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GLOBAL VORTICITY EXISTENCE OF A PERFECT INCOMPRESSIBLE FLUID IN $\mathbf{B^0_{\infty,1}}(\mathbb{R}^\mathbf{2}) \cap \mathbf{L^p}(\mathbb{R}^\mathbf{2})$

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Abstract. We prove the global (in time) vorticity existence for the 2-D Euler equations of a perfect incompressible fluid in $B^0_{\infty,1}(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$ with $1 < p < 2$. Moreover, we prove that the particle trajectory map $X(x,t)$ satisfies the following estimate: for some positive constant C

$$\|X^{\pm 1}(\,\cdot,\,t) - id(\cdot)\|_{B^1_{\infty,1}} \le Ce^{e^{Ct}},$$

where id represents the identity map on $\mathbb{R}^2$.

1. Introduction

We consider the non-stationary Euler equations of a perfect incompressible fluid:

$$\frac{\partial}{\partial t}\,u + (u,\nabla)u = -\nabla p, \qquad \text{div } u = 0, \tag{1.1}$$

where $u(x,t) = (u_1, u_2, \cdots, u_n)$ represents the *velocity* of a fluid flow, and $p(x,t)$ is the scalar *pressure*. The *vorticity* ω is the curl of the velocity vector field u: $\omega = \text{curl } u$. For example, 3-D vorticity is $\omega = (\frac{\partial}{\partial x_2}u_3 - \frac{\partial}{\partial x_3}u_2, \frac{\partial}{\partial x_3}u_1 - \frac{\partial}{\partial x_1}u_3, \frac{\partial}{\partial x_1}u_2 - \frac{\partial}{\partial x_2}u_1)$ and 2-D vorticity is the scalar function $\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$.

In [6], Pak and Park investigated local existence of the solution to the 3-D Euler equation (1.1) and proved that the vorticity ω stays locally in $B^0_{\infty,1}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$ if the initial vorticity ω_0 is in $B^0_{\infty,1}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$. In this paper, we prove the global unique vorticity existence for the

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2-dimensional Euler equations in the space $B^0_{\infty,1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. The following is our main result.

THEOREM 1.1. *Let* $1 < p < 2$. *For every initial vorticity* $\omega_0 \in B^0_{\infty,1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ *with* $\omega_0 = \operatorname{curl} u_0$ *for some divergence free vector field (of distribution)* u_0, *there exists a unique vorticity* $\omega(x,t) \in C([0,\infty); B^0_{\infty,1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2))$ *of the initial value problem for the Euler equation (1.1) with* $\omega(x,0) = \omega_0(x)$. *Moreover, the solution satisfies the following estimate:*

$$\|\omega(t)\|_{B^0_{\infty,1}\cap L^p} \le C \|\omega_0\|_{B^0_{\infty,1}\cap L^p} \exp\Big(C\|\omega_0\|_{B^0_{\infty,1}\cap L^p}\, t\Big),$$

for some positive constant C.

We point out that the $B^0_{\infty,1}$-norm of solution has an exponential upper bound - *not exponential of exponential.*

Corresponding to the Euler equations, we have a system of ordinary differential equations

$$\begin{cases} \dfrac{\partial}{\partial t} X(x,t) &= u(X(x,t),t), \\ X(x,0) &= x, \end{cases}$$

which defines *particle trajectories* $X(x,t)$ subject to the Euler flow $u(x,t)$, starting from initial positions x. It has been emphasized to observe the behavior of particle trajectories for studying regularity problem of the Euler flow(this is, what we call, the Yudovich's observation). Concerning the particle trajectories, we have the following theorem:

THEOREM 1.2. *Let* u_0 *be a divergence free vector field with* $\omega_0 = \operatorname{curl} u_0 \in B^0_{\infty,1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, $1 < p < 2$, *and let* $X(x,t)$ *be the particle trajectory map subject to the velocity* u, *where* $\omega = \operatorname{curl} u$ *is the unique vorticity* ω *of the initial value problem for the Euler equation (1.1) with* $\omega(x,0) = \omega_0(x)$. *Then the particle trajectory map* $X(x,t)$ *satisfies the estimate: for some positive constant* C,

$$\|X^{\pm 1}(\,\cdot\,, t) - id(\cdot)\|_{B^1_{\infty,1}} \le Ce^{e^{Ct}},$$

where id *represents the identity map on* $\mathbb{R}^2$.

Because the local existence of 2-D vorticity is essentially proved in [6], in this paper, we focus only on the persistence of the 2-D vorticity in $B^0_{\infty,1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. For the proof of global existence we borrow the techniques in [8, 3], especially the limiting case of Beale-Kato-Majda inequality in 2-D which was originally proved by M. Vishik. The proof of Theorem 1.2 heavily relies on the estimates introduced in [6].

2. Preliminaries

Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. We consider a nonnegative radial function $\chi \in \mathcal{S}$ satisfying $\operatorname{supp}\chi \subset \{\xi \in \mathbb{R}^n : |\xi| \le \frac{5}{6}\}$, and $\chi = 1$ for $|\xi| \le \frac{3}{5}$. Set $h_j(\xi) := \chi(2^{-j-1}\xi) - \chi(2^{-j}\xi)$, and we notice that

$$\chi(\xi) + \sum_{j=0}^{\infty} h_j(\xi) = 1, \text{ for } \xi \in \mathbb{R}^n.$$

Let φ_j and Φ be defined by $\varphi_j := \mathcal{F}^{-1}(h_j)$, $j \ge 0$ and $\Phi := \mathcal{F}^{-1}(\chi)$, where $\mathcal{F}u = \hat{u}$ denotes the Fourier transform of u on $\mathbb{R}^n$. Note that φ_j is a mollifier of φ_0, that is, $\varphi_j(x) := 2^{jn}\varphi_0(2^j x)$ (or $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$). For $f \in \mathcal{S}'$, we denote $\Delta_j f \equiv h_j(D)f = \varphi_j * f$ if $j \ge 0$, $\Delta_j f = 0$ if $j \le -2$, and $\Delta_{-1} f \equiv \chi(D)f = \Phi * f$ if $j = -1$. We also define the partial sums: $S_k f := \sum_{j=-1}^{k} \Delta_j f$ for $k \in \mathbb{Z}$. For $s \in \mathbb{R}$, the Besov spaces $B^s_{\infty,1}(\mathbb{R}^n)$ are defined by

$$B^s_{\infty,1}(\mathbb{R}^n) = \left\{ f : \sum_{j=-1}^{\infty} 2^{js} \|\Delta_j f\|_{L^p} < \infty \right\}.$$

We state a collection of a-priori estimates which is used in the proof of the main theorem. Those a-priori estimates are discussed in arbitrary dimension $n \ge 2$ including the cases of the dimension 2 and 3.

Remark 2.1. *Let $\omega = \operatorname{curl} u$. Then we have*

$$\|u\|_{B^1_{\infty,1}} \le C\left(\|\omega\|_{B^0_{\infty,1}} + \|\omega\|_{L^p}\right) := \|\omega\|_{B^0_{\infty,1} \cap L^p}.$$

The proof can be found in [6].

By virtue of Bony's para-product formula, we have the following useful estimate.

Remark 2.2. *For any differentiable divergence free vector field u and any differentiable vector field h, we have:*

$$\sum_{j=-1}^{\infty} 2^j \left\|(S_{j-2}u, \nabla)\Delta_j h - \Delta_j((u,\nabla)h)\right\|_{L^\infty} \le C\|u\|_{B^1_{\infty,1}}\|h\|_{B^1_{\infty,1}}. \tag{2.1}$$

For the proof, we refer Proposition 6 at page 1157 in [5].

3. The proofs

Let $1 < p < 2$, and we are given initial vorticity $\omega|_{t=0} = \omega_0 \in B^0_{\infty,1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ with $\omega_0 = \operatorname{curl} u_0$ for some divergence free vector field (of distribution) u_0. The 2-D vorticity equation is given by

$$\frac{\partial}{\partial t}\,\omega + (u, \nabla)\omega = 0. \tag{3.1}$$

In the following discussion, $\{X(x,t)\}$ represents the trajectory flow along u defined by the solution of

$$\begin{cases} \dfrac{\partial}{\partial t} X(x,t) & = u(X(x,t),t), \\ X(x,0) & = x. \end{cases} \tag{3.2}$$

It is well-known that the solution $\omega(x,t)$ of the 2-D vorticity equation can be represented by

$$\omega(x,t) = \omega_0(X^{-1}(x,t)), \quad x \in \mathbb{R}^2.$$

3.1. The proof of Theorem 1

We introduce the limiting case of Beale-Kato-Majda inequality in 2-D which was originally proved by M. Vishik [8].

Remark 3.1 (Logarithmic B-K-M inequality). *We have the following estimate:*

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\big(1+\log(\|\nabla_x X(\cdot,t)\|_{L^\infty}\|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty})\big)\|\omega_0\|_{B^0_{\infty,1}}.$$

Vishik's inequality(Remark 3.1) explains the exponential growth of $B^0_{\infty,1}$-norm of vorticity $\omega(t)$ as follows. The identity (from (3.2))

$$\frac{\partial}{\partial t}\nabla_x X(x,t) = (\nabla u)(X(x,t),t) \cdot \nabla_x X(x,t)$$

implies

$$\|\nabla_x X(\cdot,t)\|_{L^\infty} \leq 1 + \int_0^t \|\nabla u(X(\cdot,\tau),\tau)\|_{L^\infty}\|\nabla_x X(\cdot,\tau)\|_{L^\infty}\, d\tau.$$

Also, Gronwall's inequality, Remark 2.1, Vishik's inequality and the conservation of vorticity, $\|\omega(\cdot,\tau)\|_{L^p} = \|\omega_0\|_{L^p}$, imply that

$$\begin{aligned}
&\|\nabla_x X(\cdot,t)\|_{L^\infty} \\
&\le \exp\left\{\int_0^t \|\nabla u(\cdot,\tau)\|_{L^\infty} d\tau\right\} \\
&\le \exp\left\{C\int_0^t (\|\omega(\cdot,\tau)\|_{B^0_{\infty,1}} + \|\omega(\cdot,\tau)\|_{L^p})\, d\tau\right\} \\
&\le \exp\left\{C\|\omega_0\|_{B^0_{\infty,1}\cap L^p} \int_0^t \left(1+\log(\|\nabla_x X(\tau)\|_{L^\infty}\|\nabla_x X^{-1}(\tau)\|_{L^\infty})\right) d\tau\right\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty} \\
&\le \exp\left\{C\|\omega_0\|_{B^0_{\infty,1}\cap L^p} \int_0^t \left(1+\log(\|\nabla_x X(\tau)\|_{L^\infty}\|\nabla_x X^{-1}(\tau)\|_{L^\infty})\right) d\tau\right\}.
\end{aligned}$$

Combine these estimates together to get

$$\begin{aligned}
&\|\nabla_x X(\cdot,t)\|_{L^\infty}\ \|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty} \\
&\le \exp\left\{C\|\omega_0\|_{B^0_{\infty,1}\cap L^p} \int_0^t \left(1+\log(\|\nabla_x X(\tau)\|_{L^\infty}\|\nabla_x X^{-1}(\tau)\|_{L^\infty})\right) d\tau\right\}.
\end{aligned}$$

Or

$$\begin{aligned}
&\log(\|\nabla_x X(\cdot,t)\|_{L^\infty}\ \|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty}) \\
&\le C\|\omega_0\|_{B^0_{\infty,1}\cap L^p} \int_0^t \left(1+\log(\|\nabla_x X(\cdot,\tau)\|_{L^\infty}\|\nabla_x X^{-1}(\cdot,\tau)\|_{L^\infty})\right) d\tau.
\end{aligned}$$

Hence Gronwall's inequality implies

$$\log\left(\|\nabla_x X(\cdot,t)\|_{L^\infty}\ \|\nabla_x X^{-1}(\cdot,t)\|_{L^\infty}\right) \le \exp\left(C\|\omega_0\|_{B^0_{\infty,1}\cap L^p}\, t\right).$$

Placing this into Remark 3.1, we have

$$\begin{aligned}
\|\omega(t)\|_{B^0_{\infty,1}} &\le C\left(1+\exp\left(C\|\omega_0\|_{B^0_{\infty,1}\cap L^p}\, t\right)\right)\|\omega_0\|_{B^0_{\infty,1}} \\
&\le C\,\|\omega_0\|_{B^0_{\infty,1}} \exp\left(C\|\omega_0\|_{B^0_{\infty,1}\cap L^p}\, t\right).
\end{aligned}$$

Therefore the conservation of vorticity in 2-D, $\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}$, implies the growth rate of vorticity in time:

$$\|\omega(t)\|_{B^0_{\infty,1}\cap L^p} \le C\,\|\omega_0\|_{B^0_{\infty,1}\cap L^p} \exp\left(C\|\omega_0\|_{B^0_{\infty,1}\cap L^p}\, t\right).$$

This completes the global existence of vorticity in $B^0_{\infty,1}(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$ with $1<p<2$.

3.2. The proof of Theorem 2

We present the estimate:

$$\|X^{\pm 1}(\,\cdot,\,t) - id(\cdot)\|_{B^1_{\infty,1}} \leq Ce^{e^{Ct}},$$

and the estimate for $\nabla X(\,\cdot,\,t)$ can be obtained similarly. We set $h(x,t) := X^{-1}(x,\,t) - x$. Then we have $h(x,0) = 0$, and

$$\frac{\partial}{\partial t} h(x,t) = -(u,\nabla)h - u. \tag{3.3}$$

(We referred the formula (6.13) in [8].) Take Δ_j operator and add $(S_{j-2}u,\nabla)\Delta_j\, h$ on both sides of (3.3) to have

$$\frac{\partial}{\partial t}\,\Delta_j\, h + (S_{j-2}u,\nabla)\Delta_j h = (S_{j-2}u,\nabla)\Delta_j h - \Delta_j(u,\nabla)h - \Delta_j u.$$

Then by considering the trajectory flow $\{Y_j(x,t)\}$ along $S_{j-2}u$ defined by the solution of the ordinary differential equations

$$\begin{cases} \dfrac{\partial}{\partial t} Y_j(x,t) & = (S_{j-2}u)(Y_j(x,t),t), \\ \quad Y_j(x,0) & = x, \end{cases}$$

we get

$$\|\Delta_j\, h(t)\|_{L^\infty} \leq \int_0^t \|(S_{j-2}u,\nabla)\Delta_j h - \Delta_j((u,\nabla)h)\|_{L^\infty} + \|\Delta_j u\|_{L^\infty}\, d\tau.$$

Therefore multiplying 2^j on both sides and summing up altogether, we obtain:

$$\begin{aligned} &\|h(t)\|_{B^1_{\infty,1}} \\ &\quad \leq \int_0^t \sum_{j=-1}^{\infty} 2^j\, \|(S_{j-2}u,\nabla)\Delta_j h - \Delta_j((u,\nabla)h)\|_{L^\infty}\, d\tau + \int_0^t \|u(\tau)\|_{B^1_{\infty,1}} d\tau. \end{aligned}$$

From Remark 2.1, 2.2 and Theorem 1.1, we get

$$\|h(t)\|_{B^1_{\infty,1}} \leq Ce^{Ct} + C\int_0^t e^{Ct}\|h(\tau)\|_{B^1_{\infty,1}} d\tau. \tag{3.4}$$

By virtue of Gronwall's inequality, we finally get

$$\|h(t)\|_{B^1_{\infty,1}} \leq Ce^{e^{Ct}}.$$

This completes the proof.

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