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# CONVERGENCE THEOREM FOR KURZWEIL–HENSTOCK–PETTIS INTEGRABLE FUZZY MAPPINGS

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Abstract. In this paper, we introduce the Kurzweil-Henstock-Pettis integral of fuzzy mappings in Banach spaces in terms of the Kurzweil-Henstock-Pettis integral of set-valued mappings and obtain some properties of the Kurzweil-Henstock-Pettis integral of fuzzy mappings in Banach spaces and the convergence theorem for Kurzweil-Henstock-Pettis integrable fuzzy mappings.

## 1. Introduction

Several types of integrals of set-valued mappings were studied by Aumann [1], Di Piazza and Musial [2, 3], El Amri and Hess [4], Papageoriou [10] and others. In particular, Di Piazza and Musial [3] introduced the Kurzweil-Henstock-Pettis integral of set-valued mappings whose values are closed bounded convex subsets in Banach spaces and established some properties of the integral. Several mathematicians introduced the integrals of fuzzy mappings in terms of the integrals of set-valued mappings. Kaleva [9] introduced the integral of fuzzy mappings in $\mathbb{R}^n$ in terms of the integral of set-valued mappings in $\mathbb{R}^n$. Xue, Ha and Ma [11], Xue, Wang and Wu [12] also introduced integrals of fuzzy mappings in Banach spaces in terms of Aumann-Pettis and Aumann-Bochner integrals of set-valued mappings.

The purpose of this paper is to study the Kurzweil-Henstock-Pettis integral of fuzzy mappings in Banach spaces. We introduce the Kurzweil-Henstock-Pettis integral of fuzzy mappings in Banach spaces in terms of the Kurzweil-Henstock-Pettis integral of set-valued mappings and obtain some

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properties of the Kurzweil-Henstock-Pettis integral of fuzzy mappings in Banach spaces and the convergence theorem for Kurzweil-Henstock-Pettis integrable fuzzy mappings.

## 2. Prelininaries

Throughout this paper, $\mathcal{L}$ denotes the family of all Lebesgue measurable subsets of $[a, b]$ and $X$ a real separable Banach space with dual $X^*$. $CL(X)$ denotes the family of all nonempty closed subsets of $X$ and $CWK(X)$ the family of all nonempty convex weakly compact subsets of $X$. For $A \subseteq X$ and $x^* \in X^*$, let $s(x^*, A) = \sup\{x^*(x) : x \in A\}$, the support function of $A$. For closed bounded subsets $A, B$ of $X$, let $H(A, B)$ denote the Hausdorff metric of $A$ and $B$ defined by

$$H(A, B) = \max\left(\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right),$$

where $d(a, B) = \inf_{b\in B} \|a - b\|$ and $d(b, A) = \inf_{a\in A} \|a - b\|$. Especially,

$$H(A, B) = \sup_{\|x^*\|\leq 1} |s(x^*, A) - s(x^*, B)|$$

whenever $A, B$ are convex sets. The number $\|A\|$ is defined by

$$\|A\| = H(A, \{0\}) = \sup_{x\in A} \|x\|.$$

Let $u : X \to [0, 1]$. We denote $[u]^r = \{x \in X : u(x) \geq r\}$ for $r \in (0, 1]$ and $[u]^0 = cl\{x \in X : u(x) > 0\}$. $u$ is called a *generalized fuzzy number* on $X$ if for each $r \in (0, 1]$, $[u]^r \in CWK(X)$. Let $\mathcal{F}(X)$ denote the set of all generalized fuzzy numbers on $X$. The addition and scalar multiplication in $\mathcal{F}(X)$ are defined according to Zadeh's extension principle. For $u, v \in \mathcal{F}(X)$ and $\lambda \in \mathbb{R}$, $[u + v]^r = [u]^r + [v]^r$ and $[\lambda u]^r = \lambda[u]^r$ for each $r \in (0, 1]$. Hence $u + v, \lambda u \in \mathcal{F}(X)$. For $u, v \in \mathcal{F}(X)$, we define $u \leq v$ as follows:

$$u \leq v \ \text{ if } \ u(x) \leq v(x) \ \text{ for all } x \in X.$$

For $u, v \in \mathcal{F}(X)$, $u \leq v$ if and only if $[u]^r \subseteq [v]^r$ for each $r \in (0, 1]$. Define $D : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, +\infty]$ by the equation

$$D(u, v) = \sup_{r\in(0,1]} H([u]^r, [v]^r).$$

Then $D$ is a metric on $\mathcal{F}(X)$. The norm $\|u\|$ of $u \in \mathcal{F}(X)$ is defined by

$$\|u\| = D(u, \tilde{0}) = \sup_{r\in(0,1]} H([u]^r, \{0\}) = \sup_{r\in(0,1]} \|[u]^r\|, \text{ where } \tilde{0} = \chi_{\{0\}}.$$

A set-valued mapping $F : [a, b] \to CL(X)$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the real-valued function $s(x^*, F)$ is measurable. A set-valued mapping $F : [a, b] \to CL(X)$ is said to be *measurable* if $F^{-1}(A) = \{t \in [a, b] : F(t) \cap A \neq \emptyset\} \in \mathcal{L}$ for every $A \in CL(X)$. Note that if $F : [a, b] \to CL(X)$ is measurable then $F : [a, b] \to CL(X)$ is scalarly measurable. On the other hand, $F : [a, b] \to CWK(X)$ is measurable if and only if $F : [a, b] \to CWK(X)$ is scalarly measurable [4].

DEFINITION 2.1.([5,6]) A *K-partition* of $[a, b]$ is a finite collection $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ such that $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a, b]$ covering $[a, b]$ and $t_i \in [c_i, d_i]$ for $i = 1, 2, \cdots, n$. A *gauge* on $[a, b]$ is a function $\delta : [a, b] \to (0, \infty)$. A K-partition $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is $\delta$-fine if $[c_i, d_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$ for $i = 1, 2, \cdots, n$. A function $f : [a, b] \to X$ is said to be *Henstock integrable* on $[a, b]$ if there exists $w \in X$ with the following property: for each $\epsilon > 0$ there exists a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$\left\| \sum_{i=1}^{n} f(t_i)(d_i - c_i) - w \right\| < \epsilon$$

for each $\delta$-fine K-partition $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ of $[a, b]$. We write $w = (H) \int_a^b f(t)dt$. In case when $X$ is the real line, the function $f : [a, b] \to \mathbb{R}$ is said to be *Kurzweil-Henstock integrable* or simply *KH-integrable* on $[a, b]$ and we write $w = (KH) \int_a^b f(t)dt$.

Note that if $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable on $[a, b]$, then $f : [a, b] \to \mathbb{R}$ is KH-integrable on $[a, b]$.

$f : [a, b] \to X$ is called a *selection* of $F : [a, b] \to CL(X)$ if $f(t) \in F(t)$ for every $t \in [a, b]$. A set-valued mapping $F : [a, b] \to CL(X)$ is said to be *scalarly Kurzweil-Henstock integrable* or simply *scalarly KH-integrable* on $[a, b]$ if for each $x^* \in X^*$, $s(x^*, F)$ is KH-integrable on $[a, b]$. A set-valued mapping $F : [a, b] \to CL(X)$ is said to be *weakly Kurzweil-Henstock* *integrably bounded* or simply *weakly KH-integrably bounded* on $[a,b]$ if the real-valued function $|x^*F| : [a,b] \to \mathbb{R}$, $|x^*F|(t) = \sup\{|x^*(x)| : x \in F(t)\}$, is KH-integrable for each $x^* \in X^*$. A set-valued mapping $F : [a,b] \to CL(X)$ is said to be *Kurzweil-Henstock integrably bounded* or simply *KH-integrably bounded* on $[a,b]$ if there exists an KH-integrable real-valued function $h$ such that for each $t \in [a,b]$, $\|x\| \leq h(t)$ for all $x \in F(t)$.

DEFINITION 2.2.([3]) A set-valued mapping $F : [a,b] \to CWK(X)$ is said to be *Kurzweil-Henstock-Pettis integrable* or simply *KHP-integrable* on $[a,b]$ if $F : [a,b] \to CWK(X)$ is scalarly KH-integrable on $[a,b]$ and for each subinterval $[c,d]$ of $[a,b]$ there exists $W_{[c,d]} \in CWK(X)$ such that

$$s(x^*, W_{[c,d]}) = (KH) \int_c^d s(x^*, F(t))dt \tag{1.1}$$

for each $x^* \in X^*$. We write $W_{[c,d]} = (KHP) \int_c^d F(t)dt$.

Note that when a set-valued mapping is a function $f : [a,b] \to X$, then the set $W_{[c,d]}$ is reduced to a vector in $X$ and the equality (1.1) turns into

$$x^*(W_{[c,d]}) = (KH) \int_c^d x^* f(t)dt$$

and we say in that case that the function $f$ is *KHP-integrable on* $[a,b]$.

THEOREM 2.3. ([8]) *If* $u \in \mathcal{F}(X)$, *then*

(1) $[u]^r \in CWK(X)$ *for all* $r \in (0,1]$,

(2) $[u]^{r_1} \supseteq [u]^{r_2}$ *for* $0 < r_1 \leq r_2 \leq 1$,

(3) *if* $\{r_n\} \subseteq (0,1]$ *is a nondecreasing sequence converging to* $r \in (0,1]$, *then* $[u]^r = \cap_{n=1}^\infty [u]^{r_n}$.

*Conversely, if* $\{A_r : r \in (0,1]\} \subseteq CL(X)$ *satisfies* (1)-(3) *above, then there exists* $u \in \mathcal{F}(X)$ *such that* $[u]^r = A_r$ *for each* $r \in (0,1]$.

THEOREM 2.4. ([11]) *Let* $\{r_n\} \subseteq (0,1]$ *be a nondecreasing sequence converging to* $r \in (0,1]$, $A_{r_n}, A_r \in CWK(X)$ *and* $A_{r_n} \supseteq A_{r_{n+1}} \supseteq A_r$ $(n \in \mathbb{N})$. *Then* $\{s(x^*, A_{r_n})\}$ *converges to* $s(x^*, A_r)$ *if and only if* $A_r = \cap_{n=1}^\infty A_{r_n}$.

THEOREM 2.5. ([3]) *Let $F : [a,b] \to CWK(X)$ be a scalarly KH-integrable set-valued mapping. Then $F : [a,b] \to CWK(X)$ is KHP-integrable on $[a,b]$ if and only if each measurable selection of $F : [a,b] \to CWK(X)$ is KHP-integrable on $[a,b]$.*

## 3. Results

A mapping $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is called a *fuzzy mapping* in a Banach space $X$. In this case $\tilde{F}^r : [a,b] \to CWK(X)$ defined by $\tilde{F}^r(t) = [\tilde{F}(t)]^r$ is a set-valued mapping for each $r \in (0,1]$. A fuzzy mapping $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is said to be *measurable* (resp., *scalarly measurable*) if $\tilde{F}^r : [a,b] \to CWK(X)$ is measurable (resp., scalarly measurable) for each $r \in (0,1]$. A fuzzy mapping $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is said to be *scalarly KH-integrable on* $[a,b]$ if $\tilde{F}^r : [a,b] \to CWK(X)$ is scalarly KH-integrable on $[a,b]$ for each $r \in (0,1]$. A fuzzy mapping $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is said to be *weakly KH-integrably bounded on* $[a,b]$ if $\tilde{F}^r : [a,b] \to CWK(X)$ is weakly KH-integrably bounded on $[a,b]$ for each $r \in (0,1]$. A fuzzy mapping $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is said to be *KH-integrably bounded* on $[a,b]$ if there exists an KH-integrable real-valued function $h$ on $[a,b]$ such that for each $t \in [a,b]$, $\|x\| \le h(t)$ for all $x \in \tilde{F}^0(t)$, where $\tilde{F}^0(t) = cl(\cup_{0<r\le 1}\tilde{F}^r(t))$.

DEFINITION 3.1. A fuzzy mapping $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is said to be *Kurzweil-Henstock-Pettis integrable* or simply *KHP-integrable* on $[a,b]$ if for each subinterval $[c,d]$ of $[a,b]$ there exists $u_{[c,d]} \in \mathcal{F}(X)$ such that $[u_{[c,d]}]^r = (KHP)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0,1]$. In this case, $u_{[c,d]} = (KHP)\int_c^d \tilde{F}(t)dt$ is called the *Kurzweil-Henstock-Pettis integral* of $\tilde{F}$ over $[c,d]$.

THEOREM 3.2. *Let $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ be KHP-integrable on $[a,b]$ and $\lambda \ge 0$. Then*

(1) *$\tilde{F} + \tilde{G}$ is KHP-integrable on $[a,b]$ and for each subinterval $[c,d]$ of $[a,b]$*

$$(KHP)\int_c^d \left\{\tilde{F}(t) + \tilde{G}(t)\right\} dt$$
$$= (KHP)\int_c^d \tilde{F}(t)dt + (KHP)\int_c^d \tilde{G}(t)dt,$$

(2) *$\lambda\tilde{F}$ is KHP-integrable on $[a,b]$ and for each subinterval $[c,d]$ of $[a,b]$*

$$(KHP)\int_c^d \lambda\tilde{F}(t)dt = \lambda(KHP)\int_c^d \tilde{F}(t)dt.$$

*Proof.* The proof is straightforward. □

LEMMA 3.3. *Let $f:[a,b]\to X$ be a KHP-integrable function and $F: [a,b]\to CWK(X)$ and $G:[a,b]\to CWK(X)$ KHP-integrable set-valued mappings. Then*

(1) *if $f(t)\in F(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$*

$$(KHP)\int_c^d f(t)dt \in (KHP)\int_c^d F(t)dt;$$

(2) *if $F(t)\subseteq G(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$*

$$(KHP)\int_c^d F(t)dt \subseteq (KHP)\int_c^d G(t)dt;$$

(3) *if $F(t)=G(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$*

$$(KHP)\int_c^d F(t)dt = (KHP)\int_c^d G(t)dt.$$

*Proof.* (1) Since $f:[a,b]\to X$ is KHP-integrable on $[a,b]$, for each subinterval $[c,d]$ of $[a,b]$ and $x^*\in X^*$, $x^*f$ is KH-integrable on $[c,d]$ and

$$(KH)\int_c^d x^*f(t)dt = x^*\left((KHP)\int_c^d f(t)dt\right).$$

If $f(t)\in F(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$ and $x^*\in X^*$

$$\begin{aligned} x^*\left((KHP)\int_c^d f(t)dt\right) &= (KH)\int_c^d x^*f(t)dt \\ &\le (KH)\int_c^d s(x^*,F(t))dt \\ &= s\left(x^*,(KHP)\int_c^d F(t)dt\right). \end{aligned}$$

Since $(KHP)\int_c^d F(t)dt \in CWK(X)$, by the separation theorem

$$(KHP)\int_c^d f(t)dt \in (KHP)\int_c^d F(t)dt.$$

(2) If $F(t) \subseteq G(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$ and $x^* \in X^*$

$$(KH)\int_c^d s(x^*, F(t))dt \le (KH)\int_c^d s(x^*, G(t))dt,$$

$$s\left(x^*, (KHP)\int_c^d F(t)dt\right) \le s\left(x^*, (KHP)\int_c^d G(t)dt\right).$$

Since $(KHP)\int_c^d F(t)dt, (KHP)\int_c^d G(t)dt \in CWK(X)$, by the separation theorem

$$(KHP)\int_c^d F(t)dt \subseteq (KHP)\int_c^d G(t)dt.$$

(3) The proof is similar to (2). □

THEOREM 3.4. *Let $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ be KHP-integrable fuzzy mappings. Then*

(1) *if $\tilde{F}(t) \le \tilde{G}(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$*

$$(KHP)\int_c^d \tilde{F}(t)dt \le (KHP)\int_c^d \tilde{G}(t)dt;$$

(2) *if $\tilde{F}(t) = \tilde{G}(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$*

$$(KHP)\int_c^d \tilde{F}(t)dt = (KHP)\int_c^d \tilde{G}(t)dt.$$

*Proof.* (1) Since $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ are KHP-integrable on $[a,b]$, for each subinterval $[c,d]$ of $[a,b]$ there exist $u_{[c,d]}, v_{[c,d]} \in \mathcal{F}(X)$ such that $[u_{[c,d]}]^r = (KHP)\int_c^d \tilde{F}^r(t)dt$, $[v_{[c,d]}]^r = (KHP)\int_c^d \tilde{G}^r(t)dt$ for each $r \in (0,1]$. If $\tilde{F}(t) \le \tilde{G}(t)$ a.e. on $[a,b]$, then by Lemma 3.3 $[u_{[c,d]}]^r =$

$(KHP)\int_c^d \tilde{F}^r(t)dt \subseteq (KHP)\int_c^d \tilde{G}^r(t)\ dt = [v_{[c,d]}]^r$ for each $r \in (0,1]$ and each subinterval $[c,d]$ of $[a,b]$. Thus $(KHP)\int_c^d \tilde{F}(t)dt = u_{[c,d]} \leq v_{[c,d]} = (KHP)\int_c^d \tilde{G}(t)dt$ for each subinterval $[c,d]$ of $[a,b]$.

(2) The proof is similar to (1). □

LEMMA 3.5. *If $F : [a,b] \to CWK(X)$ and $G : [a,b] \to CWK(X)$ are KH-integrably bounded and KHP-integrable on $[a,b]$, then $H(F, G)$ is KH-integrable on $[a,b]$ and*

$$H\left((KHP)\int_a^b F(t)dt, (KHP)\int_a^b G(t)dt\right)$$
$$\leq (KH)\int_a^b H(F(t), G(t))dt.$$

*Proof.* Since $F : [a,b] \to CWK(X)$ and $G : [a,b] \to CWK(X)$ are measurable, there exist Castaing representations $\{f_n\}$ and $\{g_n\}$ for $F$ and $G$. Since $f_n$ and $g_n$ are measurable for all $n \in \mathbb{N}$,

$$H(F(t), G(t))$$
$$= \max\left(\sup_{n\geq 1}\inf_{k\geq 1}\|f_n(t) - g_k(t)\|, \sup_{n\geq 1}\inf_{k\geq 1}\|g_n(t) - f_k(t)\|\right)$$

is measurable. Since $F : [a,b] \to CWK(X)$ and $G : [a,b] \to CWK\ (X)$ are KH-integrably bounded on $[a,b]$, there exist KH-integrable real-valued functions $h_1$ and $h_2$ on $[a,b]$ such that for each $t \in [a,b]$, $\|x\| \leq h_1(t)$ for all $x \in F(t)$ and $\|x\| \leq h_2(t)$ for all $x \in G(t)$. Since $h_1$ and $h_2$ are nonnegative and KH-integrable on $[a,b]$, $h_1$ and $h_2$ are Lebesgue integrable on $[a,b]$. We have

$$H(F(t), G(t)) \leq H(F(t), \{0\}) + H(G(t), \{0\}) \leq h_1(t) + h_2(t)$$

for each $t \in [a,b]$. Therefore $H(F, G)$ is Lebesgue integrable on $[a,b]$ and so $H(F, G)$ is KH-integrable on $[a,b]$ and we have

$$H\left((KHP)\int_a^b F(t)dt, (KHP)\int_a^b G(t)dt\right)$$
$$= \sup_{\|x^*\|\leq 1}\left|s\left(x^*, (KHP)\int_a^b F(t)dt\right) - s\left(x^*, (KHP)\int_a^b G(t)dt\right)\right|$$

$$= \sup_{\|x^*\|\le 1} \left| (KH)\int_a^b s(x^*, F(t))dt - (KH)\int_a^b s(x^*, G(t))dt \right|$$
$$\le \sup_{\|x^*\|\le 1} (KH)\int_a^b |s(x^*, F(t)) - s(x^*, G(t))|dt$$
$$\le (KH)\int_a^b \sup_{\|x^*\|\le 1} |s(x^*, F(t)) - s(x^*, G(t))|dt$$
$$= (KH)\int_a^b H(F(t), G(t))dt$$

□

THEOREM 3.6. *If $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ are KH-integrably bounded and KHP-integrable on $[a,b]$, then $D(\tilde{F},\tilde{G})$ is KH-integrable on $[a,b]$ and*

$$D\left( (KHP)\int_a^b \tilde{F}(t)dt, (KHP)\int_a^b \tilde{G}(t)dt \right)$$
$$\le (KH)\int_a^b D(\tilde{F}(t), \tilde{G}(t))dt.$$

*Proof.* Since $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ are measurable, there exist Castaing representations $\{f_n^r\}$ and $\{g_n^r\}$ for $\tilde{F}^r$ and $\tilde{G}^r$ for each $r \in (0,1]$. Since $f_n^r$ and $g_n^r$ are measurable for all $n \in \mathbb{N}$,

$$H(\tilde{F}^r(t), \tilde{G}^r(t))$$
$$= \max\left( \sup_{n\ge 1}\inf_{k\ge 1} \|f_n^r(t) - g_k^r(t)\|, \sup_{n\ge 1}\inf_{k\ge 1} \|g_n^r(t) - f_k^r(t)\| \right)$$

is measurable for each $r \in (0,1]$. Hence $D(\tilde{F}(t), \tilde{G}(t)) = \sup_{k\ge 1} H(\tilde{F}^{r_k}(t), \tilde{G}^{r_k}(t))$ is measurable, where $\{r_k : k \in \mathbb{N}\}$ is dense in $(0,1]$. Since $\tilde{F} : [a,b] \to \mathcal{F}(X)$ and $\tilde{G} : [a,b] \to \mathcal{F}(X)$ are KH-integrably bounded on $[a,b]$, there exist KH-integrable real-valued functions $h_1$ and $h_2$ on $[a,b]$ such that for each $t \in [a,b]$, $\|x\| \le h_1(t)$ for all $x \in \tilde{F}^0(t)$ and $\|x\| \le h_2(t)$ for all $x \in \tilde{G}^0(t)$. Since $h_1$ and $h_2$ are nonnegative and KH-integrable on $[a,b]$, $h_1$ and $h_2$ are Lebesgue integrable on $[a,b]$. We have

$$D(\tilde{F}(t), \tilde{G}(t)) \le D(\tilde{F}(t), \tilde{0}) + D(\tilde{G}(t), \tilde{0}) \le h_1(t) + h_2(t)$$

for each $t \in [a,b]$. Therefore $D(\tilde{F},\tilde{G})$ is Lebesgue integrable and so KH-integrable on $[a,b]$. By Lemma 3.5 we have

$$H\left((KHP)\int_a^b \tilde{F}^r(t)dt, (KHP)\int_a^b \tilde{G}^r(t)dt\right)$$
$$\leq (KH)\int_a^b H(\tilde{F}^r(t),\tilde{G}^r(t))dt$$

for each $r \in (0,1]$. Hence we have

$$\begin{aligned}
&D\left((KHP)\int_a^b \tilde{F}(t)dt, (KHP)\int_a^b \tilde{G}(t)dt\right)\\
&= \sup_{r\in(0,1]} H\left(\left[(KHP)\int_a^b \tilde{F}(t)dt\right]^r, \left[(KHP)\int_a^b \tilde{G}(t)dt\right]^r\right)\\
&= \sup_{r\in(0,1]} H\left((KHP)\int_a^b \tilde{F}^r(t)dt, (KHP)\int_a^b \tilde{G}^r(t)dt\right)\\
&\leq \sup_{r\in(0,1]} (KH)\int_a^b H(\tilde{F}^r(t),\tilde{G}^r(t))dt\\
&\leq (KH)\int_a^b \sup_{r\in(0,1]} H(\tilde{F}^r(t),\tilde{G}^r(t))dt\\
&= (KH)\int_a^b D(\tilde{F}(t),\tilde{G}(t))dt.
\end{aligned}$$

□

THEOREM 3.7. *Let $\tilde{F} : [a,b] \to \mathcal{F}(X)$ be measurable, weakly KH-integrably bounded and scalarly KH-integrable on $[a,b]$. If each measurable selection of $\tilde{F}^r : [a,b] \to CWK(X)$ is KHP-integrable on $[a,b]$ for each $r \in (0,1]$, then $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is KHP-integrable on $[a,b]$.*

*Proof.* Let $[c,d]$ be a subinterval of $[a,b]$. Since $\tilde{F}^r : [a,b] \to CWK(X)$ is scalarly KH-integrable on $[a,b]$ and each measurable selection of $\tilde{F}^r : [a,b] \to CWK(X)$ is KHP-integrable on $[a,b]$ for each $r \in (0,1]$, by Theorem 2.5 $\tilde{F}^r : [a,b] \to CWK(X)$ is KHP-integrable on $[a,b]$ for each $r \in (0,1]$. Hence $A_r = (KHP)\int_c^d \tilde{F}^r(t)dt \in CWK(X)$ for each $r \in (0,1]$. For $r_1, r_2 \in (0,1]$

with $r_1 \leq r_2$, $\tilde{F}^{r_1}(t) \supseteq \tilde{F}^{r_2}(t)$ for each $t \in [a,b]$. By Lemma 3.3 $A_{r_1} = (KHP)\int_c^d \tilde{F}^{r_1}(t)dt \supseteq (KHP)\int_c^d \tilde{F}^{r_2}(t)dt = A_{r_2}$. Let $r \in (0,1]$ and $\{r_n\}$ be a sequence in $(0,1]$ such that $r_1 \leq r_2 \leq r_3 \leq \cdots$ and $\lim_{n\to\infty} r_n = r$. Then $\tilde{F}^r(t) = \cap_{n=1}^{\infty} \tilde{F}^{r_n}(t)$ for each $t \in [a,b]$. By Theorem 2.4 $\lim_{n\to\infty} s(x^*, \tilde{F}^{r_n}(t)) = s(x^*, \tilde{F}^r(t))$ for each $t \in [a,b]$ and $x^* \in X^*$. For each $n \in \mathbb{N}$, $|s(x^*, \tilde{F}^{r_n}(t))| \leq |x^*\tilde{F}^{r_1}|(t)$ for each $t \in [a,b]$ and $x^* \in X^*$. Since $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is scalarly KH-integrable and weakly KH-integrably bounded on $[a,b]$, by the Dominated Convergence Theorem for the Kurzweil-Henstock integral we have

$$\begin{aligned}\lim_{n\to\infty} s(x^*, A_{r_n}) &= \lim_{n\to\infty} (KH)\int_c^d s(x^*, \tilde{F}^{r_n}(t))dt \\ &= (KH)\int_c^d s(x^*, \tilde{F}^r(t))dt \\ &= s(x^*, A_r)\end{aligned}$$

for each $x^* \in X^*$. By Theorem 2.4 $A_r = \cap_{n=1}^{\infty} A_{r_n}$. By Theorem 2.3 there exists $u_{[c,d]} \in \mathcal{F}(X)$ such that $[u_{[c,d]}]^r = A_r = (KHP)\int_c^d \tilde{F}^r(t)\,dt$ for each $r \in (0,1]$. Hence $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is KHP-integrable on $[a,b]$. □

The following theorem is the Dominated Convergence Theorem for KHP-integrable fuzzy mappings.

Theorem 3.8. *Let $\tilde{F}_n : [a,b] \to \mathcal{F}(X)$ be KHP-integrable on $[a,b]$ for each $n \in \mathbb{N}$, $\tilde{F} : [a,b] \to \mathcal{F}(X)$ measurable and scalarly KH-integrable on $[a,b]$ and $\lim_{n\to\infty} D(\tilde{F}_n(t), \tilde{F}(t)) = 0$ on $[a,b]$. If there exists an KH-integrable real-valued function $h$ such that $\|\tilde{F}_n^0(t)\| \leq h(t)$ on $[a,b]$ for each $n \in \mathbb{N}$ and each measurable selection of $\tilde{F}^r : [a,b] \to CWK(X)$ is KHP-integrable on $[a,b]$ for each $r \in (0,1]$, then $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is KHP-integrable on $[a,b]$ and*

$$\lim_{n\to\infty} D\left((KHP)\int_a^b \tilde{F}_n(t)dt, (KHP)\int_a^b \tilde{F}(t)dt\right) = 0.$$

*Proof.* Since $\lim_{n\to\infty} D(\tilde{F}_n(t), \tilde{F}(t)) = 0$ on $[a,b]$, for each $\epsilon > 0$ and $t \in [a,b]$ there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow D(\tilde{F}_n(t), \tilde{F}(t)) < \epsilon$. For some $n \in \mathbb{N}$

with $n \geq N$,

$$\|\tilde{F}^0(t)\| = D(\tilde{F}(t), \tilde{0}) \leq D(\tilde{F}(t), \tilde{F}_n(t)) + D(\tilde{F}_n(t), \tilde{0})$$
$$< \|\tilde{F}_n^0(t)\| + \epsilon \leq h(t) + \epsilon$$

for each $t \in [a,b]$. Since $\epsilon > 0$ is arbitrary, $\|\tilde{F}^0(t)\| \leq h(t)$ on $[a,b]$. Thus $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is KH-integrably bounded and so weakly KH-integrably bounded on $[a,b]$. By Theorem 3.7, $\tilde{F} : [a,b] \to \mathcal{F}(X)$ is KHP-integrable on $[a,b]$. Since $\tilde{F}_n : [a,b] \to \mathcal{F}(X)$ is also KH-integrably bounded on $[a,b]$ for each $n \in \mathbb{N}$, by Theorem 3.6 and the Dominated Convergence Theorem for the Kurzweil-Henstock integral we have

$$D\left((KHP)\int_a^b \tilde{F}_n(t)dt, (KHP)\int_a^b \tilde{F}(t)dt\right)$$
$$\leq (KH)\int_a^b D(\tilde{F}_n(t), \tilde{F}(t))dt \to 0$$

as $n \to \infty$. Thus $\lim_{n\to\infty} D\left((KHP)\int_a^b \tilde{F}_n(t)dt, (KHP)\int_a^b \tilde{F}(t)dt\right) = 0.$ □

## References


[1] R. J. Aumann *Integrals of set-valued functions*, J. Math. Anal. Appl. **12** (1965), 1-12.

[2] L. Di Piazza and K. Musial, *A decomposition theorem for compact-valued Henstock integral*, Monatsh. Math. **148** (2006), 119-126.

[3] L. Di Piazza and K. Musial, *Set-valued Kurzweil-Henstock-Pettis integral*, Set-Valued Analysis **13** (2005), 167-179.

[4] K. El Amri and C. Hess, *On the Pettis integral of closed valued multifunctions*, Set-Valued Analysis **8** (2000), 329-360.

[5] D. H. Fremlin, *The Henstock and McShane integrals of vector-valued functions*, Illinois J. Math. **38** (1994), 471-479.

[6] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron and Henstock*, Grad. Stud. Math. 4, Amer. Math. Soc. 1994.

[7] Guoju Ye, *On Henstock-Kurzweil and McShane integrals of Banach space-valued functions*, J. Math. Anal. Appl. **330** (2007), 753-765.

[8] J. Wu and C. Wu, *The w-derivatives of fuzzy mappings in Banach spaces*, Fuzzy Sets and Systems **119** (2001) 375-381.

[9] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems **24** (1987), 301-317.

[10] N. Papageoriou, *On the theory of Banach space valued multifunctions*, J. Multivariate Anal. **17** (1985), 185-206.

[11] X. Xue, M. Ha and M. Ma, *Random fuzzy number integrals in Banach spaces*, Fuzzy Sets and Systems **66** (1994), 97-111.

[12] X. Xue, X. Wang and L. Wu, *On the convergence and representation of random fuzzy number integrals*, Fuzzy Sets and Systems, **103** (1999), 115-125.

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