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ON WEAK M-SEMICONTINUITY ON SPACES WITH MINIMAL STRUCTURES

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Abstract. We introduce the notion of weak M-semicontinuity which is a generalization of M-semicontinuity defined between spaces with minimal structures. We also investigate some properties and characterizations for such a notion.

1. Introduction

Popa and Noiri [5] introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. And they introduced the notion of M-continuous function as a function defined between spaces with minimal structures. They showed that the M-continuous functions have properties similar to those of continuous functions between topological spaces.

In [2], we introduced the notions of m-semiopen sets, m-semi-interior and m-semi-closure operators on a space with a minimal structure. For a given nonempty set, if a minimal structure is a topology, the m-semiopen sets are semiopen sets defined in [1]. We also introduced the notion of M-semicontinuous function and studied characterizations for the M-semicontinuity in [2]. In this paper, we introduce the notion of weakly M-semicontinuous function which is a generalization of M-semicontinuous function defined between spaces with minimal structures. And we investigate characterizations for such a notion and relationship between weakly M-semicontinuous function and strongly M-semiclosed graph.

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2. Preliminaries

A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* [5] on X if $\emptyset \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X. Simply we call (X, m_X) a space with a minimal structure m_X on X. Let (X, m_X) be a space with a minimal structure m_X on X. For a subset A of X, the closure of A and the interior of A are defined as the following:
$mInt(A) = \cup\{U : U \subseteq A, U \in m_X\}$;
$mCl(A) = \cap\{F : A \subseteq F, X - F \in m_X\}$.

THEOREM 2.1. [5] *Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$.*
(1) $X = mInt(X)$ and $\emptyset = mCl(\emptyset)$.
(2) $mInt(A) \subseteq A$ and $A \subseteq mCl(A)$.
(3) If $A \in m_X$, then $mInt(A) = A$ and if $X - F \in m_X$, then $mCl(F) = F$.
(4) If $A \subseteq B$, then $mInt(A) \subseteq mInt(B)$ and $mCl(A) \subseteq mCl(B)$.
(5) $mInt(mInt(A)) = mInt(A)$ and $mCl(mCl(A)) = mCl(A)$.
(6) $mCl(X - A) = X - mInt(A)$ and $mInt(X - A) = X - mCl(A)$.

Let (X, m_X) be a space with a minimal structure m_X on X and $A \subset X$. A subset A of X is called an *m-semiopen set* [2] if $A \subseteq mCl(mInt(A))$. The complement of an m-semiopen set is called an *m-semiclosed set*. Let (X, m_X) be a space with a minimal structure m_X on X. For a subset A of X, the m-semi-closure of A and the m-semi-interior of A, denoted by $msCl(A)$ and $msInt(A)$, respectively, are defined as the following:

$$msCl(A) = \cap\{F : A \subseteq F, F \text{ is } m\text{-semiclosed in } X\};$$

$$msInt(A) = \cup\{U : U \subseteq A, U \text{ is } m\text{-semiopen in } X\}.$$

THEOREM 2.2. [2] *Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then*
(1) $msInt(A) \subseteq A$ and $A \subseteq msCl(A)$.
(2) If $A \subseteq B$, then $msInt(A) \subseteq msInt(B)$ and $msCl(A) \subseteq msCl(B)$.
(3)A is m-semiopen iff $msInt(A) = A$.
(4)F is m-semiclosed iff $msCl(F) = F$.
(5) $msInt(msInt(A)) = msInt(A)$ and $msCl(msCl(A)) = msCl(A)$.
(6) $msCl(X - A) = X - msInt(A)$ and $msInt(X - A) = X - msCl(A)$.

Let $f : (X, m_X) \to (Y, m_Y)$ be a function between two spaces with minimal structures m_X and m_Y. Then f is said to be *M-continuous* [4] (resp. *M-semicontinuous* [3]) if for each x and each m-open set V containing $f(x)$, there exists an m-open (resp. m-semiopen) set U containing x such that $f(U) \subseteq V$.

3. Main Results

Definition 3.1. Let $f : (X, m_X) \to (Y, m_Y)$ be a function between two spaces with minimal structures m_X and m_Y, respectively. Then f is said to be *weakly M-semicontinuous* if for each m-open set A of Y, $f^{-1}(A) \subseteq msInt(f^{-1}(msCl(A)))$.

Every M-semicontinuous function is weakly M-semicontinuous but the converse is not always true.

Example 3.2. *Let $X = Y = \{a, b, c\}$ and let us consider two minimal structures $m_X = \{\emptyset, \{a\}, \{b\}, X\}$ and $m_Y = \{\emptyset, \{a, b\}, \{b, c\}, Y\}$. Consider a function $f : (X, m_X) \to (Y, m_Y)$ defined as $f(a) = f(b) = a$, $f(c) = c$. Then for $\{a, b\}, \{b, c\} \in m_Y$, $msCl(\{a, b\}) = msCl(\{b, c\}) = Y$ in (Y, m_Y), and so clearly f is weakly M-semicontinuous. But for $\{b, c\} \in m_Y$, $f^{-1}(\{b, c\})$ is not m-semiopen in (X, m_X). Consequently f is not M-semicontinuous.*

M-continuous $\Rightarrow$ M-semicontinuous $\Rightarrow$ weakly M-semicontinuous

Theorem 3.3. *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures m_X and m_Y. Then f is weakly M-semicontinuous if and only if for every point x and each m-open set V containing $f(x)$, there exists an m-semiopen set U containing x such that $f(U) \subseteq msCl(V)$.*

Proof. Suppose f is a weakly M-semicontinuous function. Let x be in X and V an m-open set containing $f(x)$. Then there exists an m-semiopen set B such that $f(x) \in B \subseteq V$. From the hypothesis, it follows

$$x \in f^{-1}(B) \subseteq msInt(f^{-1}(msCl(B))) \subseteq msInt(f^{-1}(msCl(V))).$$

Since $x \in msInt(f^{-1}(msCl(V)))$, there exists an m-semiopen set U such that $x \in U \subseteq f^{-1}(msCl(V))$, and hence $f(U) \subseteq msCl(V)$.

For the converse, let V be an m-open set in Y. For each $x \in f^{-1}(V)$, there exists an m-semiopen set U containing x such that $f(U) \subseteq msCl(V)$. This implies

$$f^{-1}(V) \subseteq \cup\{U : x \in f^{-1}(V)\} \subseteq f^{-1}(msCl(V)).$$

Since $\cup\{U : x \in f^{-1}(V)\}$ is an m-semiopen set containing $f^{-1}(V)$, we have $f^{-1}(V) \subseteq msInt(f^{-1}(msCl(V)))$. Hence f is weakly M-semicontinuous. □

A minimal structure m_X on a nonempty set X is said to have property $(\mathcal{B})$ [5] if the union of any family of subsets belonging to m_X belongs to m_X

Lemma 3.4. [5] *Let m_X be a minimal structure on a nonempty set X satisfying $(\mathcal{B})$. For $A \subseteq X$, the following are equivalent:*
(1) $A \in m_X$ if and only if $mInt(A) = A$.
(2) A is m-closed if and only if $mCl(A) = A$.

Theorem 3.5. *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures m_X and m_Y, respectively. If m_Y has the property $(\mathcal{B})$, then the following statements are equivalent:*
(1) f is weakly M-semicontinuous.
(2) $msCl(f^{-1}(msInt(F))) \subseteq f^{-1}(F)$ for each m-closed set F in Y.
(3) $msCl(f^{-1}(V)) \subseteq f^{-1}(mCl(V))$ for each m-open set V in Y.

Proof. (1) ⇒ (2) Let F be any m-closed set of Y. Then from Theorem 2.2, it follows

$$\begin{aligned} f^{-1}(Y - F) &\subseteq msInt(f^{-1}(msCl(Y - F))) \\ &= msInt(f^{-1}(Y - msInt(F))) \\ &= msInt(Y - f^{-1}(msInt(F))) \\ &= X - msCl(f^{-1}(msInt(F))). \end{aligned}$$

Hence we have $msCl(f^{-1}(msInt(F))) \subseteq f^{-1}(F)$.

Similarly we can prove that the implication (2) ⇒ (1) is hold.

(2) ⇒ (3) For any m-open set V in Y, by Lemma 3.4, $mCl(V)$ is m-closed. Since $V \subseteq msInt(mCl(V))$, we have the following relation:

$$msCl(f^{-1}(V)) \subseteq msCl(f^{-1}(msInt(mCl(V)))) \subseteq f^{-1}(mCl(V)).$$

Hence the statement (3) is obtained.

(3) ⇒ (1) For each an m-open set V in Y. Then from

$$V \subseteq mInt(msCl(V)),$$

it follows

$$\begin{aligned} f^{-1}(V) &\subseteq f^{-1}(mInt(msCl(V))) \\ &= X - f^{-1}(mCl(Y - msCl(V))) \\ &\subseteq X - msCl(f^{-1}(Y - msCl(V))) \\ &= msInt(f^{-1}(msCl(V))). \end{aligned}$$

Hence f is weakly M-semicontinuous. □

THEOREM 3.6. *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures m_X and m_Y. If m_Y has the property $(\mathcal{B})$, then the following statements are equivalent:*

(1) f is weakly M-semicontinuous.

(2) $msCl(f^{-1}(msInt(B))) \subseteq f^{-1}(mCl(B))$ for each set B in Y.

(3) $f^{-1}(mInt(B)) \subseteq msInt(f^{-1}(msCl(B)))$ for each set B in Y.

Proof. (1) ⇒ (2) For $B \subseteq Y$, by the property $(\mathcal{B})$, $mCl(B)$ is an m-closed set in Y. Then from Theorem 3.5 (2), it follows

$$msCl(f^{-1}(msInt(B))) \subseteq msCl(f^{-1}(msInt(mCl(B)))) \subseteq f^{-1}(mCl(B)).$$

(2) ⇒ (3) For $B \subseteq Y$, from Theorem 2.2 and (2),

$$\begin{aligned} f^{-1}(mInt(B)) &= f^{-1}(Y - mCl(Y - B)) \\ &= X - (f^{-1}(mCl(Y - B))) \\ &\subseteq X - msCl(f^{-1}(msInt(Y - B))) \\ &= msInt(f^{-1}(msCl(B))). \end{aligned}$$

This implies $f^{-1}(mInt(B)) \subseteq msInt(f^{-1}(msCl(B)))$.

(3) ⇒ (1) For each m-open set V in Y, from $B = mInt(B)$ and (3), we have $f^{-1}(B) = f^{-1}(mInt(B)) \subseteq msInt(f^{-1}(msCl(B)))$. Hence f is weakly M-semicontinuous. □

We recall the next notions: Let m_X be a minimal structure on a nonemptyset X. Then (X, m_X) is said to be

(1) *m-Urysohn* [6] if for each distinct points $x, y \in X$, there exist $U, V \in m_X$ containing x and y, respectively, such that $mCl(U) \cap mCl(V) = \emptyset$,

(2) *m-semi-T_2* [3] if for each distinct points $x, y \in X$, there exist m-semiopen sets U, V containing x and y, respectively, such that $U \cap V = \emptyset$.

DEFINITION 3.7. Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures m_X and m_Y. Then f has a *strongly M-semiclosed graph* if for each $(x, y) \in (X \times Y) - G(f)$, there exist an m-smiopen set U containing x and an m-open set V containing y such that $(U \times mCl(V)) \cap G(f) = \emptyset$.

LEMMA 3.8. *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures m_X and m_Y. Then f has a strongly M-semiclosed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist an m-smiopen set U containing x and an m-open set V containing y such that $f(U) \cap mCl(V) = \emptyset$.*

THEOREM 3.9. *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures m_X and m_Y. If f is weakly M-semicontinuous and (Y, m_Y) is m-Urysohn, then $G(f)$ is a strongly M-semiclosed graph.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$; then $f(x) \neq y$. Since Y is m-Urysohn, there are m-open sets U, V containing $f(x)$, y, respectively, such that $U \cap V = \emptyset$. This implies $mCl(V) \cap U = \emptyset$. On the one hand, from weak M-semicontinuity of f, for $f(x) \in U$, there exists an m-semiopen set G containing x such that $f(G) \subseteq msCl(U)$. This implies $f(G) \cap mCl(V) = \emptyset$, and by Lemma 3.8, $G(f)$ is strongly M-semiclosed. □

THEOREM 3.10. *Let $f : (X, m_X) \to (Y, m_Y)$ be a function on two spaces with minimal structures m_X and m_Y, respectively. If f is an weakly M-semicontinuous injection with a strongly M-semiclosed graph, then X is m-semi-T_2.*

Proof. Let x_1 and x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since the graph $G(f)$ is strongly M-semiclosed, there exist an m-semiopen set U containing x_1 and $V \in m_Y$ containing $f(x_2)$ such that $f(U) \cap mCl(V) = \emptyset$. Since f is weakly M-semicontinuous, for $V \in m_Y$ containing $f(x_2)$, there exists an m-semiopen set W containing x_2 in X such that $f(W) \subseteq msCl(V)$. Finally, we have $f(U) \cap f(W) = \emptyset$ because of $msCl(V) \subseteq mCl(V)$. Hence $U \cap W = \emptyset$, and X is m-semi-T_2. □

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