JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 1, March 2010

THE M_α-INTEGRAL

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ABSTRACT. In this paper, we define the M_α-integral and investigate properties of the M_α-integral.

1. Introduction and preliminaries

It is well-known [12] that a function $f : [a, b] \to R$ is C-integrable on $[a, b]$ if and only if there exists an ACG_c function F such that $F' = f$ almost everywhere on $[a, b]$.

In this paper, we define the M_α-integral and prove that a function $f : [a, b] \to R$ is M_α-integrable on $[a, b]$ if and only if there exists an ACG_α function F such that $F' = f$ almost everywhere on $[a, b]$.

Throughout this paper, $I_0 = [a, b]$ is a compact interval in R. Let D be a finite collection of interval-point pairs $\{(I_i, \xi_i)\}_{i=1}^n$, where $\{I_i\}_{i=1}^n$ are non-overlapping subintervals of I_0 and let δ be a positive function on I_0, i.e. $\delta : I_0 \to R^+$. We say that $D = \{(I_i, \xi_i)\}_{i=1}^n$ is

(1) a partial tagged partition of I_0 if $\cup_{i=1}^n I_i \subset I_0$,

(2) a tagged partition of I_0 if $\cup_{i=1}^n I_i = I_0$,

(3) a δ-fine McShane partition of I_0 if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in I_o$ for all $i = 1, 2, ..., n$,

(4) a δ-fine M_α-partition of I_0 for a constant $\alpha > 0$ if it is a δ-fine McShane partition of I_0 and satisfying the

$$\sum_{i=1}^{n} dist(\xi_i, I_i) < \alpha,$$

where $\text{dist}(\xi_i, I_i) = inf\{|t - \xi_i| : t \in I_i\}$,

Received November 10, 2009; Accepted November 25, 2009.
2000 Mathematics Subject Classification: Primary 26A39.
Key words and phrases: M_α-partition, M_α-integral, ACG_α function.
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(5) a δ-fine Henstock partition of I_0 if $\xi_i \in I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, 2, ..., n$.

Given a $\delta-$fine partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ we write

$$S(f, D) = \sum_{i=1}^{n} f(\xi_i)|I_i|$$

for integral sums over D, whenever $f : I_0 \to R$.

2. Properties of the M_α-integral

DEFINITION 2.1. Let $\alpha > 0$ be a constant. A function $f : I_0 \to R$ is M_α-integrable if there exists a real number A such that for each $\epsilon > 0$ there is a positive function $\delta : I_0 \to R^+$ such that

$$|S(f, D) - A| < \epsilon$$

for each $\delta-$fine M_α-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0. A is called the M_α-integral of f on I_0. and we write $A = \int_{I_0} f$ or $A = (M_\alpha) \int_{I_0} f$.

The function f is M_α-integrable on the set $E \subset I_0$ if the function $f\chi_E$ is M_α-integrable on I_0, and we write $\int_E f = \int_{I_0} f\chi_E$.

THEOREM 2.2. *A function $f : I_0 \to R$ is M_α-integrable if and only if for each $\epsilon > 0$ there is a positive function $\delta : I_0 \to R^+$ such that*

$$|S(f, D_1) - S(f, D_2)| < \epsilon$$

for any δ-fine M_α-partitions D_1 and D_2 of I_0.

Proof. Assume that $f : I_0 \to R$ is M_α-integrable on I_0. For each $\epsilon > 0$ there is a positive function $\delta : I_0 \to R^+$ such that

$$|S(f, D) - \int_{I_0} f| < \frac{\epsilon}{2}$$

for each δ-fine M_α-partition D of I_0. If D_1 and D_2 are δ-fine M_α-partitions, then

$$\begin{aligned} |S(f, D_1) - S(f, D_2)| &\leq |S(f, D_1) - \int_{I_0} f| + |\int_{I_0} f - S(f, D_2)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Conversely, assume that for each $\epsilon > 0$, there is a positive function $\delta : I_0 \to R^+$ such that $|S(f, D_m) - S(f, D_k)| < \epsilon$ for any δ-fine M_α-partitions D_m, D_k of I_0. For each $n \in N$, choose $\delta_n : I_0 \to R^+$ such that $|S(f,D_1)-S(f,D_2)|<\frac{1}{n}$ for any δ_n-fine M_α-partitions D_1 and D_2 of I_0. Assume that $\{\delta_n\}$ is decreasing. For each $n\in N$, let D_n be a δ_n-fine M_α-partition of I_0. Then $\{S(f,D_n)\}$ is a Cauchy sequence. Let $L=lim_{n\to\infty}S(f,D_n)$ and let $\epsilon>0$. Choose N such that $\frac{1}{N}<\frac{\epsilon}{2}$ and $|S(f,D_n)-L|<\frac{\epsilon}{2}$ for all $n\geq N$. Let D be a δ_N-fine M_α-partition of I_0. Then

$$\begin{aligned}|S(f,D)-L|&\leq|S(f,D)-S(f,D_N)|+|S(f,D_N)-L|\\&<\frac{1}{N}+\frac{\epsilon}{2}<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Hence f is M_α-integrable on I_0, and $\int_{I_0}f=L$. □

We can easily get the following theorems.

THEOREM 2.3. *Let $f:I_0\to R$. Then*

(1) If f is M_α-integrable on I_0, then f is M_α-integrable on every subinterval of I_0.

(2) If f is M_α-integrable on each of the intervals I_1 and I_2, where I_1 and I_2 are non-overlapping and $I_1\cup I_2=I_0$, then f is M_α-integrable on I_0 and $\int_{I_1}f+\int_{I_2}f=\int_{I_0}f$.

THEOREM 2.4. *Let f and g be M_α-integrable functions on I_0. Then*
(1) kf is M_α-integrable on I_0 and $\int_{I_0}kf=k\int_{I_0}f$ for each $k\in R$,
(2) $f+g$ is M_α-integrable on I_0 and $\int_{I_0}(f+g)=\int_{I_0}f+\int_{I_0}g$.

LEMMA 2.5. (Saks-Henstock Lemma) *Let $f:I_0\to R$ be M_α-integrable on I_0. Let $\epsilon>0$. Suppose that δ is a positive function on I_0 such that*

$$|S(f,D)-\int_{I_0}f|<\epsilon$$

for each δ-fine M_α-partition $D=\{(I,\xi)\}$ of I_0. If $D'=\{(I_i,\xi_i)\}_{i=1}^m$ is a δ-fine partial M_α-partition of I_0, then

$$|S(f,D')-\sum_{i=1}^m\int_{I_i}f(\xi_i)|\leq\epsilon.$$

Proof. Assume that $D'=\{(I_i,\xi_i)\}_{i=1}^m$ is an arbitrary δ-fine partial M_α-partition of I_0. Let $\overline{I_0-\cup_{i=1}^m I_i}=\cup_{j=1}^k I'_j$

Let $\eta>0$. Since f is M_α-integrable on each I'_j, there exists a positive function $\delta_j:I'_j\to R^+$ such that

$$|S(f,D_j)-\int_{I'_j}f|<\frac{\eta}{k}.$$

for each δ_j-fine M_α-partition of I'_j.

Assume that $\delta_j(\xi) \leq \delta(\xi)$ for all $\xi \in D_0$. Let $D_0 = D' + D_1 + D_2 + ... + D_k$. Then D_0 is a δ-fine M_α-partition of I_0 and we have

$$|S(f, D_0) - \int_{I_0} f| = |S(f, D') + \sum_{j=1}^{k} S(f, D_j) - \int_{I_0} f| < \epsilon.$$

Consequently, we obtain

$$\begin{aligned}
&|S(f, D') - \sum_{i=1}^{m} \int_{I_i} f| \\
&= |S(f, D_0) - \sum_{j=1}^{k} S(f, D_j) - (\int_{I_0} f - \sum_{j=1}^{k} \int_{I'_j} f)| \\
&\leq |S(f, D_0) - \int_{I_0} f| + \sum_{j=1}^{k} |S(f, D_j) - \int_{I'_j} f| \\
&< \epsilon + \frac{k\eta}{k} = \epsilon + \eta.
\end{aligned}$$

Since $\eta > 0$ was arbitrary, we have $|S(f, D') - \sum_{i=1}^{m} \int_{I_i} f| \leq \epsilon$. □

Now we recall the definition of the derivative of a function.

DEFINITION 2.6. *A function $F : I_0 \to R$ is differentiable at $\xi \in I_0$ if*

$$\lim_{\mu \to 0} \frac{F(\xi + \mu) - F(\xi)}{\mu}$$

exists. The limit in case it exists, is called the derivative of F at ξ, and is denoted by $F'(\xi)$.

THEOREM 2.7. *If the function $F : I_0 \to R$ is differentiable on I_0 with $f(\xi) = F'(\xi)$ for each $\xi \in I_0$, then $f : I_0 \to R$ is M_α-integrable.*

Proof. By the definition of derivative, for each $\xi \in I_0$ there is a positive function $\delta : I_0 \to R^+$ such that

$$\Big|\frac{F(\zeta) - F(\xi)}{\zeta - \xi} - f(\xi)\Big| < \frac{\alpha}{2(\epsilon + |I_0|)}$$

for all $\zeta \in I_0$ with $|\zeta - \xi| < \delta(\xi)$. Assume that $D = \{(I_i, \xi_i)\}_{i=1}^n$ is a δ-fine M_α-partition of I_0. Then we have

$$\begin{aligned} \Big|\sum_{i=1}^n [f(\xi_i)|I_i| - F(I_i)]\Big| &\le \sum_{i=1}^n |f(\xi_i)|I_i| - F(I_i)| \\ &< \frac{\alpha}{\epsilon + |I_0|}\sum_{i=1}^n (dist(\xi_i, I_i) + |I_i|) \\ &< \frac{\alpha}{\epsilon + |I_0|}(\alpha + |I_0|) = \epsilon \end{aligned}$$

Hence $f : I_0 \to R$ is M_α-integrable on I_0. □

Definition 2.8. *Let $\alpha > 0$ be a constant. Let $F : I_0 \to R$ and let E be a subset of I_0.*

(a) F is said to be AC_α on E if for each $\epsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta : I_0 \to R^+$ such that $|\sum_i F(I_i)| < \epsilon$ for each δ-fine partial M_α-partition $D = \{(I_i, \xi_i)\}$ of I_0 satisfying $\xi_i \in E$ and $\sum_i |I_i| < \eta$.

(b) F is said to be ACG_α on E if E can be expressed as a countable union of sets on each of which F is AC_α.

By considering positive and negative parts, it is clear that there is no change if the part $|\sum_i F(I_i)| < \epsilon$ of the above definition is written as $\sum_i |F(I_i)| < \epsilon$.

Theorem 2.9. *If a function $f : I_0 \to R$ is M_α-integrable on I_0 with the primitive F, then F is ACG_α on I_0.*

Proof. By the definition of M_α-integral and the Saks-Henstock Lemma, for each $\epsilon > 0$ there is a positive function $\delta : I_0 \to R^+$ such that

$$\Big|\sum_{i=1}^n [f(\xi_i)|I_i| - F(I_i)]\Big| \le \epsilon$$

for each δ-fine partial M_α-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0.

Assume that $E_n = \{\xi \in I_0 : n - 1 \le |f(\xi)| < n\}$ for each $n \in \mathbb{N}$. Then we have $I_0 = \cup E_n$. To show that F is AC_α on each E_n, fix n and take a δ-fine partial M_α-partition $D_0 = \{(I_i, \xi_i)\}$ of I_0 with $\xi_i \in E_n$ for all i. If $\sum_i |I_i| < \frac{\epsilon}{n}$, then

$$\begin{aligned}\Big|\sum_i F(I_i)\Big| &\le \Big|\sum_i [F(I_i) - f(\xi_i)\cdot|I_i|]\Big| + \Big|\sum_i f(\xi_i)|I_i|\Big| \\ &\le \Big|\sum_i [F(I_i) - f(\xi_i)|I_i|]\Big| + \sum_i |f(\xi_i)|\cdot|I_i| \\ &\le \epsilon + n\sum_i |I_i| < 2\epsilon.\end{aligned}$$

□

Now we recall the definitions of the McShane and Henstock integrals.

A function $f : I_0 \to R$ is McShane integrable if there exists a real number A such that for each $\epsilon > 0$ there is a positive function $\delta : I_0 \to R^+$ such that

$$|S(f, D) - A| < \epsilon$$

for each δ-fine McShane partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0.

A function $f : I_0 \to R$ is Henstock integrable if there exists a real number A such that for each $\epsilon > 0$ there is a positive function $\delta : I_0 \to R^+$ such that

$$|S(f, D) - A| < \epsilon$$

for each δ-fine Henstock partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0.

Since every Henstock partition is an M_α-partition and every M_α-partition is a McShane partition, we get the following theorem.

THEOREM 2.10. *Let $f : I_0 \to R$ be a function.*
(a) If f is McShane integrable on I_0, then f is M_α-integrable on I_0.
(b) If f is M_α-integrable on I_0, then f is Henstock integrable on I_0.

A function $f : I_0 \to R$ is M_α-integrable on I_0 if and only if there exists on ACG_α function F on I_0 such that $F' = f$ almost everywhere on I_0. To prove this , we need the following two lemmas.

LEMMA 2.11. *Suppose that $f : [a, b] \to R$ and let $E \subseteq [a, b]$. If $\mu(E) = 0$, then for each $\epsilon > 0$ there exists a positive function δ on E such that $S(|f|, D) < \epsilon$ for every δ-fine partial M_α-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of $[a, b]$ with $\xi_i \in E$.*

Proof. For each n, let $E_n = \{x \in E : n - 1 \le |f(x)| < n\}$ and let $\epsilon > 0$. Then $E = \cup E_n$. Since $\mu(E_n) = 0$ for each n, we can choose an open set $O_n \supset E_n$ with $\mu(O_n) < \frac{\epsilon}{n \cdot 2^n}$.

Define $\delta(x) = \rho(x, O_n{}^c)$ for $x \in E_n$. Suppose that D is a δ-fine partial M_α-partition of $[a, b]$. Let D_n be a subset of D that has tags in E_n and let $\pi = \{n \in \mathbb{Z}^+ : D_n \neq \phi\}$. Then

$$\begin{aligned} S(|f|, D) &= \sum_{n\in\pi} S(|f|, D_n) \leq \sum_{n\in\pi} n \cdot |I_i| \\ &< \sum_{n\in\pi} n\mu(O_n) < \sum_{n\in\pi} n \cdot \frac{\epsilon}{n \cdot 2^n} = \epsilon. \end{aligned}$$

□

LEMMA 2.12. *Suppose that $F : I_0 \to R$ is ACG_α on I_0 and let $E \subseteq I_0$. If $\mu(E) = 0$, then for each $\epsilon > 0$ there exists a positive function δ on E such that $\sum_{i=1}^n |F(I_i)| < \epsilon$ for every δ-fine partial M_α-partition $D = \{(I_i, \xi_i)\}_{i=1}^n$ of I_0 with $\xi_i \in E$ for all $i = 1, 2, ...n$.*

Proof. Let $E = \cup_{n=1}^\infty E_n$ where F is AC_α on each E_n. Let $\epsilon > 0$. For each n, there exists a positive function $\delta_n : E_n \to R^+$ and a positive number $\eta_n > 0$ such that $\Sigma_{i=1}^n |F(I_i)| < \frac{\epsilon}{2^n}$ for each δ_n-fine partial M_α-partition of I_0 with $\xi_n \in E_n$ and $\sum_{i=1}^n |I_i| < \eta_n$. For each n, choose an open set $O_n \supset E_n$ and $\mu(O_n) < \eta_n$. Define $\delta(x) = min\{\delta_n(x), \rho(x, O_n{}^c)\}$ for $x \in E_n$. Suppose that $D = \{(I_i, \xi_i)\}$ is a δ-fine partial M_α-partition of I_0 with $\xi_i \in E$. Let D_n be subset of D that has tags in E_n and note that $(D_n) \sum_{i=1}^n |I_i| < \mu(O_n) < \eta_n$. Hence,

$$\sum_{i=1}^n |F(I_i)| \leq \sum_n (D_n) \sum_{i=1}^n |F(I_i)| < \sum_n \frac{\epsilon}{2^n} = \epsilon.$$

□

THEOREM 2.13. *If a function $f : I_0 \to R$ is M_α-integrable on I_0 if and only if there is an ACG_α function F on I_0 such that $F' = f$ almost everywhere on I_0.*

Proof. Suppose that f is M_α-integrable on I_0 and let $F(x) = \int_a^x f$ for each $x \in I_0$. Then by Theorem 2.9, F is ACG_α on I_0. Since f is Henstock integrable on I_0, $F' = f$ almost everywhere on I_0 by [8, Theorem 9.12].

Conversely, suppose that there is an ACG_α function F such that $F = f'$ almost everywhere on I_0. Let $E = \{x \in I_0 : F'(x) \neq f(x)\}$ and let $\epsilon > 0$. Then $\mu(E) = 0$. For each $x \in I_0 - E$, choose $\delta(x) > 0$ such that

$$|F(y) - F(x) - f(x)(y - x)| < \frac{\epsilon}{6(\alpha + |I_0|)} |y - x|$$

whenever $|y - x| < \delta(x)$ and $y \in I_0$. By Lemma 2.11 and 2.12, we can find $\delta(x) > 0$ on E such that $|\sum f(\xi)|I_i|| < \frac{\epsilon}{3}$ and $|\sum F(I_i)| < \frac{\epsilon}{3}$, whenever $D = \{(I_i, \xi_i)\}$ is a δ-fine M_α-partial partition of I_0 with $\xi_i \in E$.

Suppose that $D = \{(I_i, \xi_i)\}$ is a δ-fine M_α-partial partition of I_0. Let D_1 be the subset of D that has tags in E and let $D_2 = D - D_1$ then

$$\begin{aligned}
&\left|(D)\sum f(\xi)|I_i| - (D)\sum F(I_i)\right| \\
&= \left|(D_2)\sum f(\xi)|I_i| - (D_2)\sum F(I_i)\right| + \left|(D_1)\sum f(\xi)|I_i|\right| \\
&\qquad\qquad + \left|(D_1)\sum F(I_i)\right| \\
&\le (D_2)\sum |f(\xi)|I_i| - F(I_i)| + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\le \frac{\epsilon}{3(\alpha + |I_0|)}\sum (dist(\xi_i, I_i) + |I_i|) + \frac{2}{3}\epsilon \\
&\le \frac{\epsilon}{3(\alpha + |I_0|)}(\alpha + |I_0|) + \frac{2}{3}\epsilon \\
&= \frac{\epsilon}{3} + \frac{2}{3}\epsilon = \epsilon.
\end{aligned}$$

Hence f is M_α-integrable on I_0. □

The following examples show that the converse of Theorem 2.10 is not true.

Example 2.14. (1) Let f be a function defined by

$$f(x) = \begin{cases} 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2} & \text{if} \quad 0 < x \le 1, \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Then it is easy to show that the primitive of f is

$$F(x) = \begin{cases} x^2\sin\frac{1}{x^2} & \text{if} \quad 0 < x \le 1, \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Since $F(x)$ is differentiable and $F'(x) = f(x)$ everywhere on $[0,1]$, $f(x)$ is M_α-integrable from Theorem 2.7. But $F(x)$ is not absolutely continuous on $[0,1]$ and therefore $f(x)$ is not McShane integrable on $[0,1]$.

(2) The function F defined by

$$F(x) = \begin{cases} x\sin\frac{1}{x^2} & \text{if} \quad 0 < x \le 1, \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is differentiable nearly everywhere on $[0,1]$. By [8,Theorem 9.6], F' is Henstock integrable on $[0,1]$. But we can show that F is not ACG_α on $[0,1]$. To show this, suppose that F is ACG_α. Then there exists a set $E \subset [0,1]$ such that $0 \in E$ and F is AC_α on E.

For $\epsilon = \frac{\alpha}{2}$, there exist a positive function $\delta : [0,1] \to R^+$ and a positive number $\eta > 0$ such that $|\sum_{i=1}^n F(I_i)| < \frac{\alpha}{2}$, whenever $D = \{(I_i, x_i)\}_{i=1}^n$ is a δ-fine partial M_α-partition of $[0,1]$ with $x_i \in E$ and $\sum_{i=1}^n |I_i| < \eta$.

Let $a_n = \frac{1}{\sqrt{(2n+\frac{1}{2})\pi}}$ and $b_n = \frac{1}{\sqrt{2n\pi}}$ for each positive integer n. Then $a_n < b_n < 1$ and $\sum_{n=1}^\infty a_n = \infty$. Choose a δ-fine partial partition $D = \{([a_i, b_i], 0) : N \le i \le M\}$ such that $\frac{\alpha}{2} < \sum_{i=N}^M a_i < \alpha$ and $b_N < min\{\delta(0), \eta\}$. Then $0 \in E$, $\sum_{i=N}^M (b_i - a_i) < \eta$, and $\sum_{i=N}^M dist(0, [a_i, b_i]) = \sum_{i=N}^M a_i < \alpha$.

Hence, D is a δ-fine M_α-partial partition of $[0,1]$. But we have

$$\Big|\sum_{i=N}^M F([a_i, b_i])\Big| = \Big|\sum_{i=N}^M \Big(F(b_i) - F(a_i)\Big)\Big| = \sum_{i=N}^M a_i > \frac{\alpha}{2}.$$

This contradiction shows that F is not ACG_α on $[0,1]$.

Hence, F' is not M_α-integrable on $[0,1]$.

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