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SYMMETRY OF MINIMAL GRAPHS

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Abstract. In this article, we consider a minimal graph in $\mathbf{R}^3$ which is bounded by a Jordan curve and a straight line. Suppose that the boundary is symmetric with the reflection under a plane, then we will prove that the minimal graph is itself symmetric under the reflection through the same plane.

1. Introduction

In 1956, A. D. Alexandrov [1] proved that a closed embedded hypersurface of constant mean curvature in Euclidean space is a standard sphere. Besides the importance of this result in differential geometry, the method employed in its proof has been used on a variety of problems in partial differential geometry. For example, in 1983, R. M. Schoen [7] used Alexandrov reflection technique to minimal surfaces and showed that: If Γ is any boundary consisting of convex curves lying in a pair of parallel planes which is invariant under reflection through some orthogonal plane which intersects each component of Γ, then every minimal surface spanning Γ is embedded and invariant under reflection through the same plane. It leaded him to prove that the plane and the catenoid are the only embedded minimal surface of finite total curvature with at most two ends.

On the other hand, M. Shiffman [8] proved in 1956 three elegant theorems about a minimal annulus A lying on a slab $S(-1,1)$, where

$$S(a,b) := \{(x_1,x_2,x_3) \in \mathbf{R}^3 \,|\, a \le x_3 \le b,\ a,b \in \mathbf{R}\},$$

such that the boundary curves are continuous convex Jordan curves contained in parallel planes at height ± 1, respectively, as follows:

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1. For all $-1 < t < 1$, the intermediate curve $A \cap P_t$ contained on the horizontal plane P_t is a strictly convex Jordan curve where
$$P_t := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_3 = t\}.$$
In particular, the minimal annulus A is an embedding.
2. If ∂A is a union of circles, then $A \cap P_t$ is a circle for all $-1 \leq t \leq 1$.
3. If ∂A is symmetric with respect to a plane perpendicular to x_1x_2-plane, then A is symmetric with respect to the same plane.

In 1990 Y. Fang [3] gave a generalization of the first two results as that: Suppose $A \subset S(-1, 1)$ is a minimal annulus in a slab and let
$$\partial A = \Gamma_{-1} \cup \Gamma_1$$
where $\Gamma_{-1} \subset P_{-1}$ and $\Gamma_1 \subset P_1$.

1′: If Γ_{-1} is a straight line and Γ_1 is a (smooth) convex Jordan curve, then the intermediate curve $A \cap P_t$ is a strictly convex Jordan curve for all $-1 < t < 1$. In particular A is embedded.

2′: Let Γ_{-1} and Γ_1 be circles, straight lines, or one of them is a circle and the other is a straight line, then $A \cap P_t$ is a circle for $-1 < t < 1$.

In this paper, we will try to extend the third result of Shiffman and prove that:

Theorem 1.1. *Let M be a minimal graph in a slab whose boundary consists of a Jordan curve and a straight line lying on parallel horizontal planes. Suppose that the boundary ∂M is symmetric with the reflection under a vertical plane, then we will prove that M is itself symmetric under the reflection through the same plane.*

2. Main results

Recall $P_0 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_3 = 0\}$ is a horizontal plane. Let $\Omega \subset P_0 = \mathbf{R}^2$ be an unbounded domain whose boundary is a Jordan curve γ and a straight line ℓ where $\gamma \cap \ell = \emptyset$. We may assume that ℓ is the x_2-axis and γ is invariant under the reflection through the x_1-axis of P_0.

Let us denote the minimal graph
$$u : \Omega \to \mathbf{R}$$
such that $u|_\ell \equiv 0$ and $u|_\gamma \equiv 1$. We call the image of γ by Γ, i.e.,
$$\Gamma := \{(x_1, x_2, 1)) \,|\, (x_1, x_2, 0) \in \gamma\}.$$

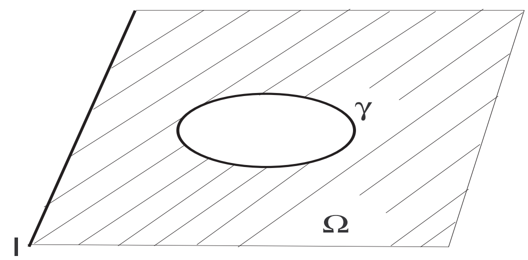

FIGURE 1.

Let $M := u(\Omega)$ be the minimal graph of u, then $M \subset S(0,1)$ and

$$\partial M = \Gamma \cup \ell$$

which is symmetric with the reflection under the x_1x_3-plane.

A minimal surface that contains a straight line segment is invariant under the rotation of angle 180^o around the line by the Schwarz reflection principle. We denote the rotation of angle 180^o around the straight line ℓ by Rot_ℓ. It leads us to get the extended minimal surface

$$\tilde{M} := M \cup Rot_\ell(M) \cup \ell$$

which is the image of the extended minimal graph

$$\tilde{u} : \tilde{\Omega} \to \mathbf{R}$$

defined on the domain

$$\tilde{\Omega} := \Omega \cup Rot_\ell \Omega \cup \ell$$

such that

$$\tilde{u}(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } (x_1, x_2) \in \Omega \\ -u(x_1, x_2) & \text{if } (-x_1, x_2) \in \Omega. \end{cases}$$

Observe that the asymptotic behavior of a complete, embedded minimal surface in $\mathbf{R}^3$ with finite total curvature, like $\tilde{M}$, is well-understood. Particularly, R. Schoen [7] demonstrated that, after a rotation, each embedded end of a complete minimal surface with finite total curvature can be parametrized as a graph over the exterior of a disk in the (x_1, x_2)-plane with the height function h by asymptotic behavior;

$$h(x_1, x_2) = \beta \log r + \frac{\alpha_1 x_1 + \alpha_2 x_2}{r^2} + O(r^{-2}) \qquad (2.1)$$

for $r = (x_1^2 + x_2^2)^{1/2}$ sufficiently large, where α_1 and α_2 are real constants and $O(r^{-2})$ denotes a function such that $r^2 O(r^{-2})$ is bounded as $r \to \infty$.

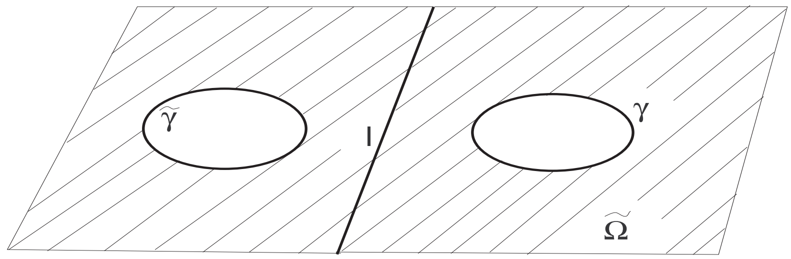

FIGURE 2.

In our case $\tilde{u}$ becomes the height function. Since the minimal surface $\tilde{M}$ lying in a slab $S(-1,1)$, the graph $\tilde{u}$ is bounded. So the number β in (2.1) must be zero. It follows that $\tilde{M}$ has a planar end near the infinity. Moreover $\tilde{M}$ meets the horizontal plane P_0 at the x_2-axis ℓ, which follows that

$$\begin{aligned} h^{-1}(0) &= \left\{ (x_1, x_2) \;\middle|\; \frac{\alpha_1 x_1 + \alpha_2 x_2}{r^2} + O(r^{-2}) = 0 \right\} \\ &= \{(x_1, x_2) \mid x_1 = 0\}\,. \end{aligned}$$

Thus $\alpha_2 = 0$ in (2.1). Therefore we have

$$\tilde{u}(x_1, x_2) = \frac{\alpha x_1}{r^2} + O(r^{-2}) \tag{2.2}$$

for some $\alpha \in \mathbf{R}$ and large $r > 0$. Consider the reflection of $\tilde{M}$ along the x_1x_3-plane, denoted by $Ref(\tilde{M})$, i.e.,

$$Ref(\tilde{M}) := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, (x_1, -x_2, x_3) \in \tilde{M}\}.$$

Take another minimal graph $\tilde{v} : \tilde{\Omega} \to \mathbf{R}$ such that

$$\tilde{v}(x_1, x_2) = \tilde{u}(x_1, -x_2)$$

then the image of $\tilde{v}$ is $Ref(\tilde{M})$. By the definition of $\tilde{v}$ and (2.2), it is clear that

$$\tilde{v}(x_1, x_2) = \frac{\alpha x_1}{r^2} + O(r^{-2}).$$

Now we can conclude that

$$\tilde{u} - \tilde{v} = O(r^{-2}) \tag{2.3}$$

near the infinity. Since every minimal graph is harmonic,

$$\Delta(\tilde{u} - \tilde{v}) = 0 \quad \text{ on } \tilde{\Omega}$$

where Δ is the Laplace operator. Additionally,

$$\tilde{u} \equiv \tilde{v} \quad \text{on } \partial\tilde{\Omega}. \tag{2.4}$$

since $\partial\tilde{M}$ is invariant under the reflection along the x_1x_3-plane.

On the other hand, the unbounded domain $\tilde{\Omega}$ is conformally equivalent to a punctured annulus $\hat{\Omega} - \{p\} \subset \mathbf{C}$, where

$$\hat{\Omega} = \left\{ z \in \mathbf{C} \;\middle|\; \frac{1}{R} < |z| < R \right\}, \; p \in \hat{\Omega}$$

for some $R > 1$. Let

$$\phi : \hat{\Omega} \to \tilde{\Omega} \cup \{\infty\}$$

be a conformal mapping such that $\phi(p) = \infty$.

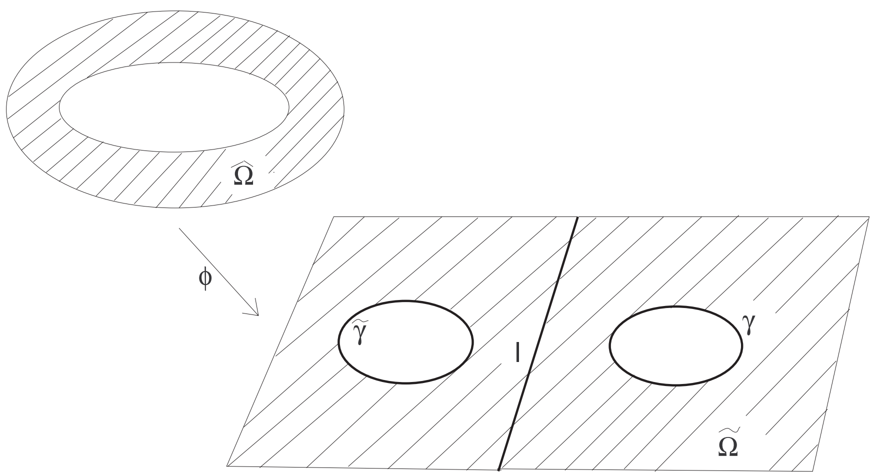

FIGURE 3.

By (2.3) the harmonic function

$$(\tilde{u} - \tilde{v}) \circ \phi : \hat{\Omega} \setminus \{p\} \to \mathbf{R}$$

has the removable singularity at the puncture p, so we can define the extending harmonic function ψ on the whole domain $\hat{\Omega}$ such that

$$\psi(z) = \begin{cases} (\tilde{u} - \tilde{v})(\phi(z)) & \text{if } z \in \hat{\Omega} - \{p\} \\ 0 & \text{if } z = p \end{cases}$$

Observe that

$$\psi|_{\partial\hat{\Omega}} \equiv 0$$

by (2.4). Therefore the harmonic function ψ is equal to zero on the whole domain $\hat{\Omega}$, and we can say that

$$\tilde{u} \equiv \tilde{v} \quad \text{on } \tilde{\Omega}.$$

It shows that the minimal surface $\tilde{M}$ is symmetric under the reflection along the x_1x_3-plane. So is M.

We completes the proof of theorem 1; If a minimal graph is bounded by a straight line ℓ and a Jordan curve Γ which lie on parallel planes, then some symmetry of the boundary can be extended to the surface itself. In the proof, the planar condition of Γ is not essential and so we can say the following theorem holds, too.

THEOREM 2.1. *Let M be a minimal graph defined on a planar domain Ω whose boundary $\partial\Omega$ consists of a straight line and a disjoint Jordan curve. If ∂M is symmetric under the reflection through a plane, then so is M.*

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