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SUB-HYPERELLIPTIC CURVES $X_1^+(N)$

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Abstract. In this work, we determine all the curves $X_1^+(N)$ which are sub-hyperelliptic.

1. Introduction

A smooth, projective curve X is called *sub-hyperelliptic* if it admits a map of degree 2 from X to the projective line $\mathbb{P}^1$. Moreover, if the genus $g(X) \geq 2$, then X is called *hyperelliptic*. If X is a rational curve, then there exists a map $\phi : X \to \mathbb{P}^1$ of degree n for any positive integer n, and so X is sub-hyperelliptic. If X is an elliptic curve, then the projection to x-axis is a map of degree 2 to $\mathbb{P}^1$, and X is sub-hyperelliptic too.

Ogg [8] determined all the sub-hyperelliptic modular curves $X_0(N)$. Later Mestre [7] determined all the modular curves $X_1(N)$ which are sub-hyperelliptic. The modular curves $X_0(N)$ carry the action of the Atkin-Lehner involutions W_d for any $d\|N$ which denote a positive integer d dividing N with $(d, N/d) = 1$. Let $X_0^{+d}(N)$ and $X_0^*(N)$ be the quotient of $X_0(N)$ by a W_d and the W_d's for all $d\|N$ respectively. Furumoto and Hasegawa [1], and Hasegawa [2] determined all the modular curves $X_0^{+d}(N)$ and $X_0^*(N)$ which are sub-hyperelliptic respectively.

Apart from $X_0(N)$, some Atkin-Lehner involutions W_d cannot act on $X_1(N)$. But the full Atkin-Lehner involution W_N always acts on $X_1(N)$, and the quotient of $X_1(N)$ by W_N is denoted by $X_1^+(N)$. In this paper, we determine all the sub-hyperelliptic curves $X_1^+(N)$.

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2. Preliminaries

Let $\Gamma(1) = SL_2(\mathbb{Z})$ be the full modular group. For any integer $N \geq 1$, we have subgroups $\Gamma_1(N)$, $\Gamma_0(N)$ of $\Gamma(1)$ defined by matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ congruent modulo N to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ respectively. We let $X_1(N)$, $X_0(N)$ be the modular curves defined over $\mathbb{Q}$ associated to $\Gamma_1(N)$, $\Gamma_0(N)$ respectively. The X's are compact Riemann surfaces.

For each divisor $d||N$, consider the matrices of the form $\begin{pmatrix} dx & y \\ Nz & dw \end{pmatrix}$ with $x, y, z, w \in \mathbb{Z}$ and determinant d. Then these matrices define a unique involution on $X_0(N)$ which is denoted by W_d. Sometimes we regard W_d as a matrix.

Now we fix a matrix W_d. By [5] W_d belongs to the normalizer of $\Gamma_1(N)$ in $PSL_2(\mathbb{R})$, and therefore defines an automorphism of $X_1(N)$. Furthermore W_d, in general, does not give an involution on $X_1(N)$. But when $d = N$, W_N still gives an involution of $X_1(N)$.

We present a genus formula for estimating the genus of the quotient $v\backslash X$ of a curve X by an involution on X.

Proposition 2.1. *Let v be any involution on the compact Riemann surface X, and let $\#$ denote the number of fixed points of v on X. Then we have the following genus formula:*

$$g(v\backslash X) = \frac{1}{4}\left(2g(X) + 2 - \#\right)$$

Proof. This follows from the Hurwitz formula. □

3. Rational and elliptic curves $X_1^+(N)$

A smooth projective curve X over an algebraically closed field $\overline{k}$ is called *d-gonal* if there exists a finite morphism $f : X \to \mathbb{P}^1$ over $\overline{k}$ of degree d. For $d = 4$ we say that the curve is *tetragonal*.

If $X_1^+(N)$ is sub-hyperellitic, then $X_1(N)$ is tetragonal, since there exists a map of degree 2 from $X_1(N)$ to $X_1^+(N)$. The author, Kim and Park [4] showed that $X_1(N)$ is tetragonal only for $N = 1 - 18, 20, 21, 22, 24$. Thus it suffices to consider $X_1^+(N)$ for such N. By using the genus formula in [6], one can calculate the genera of $g_1^+(N)$ for $N = 1 - 25$ which are in the following table:

N	$g_1^+(N)$	N	$g_1^+(N)$	N	$g_1^+(N)$	N	$g_1^+(N)$	N	$g_1^+(N)$
1	0	6	0	11	0	16	1	21	2
2	0	7	0	12	0	17	2	22	3
3	0	8	0	13	1	18	1	23	5
4	0	9	0	14	0	19	3	24	2
5	0	10	0	15	0	20	1	25	6

From the table, we have the following results:

THEOREM 3.1. *The curve $X_1^+(N)$ is rational if and only if $N = 1-12, 14, 15$.*

THEOREM 3.2. *The curve $X_1^+(N)$ is elliptic if and only if $N = 13, 16, 18, 20$.*

4. Hyperelliptic curves $X_1^+(N)$

In this section, we determine all the hyperelliptic curves $X_1^+(N)$. It suffices to determine whether the curve $X_1^+(N)$ with $N = 17, 21, 22, 24$ is hyperelliptic or not. Since every curve of genus 2 is hyperelliptic, the curves $X_1^+(17), X_1^+(21)$ and $X_1^+(24)$ are hyperelliptic. Now we prove that $X_1^+(22)$ is hyperelliptic. The author and Kim [3] proved that $X_1(22)$ is bielliptic and W_2 is the unique bielliptic involution. Thus W_2 is defined over $\mathbb{Q}$ and contained in the center of the automorphism group of $X_1(22)$. Thus W_2 defines an involution on $X_1^+(22)$. Now we will show that the quotient $W_2\backslash X_1^+(22)$ is a rational curve. By Proposition 2.1, the number of fixed points of W_2 on $X_1(22)$ is 10. Let $P_1, P_2, \ldots, P_{10}$ denote the fixed points. Then $W_2(P_i)$'s are also fixed by W_2. Thus the number of fixed points of W_2 on $X_1^+(22)$ is at least 5. From Proposition 2.1 again, we know that the genus of $W_2\backslash X_1^+(22)$ should be zero.

Finally, we get the following:

THEOREM 4.1. *The curve $X_1^+(N)$ is hyperelliptic if and only if $N = 17, 21, 22, 24$.*

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