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A CONVERGENCE THEOREM ON QUASI–ϕ–NONEXPANSIVE MAPPINGS

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Abstract. In an infinite-dimensional Hilbert space, the normal Mann iteration has only weak convergence, in general, even for nonexpansive mappings. The purpose of this paper is to modify the normal Mann iteration to have strong convergence for a closed quasi-ϕ-nonexpansive mapping in the framework of Banach spaces.

1. Introduction

Let E be a real Banach space, let C be a nonempty subset of E and let $T : C \to C$ be a nonlinear mapping. Recall that T is *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Recall that the normal Mann's iterative process was introduced by Mann [6] in 1953. Since then, construction of fixed points for nonexpansive mappings via the normal Mann's iterative process has been extensively investigated by many authors. The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1, \tag{1.1}$$

where the sequence $\{\alpha_n\}$ is in the interval $(0, 1)$.

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If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.1) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [13]). It is well known that, in an infinite-dimensional Hilbert space, the normal Mann's iterative algorithm has only weak convergence, in general, even for nonexpansive mappings.

Attempts to modify the normal Mann iteration method (1.1) for nonexpansive mappings by hybrid algorithms so that strong convergence is guaranteed have recently been made (see [7-10] and the references therein).

Nakajo and Takahashi [8] proposed the following modification of the Mann iteration for a single nonexpansive mapping T in a Hilbert space. To be more precise, they proved the following theorem:

THEOREM NT. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\alpha_n \le 1-\delta$ for some $\delta \in (0,1]$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$(1.2) \qquad \begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0. \end{cases}$$

Then $\{x_n\}$ converges in norm to $P_{F(T)}x_0$.

Recently, Matsushita and Takahashi [7] improved the results of Nakajo and Xu [9] from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem:

THEOREM MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that*

$\{x_n\}$ is given by

$$(1.3) \qquad \begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad \forall n \geq 0, \end{cases}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

The purpose of this paper is to introduce a modified hybrid algorithm which is different from Matsushita and Takahashi [7] to modify normal Mann iteration to have strong convergence for closed quasi-ϕ-nonexpansive mappings in the framework of Banach spaces.

2. Preliminaries

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single-valued and if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided $\lim\limits_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

In a smooth Banach space, consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x-y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y\in C} \phi(y, x) \tag{2.2}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see [1, 2, 5, 9, 15]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{2.3}$$

Remark 2.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [5, 15]) for more details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an *asymptotic fixed point* of T [14] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be *relatively nonexpansive* [3, 4, 7] if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [3, 4].

T is said to be *ϕ-nonexpansive* [11, 12], if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$. T is said to be *quasi-ϕ-nonexpansive* [11, 12] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 2.2. The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings which requires the strong restriction: $F(T) = \widetilde{F(T)}$.

Next, we give two examples which are closed quasi-ϕ-nonexpansive.

Example 2.3. Let E be a uniformly smooth and strictly convex Banach space and let $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then $J_r = (J + rA)^{-1}$ is a closed quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.4. Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

We need the following lemmas for the proof of our main results.

Lemma 2.5. (Kamimura and Takahashi [9]) *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.6. (Alber [2]) *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.7. (Alber [2]) *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

LEMMA 2.8. (Qin et al. [12]) *Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E and let T be a closed quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

3. Main results

Now, we are ready to give our main results.

THEOREM 3.1. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping with a fixed point. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$(3.1) \qquad \begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}[\alpha_n J x_n + (1-\alpha_n) J T x_n], \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0. \end{cases}$$

If the control sequence $\{\alpha_n\}$ satisfies the restrictions: $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

Proof. First, we show that C_n is closed and convex for all $n \ge 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some k. For $z \in C_k$, we obtain that $\phi(z, y_k) \le \phi(z, x_k)$ is equivalent to

$$2\langle z, Jx_k - Jy_k\rangle \le \|x_k\|^2 - \|y_k\|^2.$$

It is easy to see that C_{k+1} is closed and convex. Then, for all $n \ge 1$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}} x_0$ is well defined. Next, we prove $F(T) \subset C_n$ for all $n \ge 1$. $F(T) \subset C_1 = C$ is obvious. Suppose that

$F(T) \subset C_k$ for some k. Then, for all $w \in F(T) \subset C_k$, we have

$$\begin{aligned}\phi(w, y_k) &= \phi(w, J^{-1}[\alpha_k Jx_k + (1-\alpha_k)JTx_k]) \\ &= \|w\|^2 - 2\langle w, \alpha_k Jx_k + (1-\alpha_k)JTx_k\rangle \\ &\quad + \|\alpha_k Jx_k + (1-\alpha_k)JTx_k\|^2 \\ &\le \|w\|^2 - 2\alpha_k\langle w, Jx_k\rangle + 2(1-\alpha_k)\langle w, JTx_k\rangle \\ &\quad + \alpha_k\|x_k\|^2 + (1-\alpha_k)\|Tx_k\|^2 \\ &= \alpha_k\phi(w, x_k) + (1-\alpha_k)\phi(w, Tx_k) \\ &\le \alpha_k\phi(w, x_k) + (1-\alpha_k)\phi(w, x_k) \\ &= \phi(w, x_k),\end{aligned}$$

which shows $w \in C_{k+1}$. This implies that $F(T) \subset C_n$ for all $n \ge 1$. From $x_n = \Pi_{C_n} x_0$, we see that

$$\langle x_0 - x_n, Jx_n - Jz\rangle \ge 0, \quad \forall z \in C_n. \tag{3.2}$$

Since $F(T) \subset C_n$ for all $n \ge 1$, we arrive at

$$\langle x_0 - x_n, Jx_n - Jw\rangle \ge 0, \quad \forall w \in F(T). \tag{3.3}$$

From Lemma 2.7, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(w, x_0) - \phi(w, x_n) \le \phi(w, x_0)$$

for each $w \in F(T) \subset C_n$ and for all $n \ge 1$. Therefore, the sequence $\phi(x_n, x)$ is bounded. On the other hand, noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0)$$

for all $n \ge 0$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = P_{C_m} x_0 \in C_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\le \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0).\end{aligned} \tag{3.4}$$

Letting $m, n \to \infty$ in (3.4), one has $\phi(x_n, x_m) \to 0$. It follows from Lemma 2.5 that $x_n - x_m \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \to p \in C$ as $n \to \infty$.

Next, we show that $p = \Pi_{F(T)} x_0$. To end this, we first show that $p \in F(T)$. By taking $m = 1$ in (3.4), we arrive at

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.5}$$

From Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Noticing that $x_{n+1} \in C_{n+1}$, we obtain that

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n).$$

It follows from (3.5) that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

From Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.7}$$

Notice that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

It follows from (3.6) and (3.7) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.8}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.9}$$

On the other hand, from the definition of y_n, we have

$$\|Jy_n - Jx_n\| = (1 - \alpha_n)\|JTx_n - Jx_n\|.$$

By the assumption on $\{\alpha_n\}$ and (3.9), one sees that $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

From the closedness of T, we obtain that $p = Tp$.

Finally, we show that $p = \Pi_{F(T)} x_0$. From $x_n = \Pi_{C_n} x_0$, we have

$$\langle Jx_0 - Jx_n, x_n - w \rangle \geq 0, \quad \forall w \in F(T) \subset C_n. \tag{3.10}$$

Taking the limit as $n \to \infty$ in (3.10), we obtain that

$$\langle Jx_0 - Jp, p - w \rangle \geq 0, \quad \forall w \in F(T)$$

and hence $p = \Pi_{F(T)} x_0$ by Lemma 2.6. This completes the proof. □

Remark 3.2. From computational point of view, the algorithm studied in Theorem 3.1 is more convenient than the one given by Matsushita and Takahashi [7]. The mapping in Theorem 3.1 is also more general than relatively nonexpansive mappings which require the strong restriction $\widetilde{F(T)} = F(T)$. In the framework of Hilbert space, Theorem 3.1 is reduced to Theorem 3.3 of Takahashi, Takeuchi and Kubota [16].

References

[1] Ya. I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operatorequations in Banach spaces,* Panamer. Math. J. **4** (1994), 39–54.

[2] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces,* in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996.

[3] D. Butnariu, S. Reich and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces,* J. Appl. Anal. **7** (2001), 151–174.

[4] Y. Censor and S. Reich, *Iterationsof paracontractionsand firmly nonexpansive operatorswith applications to feasibility and optimization,* Optim. **37** (1996), 323–339.

[5] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems,* Kluwer, Dordrecht, 1990.

[6] W. R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc. **4** (1953), 506–510.

[7] S. Y. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space,* J. Approx. Theory **134** (2005), 257–266.

[8] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups,* J. Math. Anal. Appl. **279** (2003), 372–379.

[9] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space,* SIAM J. Optim. **13** (2002), 938–945.

[10] X. Qin and Y. Su, *Strong convergence theorems for relatively nonexpansive mappings in a Banach space,* Nonlinear Anal. **67** (2007), 1958–1965.

[11] X. Qin, Y. J. Cho, S. M. Kang and H. Zhou, *Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings,* Appl. Math. Lett. **22** (2009), 1051–1055.

[12] X. Qin, Y. J. Cho and S. M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces,* J. Comput. Appl. Math. **225** (2009), 20–30.

[13] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces,* J. Math. Anal. Appl. **67** (1979), 274–276.

[14] S. Reich, *A weak convergence theorem for the alternating method with Bregman distance,* in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996.

[15] W. Takahashi, *Nonlinear Functional Analysis,* Yokohama Publishers, 2000.

[16] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl. **341** (2008), 276–286.

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