b]$.

THEOREM 2.5. *Let $g \in BV^k[a,b]$. If (i) $\{(f_n,\, g)\}$ is said to be equi-GR_k-integrable (ii) $f_n \to f$ a.e on $[a,b]$, then $(f,g) \in GR_k[a,b]$ and $\int_a^b f\, dg = \lim_{n\to\infty} \int_a^b f_n\, dg$.*

Proof. We may assume that $f_n(x) \to f(x)$ for each $x \in [a,b]$. Since $\{(f_n, g)\}$ is said to be equi-GR_k-integrable, for $\epsilon > 0$ there exists $\delta_1(x) > 0$ for $x \in [a,b]$ independent n such that $|s(f_n,\, g; D) - A_n| < \epsilon$, for all δ_1^k-fine division $D = \{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\cdots,n-k}$ of $[a,b]$ where $A_n = \int_a^b f_n\, dg$. Also since $g \in BV^k[a,b]$, there exists $\delta_2(x) > 0$ for $x \in [a,b]$ and $M > 0$ such that $\sum_{i=0}^{n-k} |g(x_i, \cdots, x_{i+k})| < M$ for any δ_2^k-fine partial division $D = \{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\cdots,n-k}$ of $[a,b]$. Define $\delta(x) = \min\{\delta_1(x), \delta_2(x) : x \in [a,b]\}$. Let $D = \{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\cdots,n-k}$ of $[a,b]$ be a δ^k-fine division of $[a,b]$ and let $\Lambda = \{\xi_i : 0 \le i \le n-k\}$. Since $f_n(x) \to f(x)$ for each $x \in [a,b]$, we can find N such that $|f_n(\xi) - f(\xi)| < \frac{\epsilon}{M}$ for all $n > N_1$ and for all $\xi \in \Lambda$. So for all $n > N$ we have $|s(f_n,\, g; D) - s(f,\, g; D)| < \epsilon$. Now for $m,\, n > N$,

$$\begin{aligned} |A_n - A_m| &\le |s(f_m,\, g; D) - A_m| + |s(f_m,\, g; D) - s(f_n,\, g; D)| \\ &\quad + |s(f_n,\, g; D) - A_n| < 4\epsilon. \end{aligned}$$

So $\{A_n\}$ is a Cauchy sequence. Let $A = \lim_{n\to\infty} A_n$. Then there exists $K > N$ such that $|\int_a^b f_n\, dg - A| < \epsilon$ for all $n > K$. For all $n > K$, we have

$$\begin{aligned} |s(f,\, g; D) - A| &\le |s(f_n,\, g; D) - A_n| + |s(f_n,\, g; D) - s(f,\, g; D)| \\ &\quad + |A_n - A| < 3\epsilon. \end{aligned}$$

Thus $(f,\, g) \in GR_k[a,b]$ and $\int_a^b f\, dg = \lim_{n\to\infty} \int_a^b f_n\, dg$. □

THEOREM 2.6. **(Uniform Convergence Theorem)** *Let $g \in BV^k[a,b]$ and $\{f_n\}_{i=1}^{\infty}$ be a sequence of functions defined on $[a,b]$ such that $(f_n, g) \in GR_k[a,b]$ for all $n = 1, 2, \cdots$. If $\{f_n\}$ is uniformly convergent to f as $n \to \infty$, then $(f, g) \in GR_k[a,b]$ and $\int_a^b f\, dg = \lim_{n\to\infty} \int_a^b f_n\, dg$.*

Proof. Since $g \in BV^k[a,b]$, there exists $\delta_*(x) > 0$ for $x \in [a,b]$ and $M > 0$ such that $\sum_{i=0}^{n-k} |g(x_i, \cdots, x_{i+k})| < M$ for any δ_*^k-fine partial division $D = \{[x_i, x_{i+k}],\, \xi_i\}_{i=0,1,\cdots,n-k}$ of $[a,b]$. Let $A_n = \int_a^b f_n\, dg$ and let $\epsilon > 0$. Since $(f_n,\, g) \in GR_k[a,b]$ for all $n = 1, 2, \cdots$. there exists $\delta_n(x) > 0$ for $x \in [a,b]$ such that $|s(f_n,\, g; D_n) - A_n| < \epsilon$, for all δ_n^k-fine division $D_n = \{[x_i, x_{i+k}],\, \xi_i\}_{i=0,1,\cdots,n-k}$ of $[a,b]$ with $\delta_n \le \delta_*$. We may assume that $\delta_{n+1} \le \delta_n$ for all n. Also, since $\{f_n\}$ is uniformly convergent to f, there exists N such that $\sup_{a\le x\le b} |f_n(x) - f_m(x)| < \frac{\epsilon}{M}$ and $\sup_{a\le x\le b} |f_n(x) - f(x)| < \frac{\epsilon}{M}$ for $m,\, n > N$. For $m,\, n > N$ we assume that $n > m$, then we have

$$\begin{aligned} |A_n - A_m| &\le |A_n - s(f_n,\, g; D_n)| + |s(f_n,\, g; D_n) - s(f_m,\, g; D_m)| \\ &\quad + |s(f_m,\, g; D_m) - A_m| < 3\epsilon \end{aligned}$$

Thus $\{A_n\}$is a Cauchy sequence in $\mathbb{R}$ and $A = \lim_{n\to\infty} A_n$. Now we can find a positive integer $k \geq N$ such that for $n \leq k$ we have $|A_n - A| < \epsilon$. Define $\delta(x) = \delta_k(x)$ for $x \in [a,b]$. Then for any δ^k-fine division $D = \{[x_i, x_{i+k}],\, \xi_i\}_{i=0,1,\cdots,n-k}$ of $[a,b]$ we have

$$|s(f,g;D)-A| \leq |s(f,g;D)-s(f_k,g;D)|+|s(f_k,g;D)-A_k|+|A_k-A| < 3\epsilon$$

Thus $(f,g) \in GR_k[a,b]$ and $\int_a^b f\,dg = \lim_{n\to\infty} f_n\,dg$. □

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