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TRANSITIVE SETS WITH DOMINATED SPLITTING

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Abstract. Let Λ be a transitive set for f. In this paper, we show that if a f-invariant set Λ has the C^1-stably shadowing property, then Λ admits a dominated splitting.

1. Introduction

It has been a main subject in differentiable dynamical systems during last decades to understand the influence of a robust dynamic property on the behavior of the tangent map of the system. Also, the notion of pseudo-orbits often appears in the several branches of modern theory of dynamical systems, and shadowing property usually plays an important role in the investigation of stability theory and ergodic theory as well as expansivity. In [4], the authors proved that C^1-generically, if f has the C^1-stably weak shadowing property on a chain transitive set then it admits a dominated splitting. And [5], M. Lee proved that if f has the C^1-stably average shadowing property on a transitive set then it admits a dominated splitting.

Let M be a closed C^∞ manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM.

For $\delta > 0$, a sequence of points $\{x_i\}_{i\in\mathbb{Z}}$ is called a *δ-pseudo-orbit* of $f \in \mathrm{Diff}(M)$ if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Let $\Lambda \subset M$ be a closed and f-invariant set. We say that $f|_\Lambda$ has the *shadowing property* if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, there is $y \in M$ such that $d(f^n(y), x_n) < \epsilon$ for all $n \in \mathbb{Z}$. We say that f has the *shadowing property* if $M = \Lambda$ in the above definition. Note

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that f has the shadowing property if and only if f^n has the shadowing property for $n \in \mathbb{Z} \setminus \{0\}$.

We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that

$$\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda(U).$$

We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exists constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

Definition 1.1. We say that f has the *C^1-stably shadowing property* on Λ if there exists a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that:

- Λ is locally maximal,
- for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g(U)}$ has the shadowing property, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ and which is called the *continuation* of Λ.

Definition 1.2. Let Λ be a closed f-invariant set. A splitting $T_\Lambda M = E \oplus F$ is called a l-dominated splitting for a positive integer l if E and F are Df-invariant and

$$\|Df^l|_{E(x)}\| / m(Df^l|_{F(x)}) \leq \frac{1}{2},$$

for all $x \in \Lambda$, where $m(A) = \inf\{\|Av\| : \|v\| = 1\}$ denotes the mininorm of a linear map A.

We say that Λ is called a *transitive set* for f if there exists a point $x \in \Lambda$ such that $\omega(x) = \Lambda$.

Theorem A. *Let Λ be a transitive set. If f has the C^1-stably shadowing property on Λ, then Λ admits a dominated splitting.*

2. Preliminary known-results

Let M be as before, and let $f \in \text{Diff}(M)$.

Lemma 2.1. [3] *Let $\mathcal{U}(f)$ be any given C^1-neighborhood of f. Then there exists $\epsilon > 0$ and a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \cdots, x_N\}$, a neighborhood U*

of $\{x_1, x_2, \cdots, x_N\}$ *and linear maps* $L_i : T_{x_i}M \to T_{g(x_i)}M$ *satisfying* $\|L_i - D_{x_i}g\| \le \epsilon$ *for all* $1 \le i \le N$, *there exists* $\widehat{g} \in \mathcal{U}(f)$ *such that* $\widehat{g}(x) = g(x)$ *if* $x \in \{x_1, x_2, \cdots, x_N\} \cup (M \setminus U)$ *and* $D_{x_i}\widehat{g} = L_i$ *for all* $1 \le i \le N$.

We use Mãné's result which is on a uniformly family of periodic sequences of linear maps of $\mathbb{R}^n (n = \dim M)$. Let $GL(n)$ be the group of linear isomorphisms of $\mathbb{R}^n$. If a sequence $\xi : \mathbb{Z} \to GL(n)$ is *periodic* if there is $k > 0$ such that $\xi_{j+k} = \xi_j$ for $k \in \mathbb{Z}$. We call a finite subset $\mathcal{A} = \{\xi_i : 0 \le i \le k-1\} \subset GL(n)$ is a *periodic family* with period k. For a periodic family $\mathcal{A} = \{\xi_i : 0 \le i \le n-1\}$, we denote $\mathcal{C}_\mathcal{A} = \xi_{n-1} \circ \xi_{n-2} \circ \cdots \circ \xi_0$.

Definition 2.2. We say that the periodic family $\mathcal{A} = \{\xi_i : 0 \le i \le n-1\}$ admits a l-dominated splitting, if there is a splitting $\mathbb{R}^n = E \oplus F$ which satisfies:

(a) E and F are $\mathcal{C}_\mathcal{A}$ invariant, i.e., $\mathcal{C}_\mathcal{A}(E) = E$ and $\mathcal{C}_\mathcal{A}(F) = F$,
(b) For any $k = 0, 1, 2, \cdots$,

$$\frac{\|\xi_{k+l-1} \circ \cdots \circ \xi_{k+1} \circ \xi_k|_{E_k}\|}{m(\xi_{k+l-1} \circ \cdots \circ \xi_{k+1} \circ \xi_k|_{F_k})} \le \frac{1}{2},$$

where $E_k = \xi_{k-1} \circ \xi_{k-2} \circ \cdots \circ \xi_0(E)$ and $F_k = \xi_{k-1} \circ \xi_{k-2} \circ \cdots \circ \xi_0(F)$.

We know following theorems for periodic family from [2].

Theorem 2.3. *Given any* $\epsilon > 0$ *and* $K > 0$, *there is* $n_1 \ge 0$ *which satisfies the following property: Given any periodic family* $\mathcal{A} = \{\xi_i : 0 \le i \le n-1\}$ *which satisfies the period* $n \ge n_1$ *and* $\max\{\|\xi_i\|, \|\xi_i^{-1}\|\} \le K$, *for all* $i = 0, 1, \cdots, n$, *one can find a periodic family* $\mathcal{B} = \{\zeta_i : 0 \le n-1\}$ *such that* $\max\{\|\zeta_i - \xi_i\|, \|\zeta_i^{-1} - \xi_i^{-1}\|\} < \epsilon$, *for any* $i = 0, 1, \cdots, n-1$, *and* $\det(\mathcal{C}_\mathcal{A}) = \det(\mathcal{C}_\mathcal{B})$ *and the eigenvalues of* $\mathcal{C}_\mathcal{B}$ *are all real, multiplicity one and different moduli.*

Theorem 2.4. *Given any* $\epsilon > 0$ *and* $K > 0$, *there is positive integers* $n_2 \ge 0$ *and* $l \ge 0$ *which satisfies the following property: Given any periodic family* $\mathcal{A} = \{\xi_i : 0 \le i \le n-1\}$ *which satisfies the period* $n \ge n_2$ *and* $\max\{\|\xi_i\|, \|\xi_i^{-1}\|\} \le K$, *for all* $i = 0, 1, \cdots, n-1$, *if* $\mathcal{A}$ *does not admits any* l*-dominated splitting, then one can find a periodic family* $\mathcal{B} = \{\zeta_0, \zeta_1, \cdots, \zeta_{n-1}\}$ *such that* $\max\{\|\zeta_i - \xi_i\|, \|\zeta_i^{-1} - \xi_i^{-1}\|\} < \epsilon$, *for any* $i = 0, 1, \cdots, n-1$, *and* $\det(\mathcal{C}_\mathcal{A}) = \det(\mathcal{C}_\mathcal{B})$ *and the eigenvalues of* $\mathcal{C}_\mathcal{B}$ *are all real, and have same modulus.*

To prove Theorem A, we need another lemma about uniformly contracting family. Let $\mathcal{A} = \{\xi_i : 0 \le i \le k-1\} \subset GL(n)$ be a periodic family. We say the sequence $\mathcal{A}$ is *uniformly contracting family* if there is a constant $\delta > 0$ such that for any δ-perturbation of $\mathcal{A}$ are sink,.i.e, for any $\mathcal{B} = \{\zeta_i : 0 \le i \le k-1\}$ with $\|\zeta_i - \xi_i\| < \delta$, all eigenvalue of $\mathcal{C}_{\mathcal{B}}$ have moduli less than 1. Similarly, we can define the uniformly expanding periodic family.

3. Proof of Theorem A

In this section, we will use the notation of *pre-sink (pre-source)*. A periodic point p is called a *pre-sink (pre-source)* if $Df^{\pi(p)}(p)$ has an multiplicity one eigenvalue equal to $+1$ or -1 and the other eigenvalues has norm less than 1(bigger than 1).

PROPOSITION 3.1. *Let Λ be a transitive set for f. Then if f has the C^1-stably shadowing property on Λ, then we can choose natural numbers N and l such that for any $n > N$, P_n admits a l-dominated splitting.*

Proposition 3.1 can be obtained by Lemmas 3.2 and 3.5. Let Λ be a closed f-invariant set. Suppose f has the C^1-stably shadowing property on Λ. There exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ.

LEMMA 3.2. *Let Λ be a transitive set of $f \in \mathrm{Diff}(M)$, and let $\mathcal{U}(f)$ and U as in above. If f has the C^1 stably shadowing property on Λ, then for any $g \in \mathcal{U}(f)$, g has neither pre-sink nor pre-sources with the orbit staying in U.*

Proof. Suppose that f has the C^1-stably shadowing property on Λ. Then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g has the shadowing property on $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Assume that there is $g \in \mathcal{U}(f)$ such that g has a pre-sink p with $\mathcal{O}(p) \subset U$. For simplicity, we may assume p is fixed point of g(other case is similar).

By making use of the Lemma 2.1, we linearize g at p with respect to the exponential coordinates $\exp_p$, i.e, choose $\epsilon_1 > 0$ and $\alpha > 0$ with $B_\alpha(p) \subset U$ and there exists g_1 C^1-$\epsilon_1$nearby g such that

$$g_1(x) = \begin{cases} \exp_p \circ D_pg(p) \circ \exp_p^{-1}(x) & \text{if } x \in B_\alpha(p); \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

Then $g_1(p) = g(p) = p$.

Since p is pre-sink of g, D_pg has a multiplicity one eigenvalue such that $|\lambda| = 1$ and other eigenvalues of D_pg are with modulus less than 1.

Denote by E_p^c the eigenspace corresponding to λ, and E_p^s the eigenspace corresponding to the eigenvalues with modulus less than 1. Thus $T_pM = E_p^c \oplus E_p^s$.

By Theorem 2.3, and Theorem 2.4, we consider $\lambda \in \mathbb{R}$. Then $\dim E_p^c = 1$. For simplicity, we suppose that $\lambda = 1$. Since the eigenvalue $\lambda = 1$, there is a small arc $\mathcal{I}_p \subset B_\alpha(p) \cap \exp_p(E_p^c(\alpha))$ center at p such that $g_1|_{\mathcal{I}_p} = id$, where id is the identity map. Here $E_p^c(\alpha)$ is the α-ball in E_p^c center at the origin O_p. Clearly, $\mathcal{I}_p \subset \Lambda_{g_1}(U)$.

Since the above perturbation g_1 is C^1-close to f, g_1 is in $\mathcal{U}(f)$. Therefore g_1 has the shadowing property on $\Lambda_{g_1}(U)$. Let $0 < \varepsilon < \frac{1}{8}\mathrm{diam}(\mathcal{I}_p)$ be arbitrary. Then there exists $0 < \delta < \frac{\varepsilon}{4}$ such that every δ-pseudo-orbit ξ of g_1 in $\Lambda_{g_1}(U)$ is $\frac{1}{2}\varepsilon$-shadowed by some point in M.

Consider a δ-pseudo-orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ of g_1 as follows. Take the least natural number k such that $3\varepsilon < k\delta$. Fix k distinct points $p_1, p_2, \cdots, p_k$ in $\mathcal{I}_p$ such that

1. $d(p_i, p_{i+1}) < \delta$ for $i = 1, \cdots, k-1$,
2. $p_1 = p$ and $d(p_1, p_k) > 3\varepsilon$.

Define $\xi = \{x_i\}_{i\in\mathbb{Z}}$ by $x_{ki+j} = p_j$ for $i \in \mathbb{Z}$ and $j = 1, \cdots, k-1$.

Since g_1 has the shadowing property on $\Lambda_{g_1}(U)$, $g_1|_{\mathcal{I}_p}$ has the shadowing property. Therefore we can find a point in M shadowing the pseudo-orbit ξ for g_i. And the shadowing point is in $\mathcal{I}_p$ or in $M \setminus \mathcal{I}_p$. Note that the identity map on small arc does not have the shadowing property. In case that the shadowing point is in $\mathcal{I}_p$, it is easily checked that the $g_1|_{\mathcal{I}_p}$ does not have the shadowing property.

So the shadowing point should be contained in $M \setminus \mathcal{I}_p$. Since p is pre-sink, $g_1^k(z) \to \mathcal{I}_p$, as $k \to \infty$, for any $z \in M \setminus \mathcal{I}_p$. Let $z \in M \setminus \mathcal{I}_p$ be a shadowing point of the pseudo-orbit ξ of g_1. Then we can choose a point $w \in \mathcal{I}_p$ and $n > 0$ such that $g^{n_1}(z) \in B_{\frac{\delta}{2}}(w)$ for $n_1 \geq n$. Since the δ-pseudo-orbit ξ of g_1 is a subset of $\mathcal{I}_p$, we can choose $m > 0$ such that $d(g^m(z), x_i) \geq \frac{1}{2}\varepsilon$ for some $x_i \in \xi$. Thus g_1 does not have the shadowing property on $\Lambda_{g_1}(U)$. This is a contradiction. □

Remark 3.3. Let Λ be a transitive set. There exist a sequence $\{g_n\}_{n\in N}$ of diffeomorphism and a periodic orbit P_n of g_n with period $\pi(P_n) \to \infty$ as $n \to \infty$ such that $g_n \to f$ in the C^1-topology and $\lim_H P_n = \Lambda$, where $\lim_H$ is Hausdorff limit and $\pi(P_n)$ is period of P_n.

By Remark 3.3, we can choose $p_n \in P_n$ such that we get a periodic family $\mathcal{A}_n = \{D_{p_n}f, D_{f(p_n)}f, \cdots, D_{f^{\pi(p_n)-1}(p_n)}f\}$.

LEMMA 3.4. [4] *Let* Λ, P_n *be as in Remark 3.3, and* $\mathcal{A}_n$ *be given in above. Then for any* $\epsilon > 0$ *there exists a* $n_0(\epsilon) > 0$ *such that for any* $n > n_0(\epsilon)$, $\mathcal{A}_n$ *is neither* ϵ*-uniformly contracting nor* ϵ*-uniformly expanding.*

Let $\mathcal{U}_0(f)$ be given by Lemma 3.2, and let $g \in \mathcal{U}_1(f)$. We consider a periodic family of linear maps $\mathcal{A} = \{D_p g : \text{for any } p \in P(g) \cap \Lambda_g(U)\}$. Let $\mathcal{B} = \{\xi_p : \text{ for any } p \in P(g) \cap \Lambda_g(U)\}$ be a family of periodic sequence of linear maps close to $\mathcal{A}$, and for any $p \in P(g) \cap \Lambda_g(U)$, consider the linear map

$$\mathcal{C}_{\mathcal{B}} = \xi_{g^{\pi(p)-1}(p)} \circ \cdots \circ \xi_p,$$

and denote by $\lambda_s(\mathcal{C}_{\mathcal{B}})$, $\lambda_u(\mathcal{C}_{\mathcal{B}})$ its eigenvalues. And $\xi_{g^i(p)}$ is a linear map nearby $D_{g^i(p)}g$ for $0 \le i \le \pi(p) - 1$ and $|\lambda_s(\mathcal{C}_{\mathcal{B}})| \le |\lambda_u(\mathcal{C}_{\mathcal{B}})|$.

LEMMA 3.5. [4] *Let* P_n, Λ *be as in Remark 3.3. Then for any* $\epsilon > 0$ *there are* $n(\epsilon), l(\epsilon) > 0$ *such that for any* $n > n(\epsilon)$ *if* P_n *does not admits a* $l(\epsilon)$ *dominated splitting, then we can choose* g C^1*-nearby* f *and preserving the orbit of* P_n *such that* P_n *is pre-sink or pre-source respecting* g.

PROPOSITION 3.6. [1] *Let* g_n *convergent to* f *and if* Λ_{g_n} *be a closed* g_n*-invariant set of* g_n *and* $\lim \Lambda_{g_n} = \Lambda$. *Then if* Λ_{g_n} *admits a* l*-dominated splitting respecting* g_n, *then* Λ *admits a* l*-dominated splitting respecting* f.

End of the proof of Theorem A: Suppose that f has the C^1-stably shadowing property on transitive set Λ. Then by Proposition 3.1, we can choose $N, l > 0$ such that for any $n > N$, P_n admits a dominated splitting. And by Remark 3.3 and Proposition 3.6, Λ admits a l-dominated splitting.

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