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IRROTATIONAL SCREEN HOMOTHETIC HALF LIGHTLIKE SUBMANIFOLDS

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Abstract. In this paper, we study the geometry of half lightlike submanifolds of a Lorentzian manifold. The main result is a characterization theorem for irrotational screen homothetic half lightlike submanifolds of a Lorentzian space form.

1. Introduction

It is well known that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of lightlike submanifolds M of a Lorentzian manifold $(\bar{M}, \bar{g})$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$. A codimension 2 lightlike submanifold M of rank$Rad(TM) = 1$ is called a half lightlike submanifold of $(\bar{M}, \bar{g})$ [3, 5, 6, 7, 8]. Then there exists a complementary non-degenerate distribution $S(TM)$ of $Rad(TM)$ in TM, called a *screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \tag{1.1}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Then there exist vector fields $\xi \in \Gamma(Rad(TM))$ and $u \in \Gamma(S(TM^{\perp}))$ such that

$$\bar{g}(u, u) = 1, \quad \bar{g}(\xi, v) = 0, \; \forall\, v \in \Gamma(TM^{\perp}),$$

where $S(TM^{\perp})$ is a supplementary distribution to $Rad(TM)$ in $TM^{\perp}$ of rank 1, called a *co-screen distribution* on M. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly ξ and u belong to $\Gamma(S(TM)^{\perp})$. Thus we have

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

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where $S(TM^{\perp})^{\perp}$ is a orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique vector field $N \in \Gamma(ltr(TM))$ satisfying

$$\bar{g}(\xi, N) = 1, \; \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, u) = 0, \tag{1.2}$$

for all $X \in \Gamma(S(TM))$. We call $ltr(TM)$, $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ and N the *lightlike transversal vector bundle, transversal vector bundle* and *lightlike transversal vector field* of M with respect to $S(TM)$ respectively. Therefore $T\bar{M}$ is decomposed as follows:

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM^{\perp}) \oplus_{orth} S(TM). \end{aligned} \tag{1.3}$$

The purpose of this paper is to study the geometry of half lightlike submanifolds of a Lorentzian manifold. We prove a characterization theorem for half lightlike submanifolds M of a Lorentzian space form $(\bar{M}(c), \bar{g})$, $c > 0$: If M is irrotational and screen homothetic, then (1) the induced connection ∇ on M is a metric one and (2) M is totally umbilical and locally a product manifold $M = L \times M^*$, where L is a lightlike curve and M^* is a totally geodesic Riemannian space form which is isometric to a sphere (Theorem 2.3).

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). Then the local Gauss and Weingartan formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)u, \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)u, \tag{1.5}$$

$$\bar{\nabla}_X u = -A_u X + \phi(X)N, \tag{1.6}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{1.7}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{1.8}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, B and D are called the local fundamental forms of M, C is called the local second fundamental form on $S(TM)$. A_N, A^*_ξ and A_u are linear operators on $\Gamma(TM)$ and τ, ρ and ϕ are 1-forms on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and both B and D are symmetric. From the facts that $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X, Y) = \bar{g}(\bar{\nabla}_X Y, u)$, we know that B and D are independent of the choice of a screen distribution and

$$B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X), \; \forall X \in \Gamma(TM). \tag{1.9}$$

The induced connection ∇ on M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y), \tag{1.10}$$

for all $X, Y, Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X, N), \ \forall\, X \in \Gamma(TM). \tag{1.11}$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms on TM and $S(TM)$ are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.12}$$

$$C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0, \tag{1.13}$$

$$D(X, PY) = g(A_u X, PY), \quad \bar{g}(A_u X, N) = \rho(X), \tag{1.14}$$

$$D(X, Y) = g(A_u X, Y) - \phi(X)\eta(Y). \tag{1.15}$$

From (1.12), A^*_ξ is a self-adjoint operator on $\Gamma(TM)$ satisfying

$$A^*_\xi \xi = 0. \tag{1.16}$$

We denote by $\bar{R}$, R and R^* the curvature tensors of $\bar{\nabla}$, ∇ and ∇^* respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z, PW) = g(R(X,Y)Z, PW) & \\ + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW) & \\ + D(X,Z)D(Y,PW) - D(Y,Z)D(X,PW), & \end{aligned} \tag{1.17}$$

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z, \xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) & \\ + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) & \\ + D(Y,Z)\phi(X) - D(X,Z)\phi(Y), & \end{aligned} \tag{1.18}$$

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z, N) = \bar{g}(R(X,Y)Z, N) & \\ + D(X,Z)\rho(Y) - D(Y,Z)\rho(X), & \end{aligned} \tag{1.19}$$

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z, u) = (\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) & \\ + B(Y,Z)\rho(X) - B(X,Z)\rho(Y), & \end{aligned} \tag{1.20}$$

$$\begin{aligned} \bar{g}(R(X,Y)PZ, PW) = g(R^*(X,Y)PZ, PW) & \\ + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), & \end{aligned} \tag{1.21}$$

$$\begin{aligned} g(R(X,Y)PZ, N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) & \\ + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X). & \end{aligned} \tag{1.22}$$

The induced Ricci type tensor $R^{(0,\,2)}$ of M is given by

$$R^{(0,\,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}, \quad \forall\, X, Y \in \Gamma(TM). \tag{1.23}$$

In general, $R^{(0,2)}$ is not symmetric. A tensor field $R^{(0,2)}$ is called its *induced Ricci tensor*, denoted by *Ric*, if it is symmetric.

2. Irrotational screen homothetic half lightlike submanifolds

DEFINITION 2.1. (a) A half lightlike submanifold $(M, g, S(TM))$ of a Lorentzian manifold $(\bar{M}, \bar{g})$ is called *screen homothetic* [1] if $A_N = b\, A^*_\xi$ for a non-zero constant b on M, or equivalently,

$$C(X, PY) = b\, B(X, Y) \quad \forall X,\, Y \in \Gamma(TM). \tag{2.1}$$

(b) M is *irrotational* [9] if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

NOTE 2.2. If M is screen homothetic, then C is symmetric on $S(TM)$. Therefore $S(TM)$ is integrable [4]. Thus M is locally a product manifold $L \times M^*$, where L is a lightlike curve and M^* is a leaf of $S(TM)$ [4]. If M is irrotational, then, from (1.4) and (1.9), we show that $D(X, \xi) = 0$ and $\phi(X) = 0$ for all $X \in \Gamma(TM)$.

THEOREM 2.3. *Let $(M, g, S(TM))$ be an irrotational screen homothetic half lightlike submanifold of $(\bar{M}(c), \bar{g})$. Then the induced Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor of M.*

Proof. Since $(\bar{M}(c), \bar{g})$ is a Lorentzian space form, we obtain [5, 6]

$$\begin{aligned} R^{(0,2)}(X, Y) &= mcg(X, Y) + B(X, Y)trA_N + D(X, Y)trA_u \\ &\quad - g(A_N X, A^*_\xi Y) - g(A_u X, A_u Y) + \rho(X)\phi(Y). \end{aligned}$$

From this and the facts $A_N = b\, A^*_\xi$ and $\phi = 0$, we have our assertion. □

THEOREM 2.4. [5] *Let $(M, g, S(TM))$ be a half lightlike submanifold of a Lorentzian manifold $(\bar{M}, \bar{g})$. If $R^{(0,2)}$ is symmetric, then there exists a pair $\{\xi, N\}$ on $\mathcal{U}$ such that the 1-form τ vanishes on M.*

We call a pair $\{\xi, N\}$ on $\mathcal{U}$ such that the 1-form τ vanishes on M the *distinguished null pair* of M. Although, in general, $S(TM)$ is not unique, it is isomorphic to the factor bundle $S(TM)^* = TM/Rad(TM)$ considered by Kupeli [9]. Thus all $S(TM)$ are mutually isomorphic. For this reason, in the sequel, let $(M, g, S(TM))$ be an irrotational screen homothetic half lightlike submanifoldof of a Lorentzian space form $(\bar{M}(c), \bar{g})$ equipped with the distinguished null pair $\{\xi, N\}$.

THEOREM 2.5. *Let $(M, g, S(TM))$ be an irrotational screen homothetic half lightlike submanifold of a Lorentzian space form $(\bar{M}(c), \bar{g})$. If $\dim M > 3$, then we have the following two assertions:*

(1) *The induced connection* ∇ *on* M *is a metric connection.*
(2) M *is totally umbilical and locally a product manifold* $L \times M^*$*, where* L *is a null curve and* M^* *is a totally geodesic Riemannian space form which is isometric to a sphere.*

Proof. Since $(\bar{M}(c), \bar{g})$ is a space of constant curvature c and M is an irrotational screen homothetic, from (1.18), we obtain

$$(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z) \tag{2.2}$$

for all $X, Y, Z \in \Gamma(TM)$. Using this, (1.19), (1.22) and (2.1), we get

$$\begin{aligned} &\rho(Y)D(X, PZ) - \rho(X)D(Y, PZ) \\ &= c\,\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}. \end{aligned} \tag{2.3}$$

Replace Y by ξ in (2.3) and use (1.9) with $\phi = 0$, we obtain

$$\rho(\xi)D(X, PZ) = -c\,g(X, PZ). \tag{2.4}$$

Since $c \neq 0$, we have $\rho(\xi) \neq 0$ and $D \neq 0$. From (1.9) and (2.4), we get

$$D(X, Y) = \alpha\, g(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM), \tag{2.5}$$

where $\alpha = -c\rho(\xi)^{-1} \neq 0$. While, from (1.20), we get

$$(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) + B(Y, Z)\rho(X) - B(X, Z)\rho(Y) = 0, \tag{2.6}$$

for all $X, Y, Z \in \Gamma(TM)$. From this, (1.10) and (2.5), we have

$$\begin{aligned} &X[\alpha]\, g(Y, Z) - Y[\alpha]\, g(X, Z) + B(X, Z)\{\alpha\eta(Y) - \rho(Y)\} \\ &\quad - B(Y, Z)\{\alpha\eta(X) - \rho(X)\} = 0, \ \forall\, X,\, Y,\, Z \in \Gamma(TM). \end{aligned} \tag{2.7}$$

Replace Y by ξ in (2.7) and use (1.9), we obtain

$$B(X, Y)\{\alpha - \rho(\xi)\} = \xi[\alpha]\, g(X, Y). \tag{2.8}$$

Since $c > 0$, we have $\alpha - \rho(\xi) \neq 0$. From (2.1) and (2.8), we have

$$B(X, Y) = \beta\, g(X, Y), \ C(X, Y) = b\beta\, g(X, Y), \tag{2.9}$$

for all $X, Y \in \Gamma(TM)$, where $\beta = \xi[\alpha](\alpha - \rho(\xi))^{-1}$. Thus M and $S(TM)$ are totally umbilical. Since M is screen homothetic, by Note 1, M is locally a product manifold $L \times M^*$ where L is a lightlike curve and M^* is a leaf of $S(TM)$. Since $\bar{M}$ is a space of constant curvature c, from (1.17), (1.21) and (2.9), we have

$$R^*(X, Y)Z = (c + \alpha^2 + 2b\beta^2)\{g(Y, Z)X - g(X, Z)Y\}, \tag{2.10}$$

for all $X, Y, Z \in \Gamma(S(TM))$. Let Ric^* be the induced symmetric Ricci tensor of M^*. From (2.10), we have

$$Ric^*(X, Y) = (m - 1)(c + \alpha^2 + 2b\beta^2)\, g(X, Y), \ \forall\, X,\, Y \in \Gamma(S(TM)).$$

Thus M^* is an Einstein manifold. Since M^* is a Riemannian manifold with dim $M^* > 2$, $(c + \alpha^2 + 2b\beta^2)$ is a constant and M^* is a space of constant curvature. Differentiating the first equation of (2.9) and using (1.10), (2.2) and the first equation of (2.9), we have

$$\{X[\beta] - \beta^2\eta(X)\}g(Y, Z) = \{Y[\beta] - \beta^2\eta(Y)\}g(X, Z).$$

Replace Y by ξ in this equation, we have $\xi[\beta] = \beta^2$. Since $(c+\alpha^2+2b\beta^2)$ is a constant, $0 = \xi[c+\alpha^2+2b\beta^2] = 2\beta(c+\alpha^2+2b\beta^2)$. Since $(c+\alpha^2+2b\beta^2)$ is a constant, we have $\beta = 0$ or $c+\alpha^2+2b\beta^2 = 0$. If $c+\alpha^2+2b\beta^2 = 0$, M^* is a Euclidean space and the second fundamental form C of M^* satisfies $C = 0$ [2]. From (2.9), we have $b\beta = 0$. Thus we have $c + \alpha^2 = 0$. This means $c = 0$ and $\alpha = 0$. It is a contradiction to $c > 0$ and $\alpha \neq 0$. Consequently, we have $\beta = 0$ and $c + \alpha^2 \neq 0$. Thus $B = C = 0$; $D \neq 0$ and M^* is a totally geodesic Riemannian space form of positive constant curvature $(c + \alpha^2)$ which is isometric to a sphere. Since $B(X, Y) = 0$ for all X, $Y \in \Gamma(TM)$, we show that the induced connection ∇ is metric due to (1.10). □

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