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ON FUZZY SUBHYPERNEAR-RINGS OF HYPERNEAR-RINGS WITH t-NORMS

Jong Geol Lee* and Kyung Ho Kim**

Abstract. In this paper, we investigate some properties of T-fuzzy subhypernear-rings of a hypernear-ring.

1. Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} congress of the Scandinavian Mathematicians [16]. Marty introduced the notion of a hypergroup and then many researchers have been worked on this new field of modern algebra and developed it. A comprehensive review of the theory of hyperstructures appear [5] and [20]. The notion of the hyperfield and hyperring was studied by Krasner [14]. In [6], Dasic has introduced the notion of hypernear-rings generalizing the concept of near-ring [17]. In [11], Gontineac defined the zero-symmetric part and the constant part of a hypernear-ring and introduced a structure theorem and other properties of hypernear-rings. Davvaz in [8] introduced the notion of an H_v-near ring generalizing the notion of hypernear-ring.

In [7], Davvaz has introduced the concept of fuzzy subhypernear-rings and fuzzy hyperideals of a hypernear-ring which are a generalization of the concept of a fuzzy subnear-rings and fuzzy ideals in a near-ring. Now, in this paper, we investigate some properties of T-fuzzy subhypernear-rings In this paper, we investigate some properties of T-fuzzy subhypernear-rings of a hypernear-ring.

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Correspondence should be addressed to Kyung Ho Kim, ghkim@cjnu.ac.kr.

2. Preliminaries

Let H be a non-empty set. A *hyperoperation* $*$ on H is a mapping of $H \times H$ into the family of non-empty subsets of H.

A *hypernear-ring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:
(1) $(R, +)$ is a hypergroup i.e., in $(R, +)$ the following hold:

(i) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$;
(ii) There is $0 \in R$ such that $x + 0 = 0 + x = x$ for all $x \in R$;
(iii) For every $x \in R$ there exists one and only one $x' \in R$ such that $0 \in x + x'$, (we shall write $-x$ for x' and we call it the opposite of x);
(iv) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

If $x \in R$ and A, B are subsets of R, then by $A + B$, $A + x$ and $x + B$ we mean

$$A + B = \bigcup_{\substack{a \in A \\ b \in B}} a + b, A + x = A + \{x\}\ ,\ x + B = \{x\} + B.$$

(2) With respect to the multiplication, $(R, \cdot)$ is a semigroup having absorbing element 0 i.e., $x \cdot 0 = 0$ for all $x \in R$.
(3) The multiplication is distributive with respect to the hyperoperation $+$ on the left side i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Note that for all $x, y \in R$, we have $-(-x) = x, 0 = -0, -(x + y) = -y - x$ and $x(-y) = -xy$.

Let $(R, +, \cdot)$ be a hypernear-ring. A non-empty subset H of R is a *subhypernear-ring* if

(1) $(H, +)$ is a subhypergroup of $(R, +)$, i.e., $a, b \in H$ implies $a + b \subseteq H$, and $a \in H$ implies $-a \in H$,
(2) $ab \in H$ for all $a, b \in H$.

EXAMPLE 2.1. Consider hypernear-ring $R = \{0, a, b\}$ with two binary operations as follows:

$+$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

$\cdot$	0	a	b
0	0	0	0
a	0	a	b
b	0	a	b

Then $(R, +, \cdot)$ is a hypernear-ring and $\{0\}$ and R are subhypernear-rings of R.

A *fuzzy subset* μ in a set R is a function $\mu : R \to [0,1]$ and $\mathrm{Im}(\mu)$ denote the *image set* of μ.

DEFINITION 2.2. Let $(R,+,\cdot)$ be a hypernear-ring and μ a fuzzy subset of R. We say that μ is a *fuzzy subhypernear-ring* of R if

(H1) $\min\{\mu(x),\mu(y)\} \leq \inf_{\alpha\in x+y}\{\mu(\alpha)\}$ for all $x,y \in R$,
(H2) $\mu(x) \leq \mu(-x)$,
(H3) $\mu(xy) \geq \min\{\mu(x),\mu(y)\}$ for all $x,y \in R$.

DEFINITION 2.3. ([5]) By a *t-norm* T, we mean a function $T : [0,1]\times[0,1] \to [0,1]$ satisfying the following conditions:
(T1) $T(x,1) = x$,
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$,
(T3) $T(x,y) = T(y,x)$,
(T4) $T(x,T(y,z)) = T(T(x,y),z)$,
for all $x,y,z \in [0,1]$.

For a t-norm T on $[0,1]$, denote by Δ_T the set of element $\alpha \in [0,1]$ such that $T(\alpha,\alpha) = \alpha$, i.e., $\Delta_T := \{\alpha \in [0,1] \mid T(\alpha,\alpha) = \alpha\}$.

PROPOSITION 2.4. *Every t-norm T has a useful property:*

$$T(\alpha,\beta) \leq \min(\alpha,\beta)$$

for all $\alpha,\beta \in [0,1]$.

DEFINITION 2.5. Let T be a t-norm. A fuzzy subset μ of R is said to satisfy *idempotent property* if $\mathrm{Im}(\mu) \subseteq \Delta_T$.

3. Fuzzy subhypernear-rings of hypernear-rings with t-norms

DEFINITION 3.1. Let $(R,+,\cdot)$ be a hypernear-ring and μ a fuzzy subset of R. We say that μ is a *fuzzy subhypernear-ring of R with respect to t-norm T* (briefly, a *T-fuzzy subhypernear-ring of R*) if

(TH1) $T(\mu(x),\mu(y)) \leq \inf_{\alpha\in x+y}\{\mu(\alpha)\}$ for all $x,y \in R$,
(TH2) $\mu(x) \leq \mu(-x)$,
(TH3) $\mu(xy) \geq T(\mu(x),\mu(y))$ for all $x,y \in R$.

EXAMPLE 3.2. Let $R = \{0, a, b, c\}$ be a set with a hyperoperation "$+$" and a binary operation "$\cdot$" as follows:

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{0, a, c\}$	$\{b, c\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, a, b\}$

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Then $(R, +, \cdot)$ is a hypernear-ring. We define a fuzzy set μ in R by

$$\mu(0) = 0.7, \mu(a) = 0.5 \text{ and } \mu(b) = \mu(c) = 0.3.$$

Let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0) \text{ for all } \alpha, \beta \in [0, 1]$$

which is a t-norm. Routine calculations give that μ is a T-fuzzy subhypernear-ring of R.

PROPOSITION 3.3. *Let μ be an idempotent T-fuzzy subhypernear-ring of a hypernear-ring R. Then $\mu(x) \le \mu(0)$ for all $x \in R$.*

Proof. For any $x \in R$, we have

$$\mu(0) \ge \inf_{\alpha \in x - x} \mu(\alpha) \ge T(\mu(x), \mu(-x)) \ge T(\mu(x), \mu(x)) = \mu(x).$$

□

PROPOSITION 3.4. *Let T be an t-norm. If μ is an idempotent T-fuzzy subhypernear-ring of hyper near-ring R, then the set*

$$R^\omega = \{x \in R \mid \mu(x) \ge \mu(\omega)\}$$

is a subhypernear-ring of a hyper near-ring R.

Proof. Let $x, y \in R^\omega$. Then $\mu(x) \ge \mu(\omega)$ and $\mu(y) \ge \mu(\omega)$. Since μ is an T-fuzzy subhypernear-ring of R, it follows that

$$\inf_{\alpha \in x + y} \{\mu(\alpha)\} \ge T(\mu(x), \mu(y)) \ge T(\mu(x), \mu(\omega)) \ge T(\mu(\omega), \mu(\omega)) = \mu(\omega).$$

Hence $x + y \subseteq R^\omega$ implies $x + y \in \mathcal{P}^*(R^\omega)$. Let $x \in R^\omega$. Then we have $\mu(x) \ge \mu(\omega)$, and so $\mu(-x) \ge \mu(x) \ge \mu(\omega)$. Thus we have $-x \in \bar{R}^\omega$. Let $x, y \in R^\omega$. Then we get $\mu(xy) \ge T(\mu(x), \mu(y)) \ge T(\mu(\omega), \mu(\omega)) = \mu(\omega)$, and so $xy \in R^\omega$. This completes the proof. □

Corollary 3.5. *Let T be an t-norm. If μ is an idempotent T-fuzzy subhypernear-ring of R, then the set*

$$R^\mu = \{x \in R \mid \mu(x) = \mu(0)\}$$

is a subhypernear-ring of a hyper near-ring R.

Proof. From the Corollary 3.3, $R^\mu = \{x \in R \mid \mu(x) = \mu(0)\} = \{x \in R \mid \mu(x) \geq \mu(0)\}$, hence R^μ is a subhypernear-ring of a hyper near-ring R from the Prosition 3.4. □

Lemma 3.6. ([1]) *Let T be a t-norm. Then*

$$T(T(\alpha,\beta),T(\gamma,\delta)) = T(T(\alpha,\gamma),T(\beta,\delta))$$

for all $\alpha,\beta,\gamma,\delta \in [0,1]$.

Proposition 3.7. *If μ and ν are T-fuzzy subhypernear-rings of a hypernear-ring R, then $\mu \wedge \nu : R \to [0,1]$ defined by*

$$(\mu \wedge \nu)(x) = T(\mu(x),\nu(x))$$

for all $x \in R$ is a T-fuzzy subhypernear-ring of R.

Proof. Let $x, y \in R$. Then we have

$$\begin{aligned}
\inf_{\alpha \in x+y}\{(\mu \wedge \nu)(\alpha)\} &= \inf_{\alpha \in x+y}\{T(\mu(\alpha),\nu(\alpha))\} \\
&\geq T(\inf_{\alpha \in x+y}\{\mu(\alpha)\}, \inf_{\alpha \in x+y}\{\nu(\alpha)\}) \\
&\geq T(T(\mu(x),\mu(y)),T(\nu(x),\nu(y))) \\
&= T(T(\mu(x),\nu(x)),T(\mu(y),\nu(y))) \\
&= T((\mu \wedge \nu)(x),(\mu \wedge \nu)(y))
\end{aligned}$$

and

$$\begin{aligned}
(\mu \wedge \nu)(-x) &= T(\mu(-x),\nu(-x)) \geq T(\mu(x),\nu(x)) \\
&= (\mu \wedge \nu)(x)
\end{aligned}$$

since $\mu(-x) \geq \mu(x)$ and $\nu(-x) \geq \nu(x)$. Also, for $x, y \in R$, we have

$$\begin{aligned}
(\mu \wedge \nu)(xy) &= T(\mu(xy),\nu(xy)) \\
&= T(T(\mu(x),\mu(y)),T(\nu(x),\nu(y))) \\
&\geq T(T(\mu(x),\nu(x)),T(\mu(y),\nu(y))) \\
&= T((\mu \wedge \nu)(x),(\mu \wedge \nu)(y))
\end{aligned}$$

This completes the proof. □

Proposition 3.8. *Let H be a non-empty subset of a hypernear-ring R and let μ be a fuzzy set in R defined by*

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in H \\ t_2 & \text{otherwise,} \end{cases}$$

where $t_1 > t_2$ in $[0,1]$. Then μ is an idempotent T-fuzzy subhypernear-ring of R if and only if H is a subhypernear-ring of R.

Proof. Suppose that μ is an idempotent T-fuzzy subhypernear-ring of R. Let $x, y \in H$. Then $\inf_{\alpha \in x+y} \mu(\alpha) \geq T(\mu(x), \mu(y)) = t_1$ and so $\inf_{\alpha \in x+y} \mu(\alpha) \geq t_1$. It follows that $x + y \subseteq H$. Next, let $x \in H$. Then we have $t_1 = \mu(x) \leq \mu(-x)$, and so $\mu(-x) = t_1$, that is, $-x \in H$. Next, we have $\mu(xy) \geq T(\mu(x), \mu(y)) \geq t_1$, and so $\mu(xy) = t_1$. Hence $xy \in H$ and therefore H is a subhypernear-ring of R. Conversely suppose that H is a subhypernear-ring of R. Let $x, y \in R$. If $x \in R \setminus H$ or $y \in R \setminus H$, then $\mu(x) = t_2$ or $\mu(y) = t_2$ and so

$$\inf_{\alpha \in x+y} \mu(\alpha) \geq t_2 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) \geq t_2 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$. Assume that $x \in H$ and $y \in H$. Then $x + y \subseteq H$ and hence

$$\inf_{\alpha \in x+y} \mu(\alpha) = t_1 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and $\mu(xy) = t_1 = \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$. Since $x \in H$, we obtain $-x \in H$, which implies $\mu(x) \leq \mu(-x)$. Consequently μ is a T-fuzzy subhypernear-ring of R. □

References

[1] M. T. Abu Osman, *On some product of fuzzy subgroups,* Fuzzy Sets and Sys, **24**, (1987), 79-86.

[2] S. Abou-Zaid, *On fuzzy subnear-rings and ideals*, Fuzzy Sets and Sys. **44**, (1991), 139-146.

[3] R. Ameri and M. M. Zahedi, *T-fuzzy hyperalgebraic systems*, AFSS 2002, LNAI 2275, (2002), 509-514.

[4] J.M. Anthony and H. Shewood, *Fuzzy groups redifined*, **69** (1979), 124-130.

[5] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory,* Advanced in Mathematics, Vol. 5, Klower Academic Publishers, 2003.

[6] V. Dasic, *Hypernear-rings,* Proceedings of the Forth International Congress on A. H. A., Xanthi, Greece, World Scientific, (1990), 75-85.

[7] B. Davvaz, *On hypernear-rings and fuzzy hyperideals,* The Journal of Fuzzy Mathematics, **7** (1999), no. 3, 745-753.

[8] B. Davvaz, *H_v-Near-Rings,* Math. Japonica, **52** (2000), no. 2, 387-392.

[9] B. Davvaz, *T-fuzzy H_v-subrings of an H_v-ring,* The Journal of Fuzzy Mathematics, **11** (2003), no. 1, 215-224.
[10] B. Davvaz, *Fuzzy R-subgroups with thresholds of near-ings and implication operators,* Soft Computing (to appear)
[11] V. M. Gontineac, *On hypernear-rings and H-hypergroups,* Proceedings of the Forth International Congress on A. H. A. 1993, Hadronic Press, Inc., USA, (1994), 171-179.
[12] K. H. Kim, B. Davvaz and E. H. Roh, *On Hyper R-subgroups of Hypernear-rings,* Scientiae Mathematicae Japoniciae (submitted).
[13] K. H. Kim, B. Davvaz and E. H. Roh, *On fuzzy hyper R-subgroups of hypernear-rings,* Italian Journal of Pure and Applied Mathematics, **20** (2006), 177-192.
[14] M. Krasner, *A class of hyperrings and hyperfields,* Int. J. Math. and Math. Sci. **2** (1983), 307-312.
[15] O. Kazanci, S. Yamark and S. Yilmaz, *On intuitionistic Q-fuzzy R-subgroups of Near-rings,* International Mathematical Forum, **2** (No. 59) (2007), 2899-2910.
[16] F. Matry, *Sur une generalization de la notion de groupe,* 8^{iem} congress Math. Scandenaves, Stoockholm, (1934), 45-49.
[17] G. Pllz, *Near-rings,* North-Holland Publ., Co, 1977.
[18] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.*, **35** (1971), 512-517.
[19] B. Schweizer and A. Sklar, *Statistical metric spaces,* Pacific J. Math. **10** (1960), 313-334.
[20] T. Vougiouklis, *Hyperstructures and their representations,* Hadronic Press, Inc. 115, Palm Harber, USA, 1994.
[21] S. Willard, *General Topology,* Addison-Wesley Publ., USA, 1970.
[22] Y. H. Yon, *On intuitionistic fuzzy R-subgroups of near-rings,* Soochow Journal of Mathematics **27 (3)**, (2001), 243-253.
[23] L. Zadeh, *Fuzzy sets,* Inform. and Contral **8**, (1965), 338-353.

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Department of Mathematics
Chungju National University
Chungju 380-702, Republic of Korea
E-mail: jglee@cjnu.ac.kr

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Department of Mathematics
Chungju National University
Chungju 380-702, Republic of Korea
E-mail: ghkim@cjnu.ac.kr