JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 2, June 2010

SOME REMARKS ON THE DIMENSIONS OF THE PRODUCTS OF CANTOR SETS

Jin-Oh Kim*

Abstract. Using the properties of the concave function, we show that the Hausdorff dimension of the product $C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}$ of the same symmetric Cantor sets is greater than that of the product $C_{a,b} \times C_{a,b}$ of the same anti-symmetric Cantor sets. Further, for $1/e^2 < a, b < 1/2$, we also show that the dimension of the product $C_{a,a} \times C_{b,b}$ of the different symmetric Cantor sets is greater than that of the product $C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}$ of the same symmetric Cantor sets using the concavity. Finally we give a concrete example showing that the latter argument does not hold for all $0 < a, b < 1/2$.

1. Introduction

Recently the exact packing measure of the symmetric Cantor set and the exact packing measure of the product of the same symmetric Cantor sets were computed in [2, 3]. It is still open to compute the exact packing measure of the product of the anti-symmetric Cantor sets or the different symmetric Cantor sets. However, computing the packing dimension is easier than computing the exact packing measure of the product of the same symmetric Cantor sets. The calculating method of the Hausdorff and packing dimensions of the product of the same symmetric Cantor sets was well-known in [1, 4]. We note that the main idea of its calculating method is due to [5]. More precisely, if one(say it B) of the sets A, B making product $A \times B$ has the same Hausdorff and packing dimension, then the Hausdorff dimension of the product $A \times B$ is the sum of the Hausdorff dimension of A and the Hausdorff and packing dimension of B. Similarly, if one(say it B) of the sets A, B making

Received December 28, 2009; Accepted April 23, 2010.
2010 Mathematics Subject Classification: Primary 28A78; Secondary 28A80.
Key words and phrases: Cantor set, product, Hausdorff dimension, packing dimension.

product $A \times B$ has the same Hausdorff and packing dimension, then the packing dimension of the product $A \times B$ is the sum of the packing dimension of A and the Hausdorff dimension(or packing dimension) of B. Using the formulas, we easily have the dimensions of the product $A \times B$ where A, B are Cantor sets. In this paper, we discuss the Hausdorff and packing dimension of the product $A \times B$ where A, B are Cantor sets for the condition that the sum of the lengths of the first stage of Cantor sets making the product is fixed. There are 3 types for the first stage of Cantor sets making the product. More precisely, the three types are $C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}$, $C_{a,b} \times C_{a,b}$, $C_{a,a} \times C_{b,b}$ whose length is $2(a+b)$. This follows from the length equation $2(\frac{a+b}{2}+\frac{a+b}{2}) = 2(a+b) = (a+a)+(b+b)$. Their Hausdorff and packing dimensions are same, which follows from an easy calculation for the dimensions of their product. We note that the product $C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}$ of the same symmetric Cantor sets is the self-similar set while the products $C_{a,b} \times C_{a,b}$, $C_{a,a} \times C_{b,b}$ are the self-affine sets. Finally we give concrete examples and a counter example of our main results related to the above length conditions.

2. Preliminaries

The definitions of Hausdorff dimension of a set A denoted by $\dim(A)$ and packing dimension of a set A denoted by $\mathrm{Dim}(A)$ are well-known[1, 4]. We define $f_1(x) = ax$ and $f_2(x) = (1-b) + bx$ with $0 < a, b < 1$ and $0 < a + b \leq 1$. We define the Cantor set $C_{a,b}$ to be the attractor of the similarities $\{f_1, f_2\}$ on $\mathbb{R}$. From now on, we give some necessary propositions to compute the dimensions of the product of Cantor sets. Further we also give some lemmas for the concave function for explanation of our main theorems. This properties can be straightforward, but are postulated for easy access to the main theorems. We give the dimension formula for dimension of the product $A \times B$.

Proposition 2.1. *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be non-empty Borel sets. Then*

$$\dim(A) + \dim(B) \leq \dim(A \times B) \leq \mathrm{Dim}(A \times B) \leq \mathrm{Dim}(A) + \mathrm{Dim}(B).$$

Proof. It follows from [4] and the well-known fact that $\dim(A \times B) \leq \mathrm{Dim}(A \times B)$. $\square$

Corollary 2.2. *Let* $A \subset \mathbb{R}$ *and* $B \subset \mathbb{R}$ *be non-empty Borel sets with* $\dim(A) = \mathrm{Dim}(A), \dim(B) = \mathrm{Dim}(B)$. *Then*

$$\dim(A) + \dim(B) = \dim(A \times B) = \mathrm{Dim}(A \times B) = \mathrm{Dim}(A) + \mathrm{Dim}(B).$$

Proof. It follows immediately from the above Proposition. □

Example 2.3. *Let* $A \subset \mathbb{R}$ *and* $B \subset \mathbb{R}$ *be self-similar Cantor sets in* $\mathbb{R}$. *Then*

$$\dim(A) + \dim(B) = \dim(A \times B) = \mathrm{Dim}(A \times B) = \mathrm{Dim}(A) + \mathrm{Dim}(B),$$

since they have the same Hausdorff dimension and packing dimension.

Lemma 2.4. *Let* $0 < s < 1$ *be in* $\mathbb{R}$. *Then for* $\lambda > 0$ *and* $\epsilon \geq 0$,

$$\lambda^s \geq \frac{(\lambda+\epsilon)^s + (\lambda-\epsilon)^s}{2}.$$

Proof. It follows easily from that $g(x) = x^s$ is a concave function for $0 < s < 1$ and $\lambda = \frac{\lambda+\epsilon}{2} + \frac{\lambda-\epsilon}{2}$. □

3. Main results

We note that $C_{\frac{a+b}{2},\frac{a+b}{2}}, C_{a,b}, C_{a,a}, C_{b,b}$ below are all self-similar Cantor sets in $\mathbb{R}$ mentioned in the above Example. So their Hausdorff dimension and packing dimensions are same, which gives that their products also have the same Hausdorff dimension and packing dimensions. We only use their Hausdorff dimension instead of using their dimensions separately.

Using the properties of the concave function, we see that the dimension of the product $C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}$ of the same symmetric Cantor sets is greater than that of the product $C_{a,b} \times C_{a,b}$ of the same anti-symmetric Cantor sets.

Theorem 3.1. *For each* $0 < a, b < 1$ *with* $0 < a + b \leq 1$, *we have*

$$\dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}) \geq \dim(C_{a,b} \times C_{a,b}).$$

Proof. Let $\frac{a+b}{2} = \lambda$. We note that $\dim(C_{\lambda,\lambda} \times C_{\lambda,\lambda}) = 2\log_\lambda \frac{1}{2}$ and $\dim(C_{a,b} \times C_{a,b}) = 2s$ where $a^s + b^s = 1$. Without loss of generality, we may assume that $0 < b \leq a < 1$ with $0 < a + b \leq 1$. There must be $\epsilon \geq 0$ such that $a = \lambda + \epsilon$ and $b = \lambda - \epsilon$. Since $f(x) = \lambda^x$ is a decreasing function for $x \in (0, 1)$, we only need to show that

$$2\lambda^s \geq (\lambda+\epsilon)^s + (\lambda-\epsilon)^s,$$

where $a^s + b^s = 1$. It follows from the above Lemma. $\square$

Using the properties of the concave function, we also see that the dimension of the product $C_{a,a} \times C_{b,b}$ of the different symmetric Cantor sets is greater than that of the product $C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}$ of the same symmetric Cantor sets for $e^{-2} = 0.135335283237... < a, b < 1/2$.

THEOREM 3.2. *For each $e^{-2} < a, b \leq 1/2$, we have*

$$\dim(C_{a,a} \times C_{b,b}) \geq \dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}).$$

Proof. We note that

$$\dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}) = 2\log_{\frac{a+b}{2}} \frac{1}{2}$$

and

$$\dim(C_{a,a} \times C_{b,b}) = \log_a \frac{1}{2} + \log_b \frac{1}{2}.$$

We note that $\phi(x) = \frac{1}{\log x}$ is a concave function on the interval $(e^{-2}, 1)$ since $\phi''(x) < 0$ on $(e^{-2}, 1)$. Therefore

$$\phi(\frac{a+b}{2}) \geq \frac{\phi(a) + \phi(b)}{2}.$$

This gives

$$\frac{1}{\log \frac{a+b}{2}} \geq \frac{\frac{1}{\log a} + \frac{1}{\log b}}{2}.$$

Multiplying the negative number $2\log\frac{1}{2}$ on each side, we have

$$\frac{2\log\frac{1}{2}}{\log\frac{a+b}{2}} \leq \frac{\log\frac{1}{2}}{\log a} + \frac{\log\frac{1}{2}}{\log b}.$$

Hence we have $2\log_{\frac{a+b}{2}} \frac{1}{2} \leq \log_a \frac{1}{2} + \log_b \frac{1}{2}$. $\square$

EXAMPLE 3.3. For $a = 1/2, b = 1/4$, we see that

$$\dim(C_{a,a} \times C_{b,b}) \geq \dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}) \geq \dim(C_{a,b} \times C_{a,b}).$$

We note that $\dim(C_{a,a} \times C_{b,b}) = \frac{3}{2} = 1.5$, $\dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}) = \log_{\frac{3}{8}} \frac{1}{4} = 1.41339...$, and $\dim(C_{a,b} \times C_{a,b}) = 2\log_{\frac{1}{2}} \frac{\sqrt{5}-1}{2} = 1.38848....$ We note that $C_{a,a}$ is the unit interval which is not a Cantor set.

EXAMPLE 3.4. If $a = 1/4, b = 1/16$, the assumption of the above theorem does not hold. But we still see that for $a = 1/4, b = 1/16$

$$\dim(C_{a,a} \times C_{b,b}) \geq \dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}) \geq \dim(C_{a,b} \times C_{a,b}).$$

We note that $\dim(C_{a,a} \times C_{b,b}) = \frac{3}{4} = 0.75$, $\dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}) = \log_{\frac{5}{32}} \frac{1}{4} = 0.74680...$, and $\dim(C_{a,b} \times C_{a,b}) = 2\log_{\frac{1}{4}} \frac{\sqrt{5}-1}{2} = 0.69424....$

From the above example, we naturally ask if $\dim(C_{a,a} \times C_{b,b}) \geq \dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}})$ holds for all $0 < a, b \leq 1/2$. We find the negative answer in the following example.

EXAMPLE 3.5. If $a = 1/8, b = 1/64$, the assumption of the above theorem does not hold. We see that for $a = 1/8, b = 1/64$

$$\dim(C_{a,a} \times C_{b,b}) \ngeq \dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}) \geq \dim(C_{a,b} \times C_{a,b}).$$

We note that $\dim(C_{a,a} \times C_{b,b}) = \frac{3}{6} = 0.5$, $\dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}}) = \log_{\frac{9}{128}} \frac{1}{4} = 0.52218...$, and $\dim(C_{a,b} \times C_{a,b}) = 2\log_{\frac{1}{8}} \frac{\sqrt{5}-1}{2} = 0.46282....$ It is false that $\dim(C_{a,a} \times C_{b,b}) \geq \dim(C_{\frac{a+b}{2},\frac{a+b}{2}} \times C_{\frac{a+b}{2},\frac{a+b}{2}})$ holds for all $0 < a, b < 1/2$. It is a counter example of such an argument.

Acknowledgments

I would like to thank Professor In-Soo Baek for the helpful comments on this paper.

References

[1] K. J. Falconer, Fractal Geometry-Mathematical Foundations and Applications, Wiley New York, 1990.

[2] D.-J. Feng, Exact packing measure of linear Cantor sets, Math. Nachr. 248-249 (2003), 102-109.

[3] B. Jia, Z. Zhou, Z. Zhu, J. Luo, The packing measure of the Cartesian product of the middle third Cantor set with itself, J. Math. Anal. Appl. **288** (2003), 424-441.

[4] P. Mattila, Geometry of sets and measures in Euclidean spaces, Cambridge University Press, 1995.

[5] C. Tricot, Two definitions of fractional dimension, Math. Proc. Camb. Phil. Soc. **91** (1982), 54-74.

*
Department of Mathematics Eduacation
Graduate School of Education
Pusan University of Foreign Studies
Pusan 608-738, Republic of Korea
E-mail: jokim@pufs.ac.kr