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A PARTICULAR SOLUTION OF THE EINSTEIN'S EQUATION IN EVEN-DIMENSIONAL UFT X_n

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Abstract. In the unified field theory(UFT), in order to find a solution of the Einstein's equation it is necessary and sufficient to study the torsion tensor. The main goal in the present paper is to obtain, using a given torsion tensor (3.1), the complete representation of a particular solution of the Einstein's equation in terms of the basic tensor $g_{\lambda\nu}$ in even-dimensional UFT X_n.

1. Introduction

Einstein ([1], 1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Characterizing Einstein's unified field theory as a set of geometrical postulates in a 4-dimensional generalized Riemannian space X_4 (i.e., space-time), Hlavatý ([9], 1957) gave the mathematical foundation of the 4-dimensional unified field theory(UFT X_4) defined by the unified field tensor $g_{\lambda\nu}$ for the first time. Generalizing X_4 to the n-dimensional generalized Riemannian manifold X_n, n-dimensional generalization of this theory, the so-called *Einstein's n-dimensional unified field theory*(UFT X_n), had been obtained by Mishra ([8], 1958). Since then many consequences of this theory has been obtained by a number of mathematicians. However, it has been unable yet to represent a general n-dimensional Einstein's connection in a surveyable tensorial form. The purpose of the present paper is to obtain a necessary and sufficient condition for the existence of a particular solution of Einstein's equation in even-dimensional UFT X_n. Next, under this condition, we shall obtain a precise tensorial representation of this solution in terms of the basic tensor $g_{\lambda\nu}$. The obtained results and

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discussions in the present paper will be useful for the even-dimensional considerations of the unified field theory.

2. Preliminary

This section is a brief collection of basic concepts, notations, and results, which are needed in our further considerations in the present paper.

Let X_n be an n-dimensional generalized Riemannian manifold covered by a system of real coordinate neighborhoods $\{U; x^\nu\}$, where, here and in the sequel, Greek indices run over the range $\{1, 2, \cdots, n\}$ and follow the summation convention. In the Einstein's usual n-dimensional unified field theory(UFT X_n), the algebraic structure on X_n is imposed by a basic real non-symmetric tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.1}$$

where we assume that

$$G = det(g_{\lambda\mu}) \neq 0, \quad H = det(h_{\lambda\mu}) \neq 0. \tag{2.2}$$

Since $det(h_{\lambda\mu}) \neq 0$, we may define a unique tensor $h^{\lambda\nu}(= h^{\nu\lambda})$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu. \tag{2.3}$$

We use the tensors $h^{\lambda\nu}$ and $h_{\lambda\mu}$ as tensors for raising and/or lowering indices for all tensors defined in UFT X_n in the usual manner. Then we may define new tensors by

$$k^\alpha{}_\mu = k_{\lambda\mu} h^{\lambda\alpha}, \quad k_\lambda{}^\alpha = k_{\lambda\mu} h^{\mu\alpha}. \tag{2.4}$$

In UFT X_n, the differential geometric structure is imposed by the tensor $g_{\lambda\mu}$ by means of a connection $\Gamma^\nu_{\lambda\mu}$ defined by the Einstein's equation:

$$\partial_\omega g_{\lambda\mu} - g_{\alpha\mu}\Gamma^\alpha_{\lambda\omega} - g_{\lambda\alpha}\Gamma^\alpha_{\omega\mu} = 0 \ \ (\partial_\nu = \frac{\partial}{\partial x^\nu}), \tag{2.5a}$$

or equivalently

$$D_\omega g_{\lambda\mu} = 2 S_{\omega\mu}{}^\alpha g_{\lambda\alpha}, \tag{2.5b}$$

where D_ω denotes the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$, and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma^\nu_{\lambda\mu}$.

In UFT X_n, the following quantities are frequently used, where $p = 1, 2, 3, \ldots$:

$$(2.6)\qquad \begin{aligned} &(a)\ \ g = \frac{G}{H}, \quad k = \frac{T}{H}, \\ &(b)\ \ K_0 = 1, \qquad K_p = k_{[\alpha_1}{}^{\alpha_1} k_{\alpha_2}{}^{\alpha_2} \ldots k_{\alpha_p]}{}^{\alpha_p}, \\ &(c)\ \ {}^{(0)}k_\lambda{}^\nu = \delta^\nu_\lambda, \quad {}^{(p)}k_\lambda{}^\nu = k_\lambda{}^{\alpha\,(p-1)}k_\alpha{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha\, k_\alpha{}^\nu, \\ &(d)\ \ \phi = {}^{(2)}k_\alpha{}^\alpha. \end{aligned}$$

It should be remarked that the tensor ${}^{(p)}k_{\lambda\nu}$ is symmetric if p is even, and skew-symmetric if p is odd.

An eigenvector Z^μ of $k_{\lambda\mu}$ which satisfies

$$(2.7)\qquad (Mh_{\lambda\mu} + k_{\lambda\mu})Z^\mu = 0,$$

where M is an arbitrary scalar, is called a basic vector of UFT X_n, and corresponding eigenvalue of $k_{\lambda\mu}$ basic scalar of UFT X_n. Furthermore the characteristic polynomial corresponding to $k_{\lambda\mu}$, that is,

$$(2.8)\qquad D(M) = Det(Mh_{\lambda\mu} + k_{\lambda\mu}),$$

will be termed basic polynomial of UFT X_n

Remark 2.1. From now on, we shall assume that

$$(2.9)\qquad T = det(k_{\lambda\mu}) \neq 0.$$

Hence there exists a unique skew-symmetric tensor $\overline{k}^{\lambda\mu}$ in X_n satisfying

$$(2.10)\qquad k_{\lambda\mu}\,\overline{k}^{\lambda\nu} = \delta^\nu_\mu.$$

Since $k_{\lambda\mu}$ is skew-symmetric, and $T \neq 0$, the dimension of X_n is even. That is, n is *even*. Hence *all our further considerations in the present paper are dealt in even-dimensional UFT X_n.*

It has been shown by Chung[4, 5, 6] that the following relations hold in UFT X_n.

$$(2.11)\qquad \begin{aligned} &(a)\ \ K_n = k, \quad K_p = 0\ (p\ is\ odd), \\ &(b)\ \ g = \sum_{s=0}^{n} K_s, \\ &(c)\ \ \sum_{s=0}^{n} K_s{}^{(n-s)}k_\lambda{}^\nu = 0. \end{aligned}$$

Here and in what follows, the index s is assumed to take the values 0, 2, 4, ... , n in the specified range.

It has been shown by Chung[5] that in UFT X_n, the basic polynomial (2.8) may be given by

$$\begin{aligned} D(M) &= H(M^n + M^{n-2}K_2 + ... + M^2 K_{m-2} + k) \\ &= H\sum_{p=0}^{n} M^{n-p}K_p, \end{aligned} \tag{2.12}$$

and hence M is a basic scalar in UFT X_n if and only if M satisfies

$$\sum_{p=0}^{n} M^{n-p}K_p = 0. \tag{2.13}$$

It has been shown by Lee[2] that in UFT X_n, the representation of the tensor $\overline{k}^{\lambda\mu}$, given by (2.10), may be given by

$$\overline{k}^{\lambda\mu} = \frac{1}{k}\sum_{s=0}^{n-2} K_s \;{}^{(n-s-1)}k^{\lambda\mu}. \tag{2.14}$$

3. A particular solution of the Einstein's equation

In this section, when a connection $\Gamma^{\nu}_{\lambda\mu}$ of the form

$$S_{\lambda\mu}{}^{\nu} = k_{\lambda\mu}Y^{\nu}, \tag{3.1}$$

for some nonzero vector Y^{ν}, is a solution of the Einstein's equation (2.5) in UFT X_n, we find its complete representation.

LEMMA 3.1. *When a connection $\Gamma^{\nu}_{\lambda\mu}$ of the form (3.1) is a solution of the Einstein's equation (2.5), (2.5) is equivalent to the following system of equations:*

$$\begin{aligned} &(a)\quad D_\omega h_{\lambda\mu} = 2\,k_{\omega(\mu}Y_{\lambda)} + 2\,k_{\omega(\mu}k_{\lambda)\alpha}Y^{\alpha}, \\ &(b)\quad D_\omega k_{\lambda\mu} = 2\,k_{\omega[\mu}Y_{\lambda]} + 2\,k_{\omega[\mu}k_{\lambda]\alpha}Y^{\alpha}, \end{aligned} \tag{3.2}$$

Proof. Substituting (2.1) and (3.1) into (2.5b), we obtain

$$D_\omega g_{\lambda\mu} = 2\,k_{\omega\mu}Y_{\lambda} + 2\,k_{\omega\mu}k_{\lambda\alpha}Y^{\alpha}. \tag{3.3}$$

The equations (3.2)(a) and (3.2)(b) follow from (3.3) and from

$$D_\omega h_{\lambda\mu} = D_\omega g_{(\lambda\mu)}, \qquad D_\omega k_{\lambda\mu} = D_\omega g_{[\lambda\mu]}.$$

Conversely, taking the sum of (3.2)(a) and (3.2)(b), we obtain (3.3). □

THEOREM 3.2. *When a connection $\Gamma^\nu_{\lambda\mu}$ of the form (3.1) is a solution of the Einstein's equation (2.5), (2.5) is equivalent to the following system of equations:*

$$\Gamma^\nu_{\lambda\mu} = \{{}_\lambda{}^\nu{}_\mu\} - 2k_{(\lambda}{}^\nu\, k_{\mu)\alpha} Y^\alpha + k_{\lambda\mu}\, Y^\nu, \tag{3.4}$$

$$\nabla_\nu\, k_{\lambda\mu} = -2k_{\nu[\lambda}\, Y_{\mu]} + 2\,{}^{(2)}k_{\nu[\lambda}\, k_{\mu]\alpha} Y^\alpha, \tag{3.5}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\{{}_\lambda{}^\nu{}_\mu\}$ defined by $h_{\lambda\mu}$.

Proof. From lemma 3.1, when a connection $\Gamma^\nu_{\lambda\mu}$ of the form (3.1) is a solution of the Einstein's equation (2.5), (2.5) is equivalent to the system of equations (3.2)(a) and (b). In virtue of relation

$$D_\omega h_{\lambda\mu} = \partial_\omega h_{\lambda\mu} - h_{\alpha\mu}\Gamma^\alpha_{\lambda\omega} - h_{\lambda\alpha}\Gamma^\alpha_{\mu\omega}, \tag{3.6}$$

and (3.1), we obtain

$$\begin{aligned} &\frac{1}{2}h^{\nu\alpha}(D_\lambda h_{\alpha\mu} + D_\mu h_{\alpha\lambda} - D_\alpha h_{\lambda\mu}) \\ &= \{{}_\lambda{}^\nu{}_\mu\} - 2S^\nu{}_{(\lambda\mu)} + S_{\lambda\mu}{}^\nu - \Gamma^\nu_{\lambda\mu} \\ &= \{{}_\lambda{}^\nu{}_\mu\} - 2\,k^\nu{}_{(\lambda} Y_{\mu)} + k_{\lambda\mu} Y^\nu - \Gamma^\nu_{\lambda\mu}. \end{aligned} \tag{3.7}$$

On the other hand, it follows from (3.2)(a) that

$$\begin{aligned} &\frac{1}{2}h^{\nu\alpha}(D_\lambda h_{\alpha\mu} + D_\mu h_{\alpha\lambda} - D_\alpha h_{\lambda\mu}) \\ &= 2\,k_{(\lambda}{}^\nu Y_{\mu)} - 2\,k_{(\lambda}{}^\nu k_{\mu)\alpha} Y\alpha. \end{aligned} \tag{3.8}$$

Comparing (3.7) with (3.8), we obtain (3.4). On the other hand, substituting (3.4) into

$$D_\nu k_{\lambda\mu} = \partial_\nu k_{\lambda\mu} + 2\,k_{\alpha\mu}\Gamma^\alpha_{\lambda\nu} - k_{\lambda\alpha}\Gamma^\alpha_{\mu\nu},$$

we obtain

$$D_\nu k_{\lambda\mu} = \nabla_\nu k_{\lambda\mu} - 2\,{}^{(2)}k_{\nu[\mu} k_{\lambda]\alpha} Y^\alpha + 2\,k_{\nu[\mu} k_{\lambda]\alpha} Y^\alpha. \tag{3.9}$$

Comparing (3.2)(b) with (3.9), we obtain (3.5). Conversely, suppose that (3.4) and (3.5) hold. Substituting (3.4) into (3.6), we obtain (3.2)(a). Similarly, substituting (3.5) into (3.9), we obtain (3.2)(b). □

REMARK 3.3. In virtue of Theorem 3.2, it is obvious that if the Einstein's equation (2.5) admits a particular solution $\Gamma^\nu_{\lambda\mu}$ of the form (3.1), it must be of the form (3.4). This reduces the investigation of the particular solution (3.4) to the study of the vector Y^ν defining (3.4).

4. The representation of a particular solution (3.4) of the Einstein's equation

From Remark(3.3), in order to know the particular solution (3.4) of the Einstein's equation it is necessary and sufficient to know the vector Y^ν defining (3.4) and satisfying (3.5), which is the main goal of this section. Our investigation is based on the skew-symmetric tensor

$$P_{\lambda\mu} = (1-\phi)k_{\lambda\mu} + {}^{(3)}k_{\lambda\mu}, \tag{4.1}$$

where ϕ is given by (2.6)(d). And the following quantities are used in our further considerations. For $s = 2, 4, ..., n+2$,

$$\Omega_0 = 0, \qquad \Omega_s = (\phi-1)\Omega_{s-2} + K_{s-2}. \tag{4.2}$$

A direct calculation shows that

$$\begin{aligned} \Omega_{n+2} &= (\phi-1)^{\frac{n}{2}} K_0 + (\phi-1)^{\frac{n-2}{2}} K_2 + (\phi-1)^{\frac{n-4}{2}} K_4 + \\ &\quad ... + (\phi-1)K_{n-2} + K_n \\ &= \sum_{p=0}^{n} \{\sqrt{\phi-1}\}^{n-p} K_p \end{aligned} \tag{4.3}$$

Lemma 4.1. *The determinant of the tensor $P_{\lambda\mu}$, given by (4.1), never vanishes, i.e.,*

$$det(P_{\lambda\mu}) \neq 0, \tag{4.6}$$

if and only if

$$Det(\sqrt{\phi-1}\, h_{\lambda\mu} + k_{\lambda\mu}) \neq 0. \tag{4.7}$$

Proof. The tensor $P_{\lambda\mu}$ can be rewritten as

$$P_{\lambda\mu} = -k_{\lambda\nu} h^{\nu\alpha}(\sqrt{\phi-1}\, h_{\beta\alpha} + k_{\beta\alpha}) h^{\beta\gamma}(\sqrt{\phi-1}\, h_{\gamma\mu} + k_{\gamma\mu}). \tag{4.8}$$

Since the determinant of a product of matrices is the product of the determinants of the matrices, and $det(A^{-1}) = 1/det(A)$ we obtain, in virtue of (2.2), (2.3) and (2.9),

$$\begin{aligned} & det(P_{\lambda\mu}) \\ &= -T(\frac{1}{H})\{Det(\sqrt{\phi-1}\, h_{\lambda\mu} + k_{\lambda\mu})\}(\frac{1}{H})\{Det(\sqrt{\phi-1}\, h_{\lambda\mu} + k_{\lambda\mu})\} \\ &= -\frac{T}{H^2}\{Det(\sqrt{\phi-1}\, h_{\lambda\mu} + k_{\lambda\mu})\}^2, \end{aligned} \tag{4.9}$$

which proves this lemma. □

Lemma 4.2. *The determinant of the tensor $P_{\lambda\mu}$ never vanishes if and only if*

$$\Omega_{n+2} \neq 0, \tag{4.10}$$

if and only if the scalar $\sqrt{\phi - 1}$ is not a basic scalar in UFT X_n.

Proof. In virtue of (2.8), (2.12), and (4.3), we obtain

$$Det(\sqrt{\phi - 1}\, h_{\lambda\mu} + k_{\lambda\mu}) = H \sum_{p=0}^{n} \{\sqrt{\phi - 1}\}^{n-p} K_p = H\Omega_{n+2}. \tag{4.11}$$

Hence from the above relation (4.11) and Lemma 4.1 we obtain, in virtue of (2.2), (2.13) and (4.3), that $det(P_{\lambda\mu}) \neq 0$, iff $\Omega_{n+2} \neq 0$, iff the scalar $\sqrt{\phi - 1}$ is not a basic scalar. □

Remark 4.3. In our further considerations in the present paper, we assume that the scalar $\sqrt{\phi - 1}$ is not a basic scalar in UFT X_n, that is $\Omega_{n+2} \neq 0$. Therefore $det(P_{\lambda\mu}) \neq 0$. For the lower-dimensional cases $n = 2, 4$, we obtain the following Table 1, in virtue of (2.6)(b) and (d), (2.11)(a) and (b), and (4.2). According to this Table 1, this assumption is automatically satisfied for the case $n = 2$.

Table 1. For $n = 2, 4$, the representations of g and Ω_{n+2}

n	g	Ω_{n+2}
2	$g = 1 + k$	$\Omega_4 = -k - 1 = -g \neq 0$
4	$g = 1 - \frac{1}{2}\phi + k$	$\Omega_6 = 2(g - k)^2 - (g - k) + k$

Remark 4.4. From Remak 4.3, since $det(P_{\lambda\mu}) \neq 0$, there exists a unique skew-symmetric tensor $Q^{\lambda\nu}$ satisfying

$$P_{\lambda\mu}\, Q^{\lambda\nu} = \delta_\mu^\nu. \tag{4.12}$$

In our further considerations in the present paper, we use the following useful abbreviations for any tensor $Z_{\lambda\nu}$, for $p, q = 1, 2, 3, ...$

$$^{(p)}Z_{\lambda\mu} = {}^{(p-1)}k_\lambda{}^\nu\, Z_{\nu\mu}. \tag{4.13}$$

We then have

$$^{(1)}Z_{\lambda\mu} = Z_{\lambda\mu}, \qquad {}^{(p)}k_\lambda{}^{\nu}\, {}^{(q)}Z_{\nu\mu} = {}^{(p+q)}Z_{\lambda\mu}. \tag{4.14}$$

Lemma 4.5. *The following recurrence relations in holds:*

$$(4.15)\qquad \begin{aligned} &(a)\ \ {}^{(4)}Q^{\mu\nu} = (\phi-1)\,{}^{(2)}Q^{\mu\nu} - h^{\mu\nu},\\ &(b)\ \ {}^{(3)}Q^{\omega\nu} = (\phi-1)\,Q^{\omega\nu} + \overline{k}^{\omega\nu},\\ &(c)\ \ {}^{(p)}Q^{\omega\nu} = (\phi-1)\,{}^{(p-2)}Q^{\omega\nu} - {}^{(p-4)}k^{\omega\nu} \quad (p=4,5,...). \end{aligned}$$

Proof. Substituting (4.1) into (4.12), we obtain (4.15)(a) in virtue of (4.13). Multiplying $\overline{k}^{\mu\omega}$ to both sides of (4.15)(a), we obtain (4.15)(b) in virtue of (2.10) and (4.13). Multiplying ${}^{(p-4)}k^{\omega}{}_{\mu}$ to both sides of (4.15)(a), we obtain the relation (4.15)(c) in virtue of (4.13). □

Theorem 4.6. *The representation of the tensor $Q^{\lambda\mu}$ in UFT X_n, given by (4.12), may be given by*

$$(4.16)\qquad Q^{\lambda\mu} = \frac{1}{\Omega_{n+2}} \sum_{s=0}^{n-2} \Omega_{s+2}\ {}^{(n-s-3)}k^{\lambda\mu},$$

where Ω_{n+2} is given by (4.3), and

$$(4.17)\qquad {}^{(-1)}k^{\lambda\mu} = -\overline{k}^{\lambda\mu}.$$

Proof. Multiplying $Q^{\nu\mu}$ to both sides of (2.11)(c), and using (4.13), we obtain

$$(4.18)\qquad \begin{aligned} &\sum_{s=0}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\mu}\\ &= K_0\ {}^{(n+1)}Q^{\lambda\mu} + K_2\ {}^{(n-1)}Q^{\lambda\mu} + \sum_{s=4}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\nu} = 0. \end{aligned}$$

Substituting ${}^{(n+1)}Q^{\lambda\mu}$ from (4.15)(c) into the first term of (4.18), and using (2.6)(b) and (4.2), we obtain

$$(4.19)\qquad \begin{aligned} &-\,{}^{(n-3)}k^{\lambda\mu} + \{(\phi-1)+K_2\}\,{}^{(n-1)}Q^{\lambda\mu} + \sum_{s=4}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\nu}\\ &= -\,{}^{(n-3)}k^{\lambda\mu} + \Omega_4\,{}^{(n-1)}Q^{\lambda\mu} + K_4\,{}^{(n-3)}Q^{\lambda\mu} + \sum_{s=6}^{n} K_s\ {}^{(n-s+1)}Q^{\lambda\nu}\\ &= 0. \end{aligned}$$

Substituting again ${}^{(n-1)}Q_{\lambda\mu}$ from (4.15)(c) into (4.19), and using (4.2), we obtain

$$
\begin{aligned}
(4.20)\qquad & -\,{}^{(n-3)}k^{\lambda\mu} - \Omega_4\;{}^{(n-5)}k^{\lambda\mu} + \{(\phi-1)\Omega_4 + K_4\}\,{}^{(n-3)}Q^{\lambda\mu} \\
& + \sum_{s=6}^{n} K_s\;{}^{(n-s+1)}Q^{\lambda\mu} \\
& = -\,{}^{(n-3)}k^{\lambda\mu} - \Omega_4\;{}^{(n-5)}k^{\lambda\mu} + \Omega_6\,{}^{(n-3)}Q^{\lambda\mu} + K_6\,{}^{(n-5)}Q^{\lambda\mu} \\
& + \sum_{s=8}^{n} K_s\;{}^{(n-s+1)}Q^{\lambda\mu} \\
& = 0.
\end{aligned}
$$

After $(n-2)/2$ steps of successive repeat substituting for ${}^{(p)}Q^{\lambda\mu}$ from (4.15)(c), we obtain

$$
(4.21)\qquad -\sum_{s=0}^{n-4} \Omega_{s+2}\;{}^{(n-s-3)}k^{\lambda\mu} + \Omega_n\;{}^{(3)}Q^{\lambda\mu} + K_n\,Q^{\lambda\mu} = 0,
$$

in virtue of (4.2). Substituting (4.15)(b) into (4.21), and using (4.15)(b) and (4.17), we obtain

$$
\begin{aligned}
(4.22)\qquad & -\sum_{s=0}^{n-4} \Omega_{s+2}\;{}^{(n-s-3)}k^{\lambda\mu} + \Omega_n\,\overline{k}^{\lambda\mu} + \{(\phi-1)\Omega_n + K_n\}\,Q^{\lambda\mu} \\
& = -\sum_{s=0}^{n-2} \Omega_{s+2}\;{}^{(n-s-3)}k^{\lambda\mu} + \Omega_{n+2}\,Q^{\lambda\mu} = 0,
\end{aligned}
$$

which is condensed to (4.16). □

TABLE 2. For $n = 2, 4$, the representations of $\overline{k}^{\lambda\mu}$ and $Q^{\lambda\mu}$

n	$\overline{k}^{\lambda\mu}$	$Q^{\lambda\mu}$
2	$\frac{1}{k}k^{\lambda\mu}$	$\frac{1}{g}\overline{k}^{\lambda\mu}$
4	$\frac{1}{k}({}^{(3)}k^{\lambda\mu} + (g-k-1)k^{\lambda\mu})$	$\frac{k^{\lambda\mu} + (g-k)\overline{k}^{\lambda\mu}}{2(g-k)^2 - (g-k) + k}$

REMARK 4.7. As useful results of Theorem 4.7, for the lower-dimensional cases $n = 2, 4$, we obtain the following Table 2, in virtue of (2.6)(b) and (d), (2.14), and Table 1.

THEOREM 4.8. *A necessary and sufficient condition for the Einstein's equation (2.5) to admit exactly one particular solution $\Gamma^\nu_{\lambda\mu}$ of the form (3.1) is that the basic tensor $g_{\lambda\mu}$ satisfies the following condition:*

$$\nabla_\nu k_{\lambda\mu} = -2(k_{\nu[\lambda} h_{\mu]\alpha} - {}^{(2)}k_{\nu[\lambda} k_{\mu]\alpha})Q^{\gamma\alpha}\nabla_\beta k_\gamma{}^\beta, \tag{4.23}$$

where $Q^{\lambda\mu}$ is given by (4.16). If this condition is satisfied, then the vector Y^ν which defines the particular solution is given by

$$Y^\alpha = Q^{\lambda\alpha}\nabla_\beta k_\lambda{}^\beta. \tag{4.24}$$

Proof. If the Einstein's equation (2.5) admits a solution of the form (3.1), then the condition (3.5) holds in virtue of Theorem 3.2. The condition (3.5) is equivalent to

$$\nabla_\nu k_\lambda{}^\mu = -k_{\nu\lambda}Y^\mu + k_\nu{}^\mu Y_\lambda + {}^{(2)}k_{\nu\lambda}k^\mu{}_\alpha Y^\alpha - {}^{(2)}k_\nu{}^\mu k_{\lambda\alpha}Y^\alpha. \tag{4.25}$$

Contracting for ν and μ in (4.25), we obtain

$$\nabla_\beta k_\lambda{}^\beta = \{(1-\phi)k_{\lambda\beta} + {}^{(3)}k_{\lambda\beta}\}Y^\alpha = P_{\lambda\beta}Y^\beta. \tag{4.26}$$

Multiplying $Q^{\lambda\alpha}$ on both sides of (4.26) and making use of (4.12), we obtain (4.24). Substituting (4.24) into (3.5), we obtain (4.23). Conversely, suppose that the condition (4.23) holds. With the vector Y^ν given by (4.24), define a connection $\Gamma^\nu_{\lambda\mu}$ by (3.4), and substitute this connection into (2.5). This connection satisfies (2.5) in virtue of our assumption (4.23). Hence it is a solution of the Einstein's equation (2.5). Assume now that the Einstein's equation (2.5) has another solution ${}^*\Gamma^\nu_{\lambda\mu}$ of the form

$$(4,27) \qquad {}^*S_{\lambda\mu}{}^\nu = k_{\lambda\mu}{}^*Y^\nu,$$

$${}^*Y^\nu \neq Y^\nu. \tag{4.28}$$

Then in virtue of the proof of Theorem 3.2, the vector ${}^*Y^\nu$ must satisfy

$$\nabla_\nu k_{\lambda\mu} = -2k_{\nu[\lambda}{}^*Y_{\mu]} + 2{}^{(2)}k_{\nu[\lambda}k_{\mu]\alpha}Y^\alpha, \tag{4.29}$$

Applying the same method used to derive (4.24), we have from (4.29)

$${}^*Y^\alpha = Q^{\lambda\alpha}\nabla_\beta k_\lambda{}^\beta = Y^\alpha,$$

which contradicts to the assumption (4.28). This proves the uniqueness of the solution of the form (3.1) under condition (4.23). □

Since we have obtained the representation of the tensor $Q_{\lambda\nu}$ in terms of the basic tensor $g_{\lambda\mu}$, it is possible for us to represent the solution $\Gamma^\nu_{\lambda\mu}$ of The Einstein's equation, of the form (3.1), in terms of $g_{\lambda\mu}$ by simply substituting (4.24) into (3.4). Without proof, we can state the following theorem.

THEOREM 4.9. *Under the condition (4.23), when a connection $\Gamma^{\nu}_{\lambda\mu}$ of the form (3.1) is a solution of the Einstein's equation (2.5), the complete representation of the solution in terms of the basic tensor $g_{\lambda\mu}$ may be given by*

(4.30)

$$\Gamma^{\nu}_{\lambda\mu} = \{\lambda{}^{\nu}{}\mu\} - \frac{1}{\Omega_{n+2}}\sum_{s=0}^{n-2}\Omega_{s+2}(2k_{(\lambda}{}^{\nu}k_{\mu)\alpha} - k_{\lambda\mu}\delta^{\nu}_{\alpha})\,{}^{(n-s-3)}k^{\gamma\alpha}\nabla_{\beta}k_{\gamma}{}^{\beta}.$$

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