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THE $\alpha\psi$-CLOSURE AND THE $\alpha\psi$-KERNEL VIA $\alpha\psi$-OPEN SETS

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Abstract. In this paper, we introduce the concept of *weakly ultra-$\alpha\psi$-separation* of two sets in a topological space using $\alpha\psi$-*open* sets. The $\alpha\psi$-*closure* and the $\alpha\psi$-*kernel* are defined in terms of this *weakly ultra-$\alpha\psi$-separation*. We also investigate some of the properties of the $\alpha\psi$-*kernel* and the $\alpha\psi$-*closure*.

1. Introduction

The notion of $\alpha\psi$-*closed* set was introduced and studied by R. Devi et al.[2]. In this paper, we define that a set A is *weakly ultra-$\alpha\psi$-separated* from B if there exists an $\alpha\psi$-*open* set G containing A such that $G \cap B = \phi$. Using this concept, we define the $\alpha\psi$-*closure* and the $\alpha\psi$-*kernel*. Also we define the $\alpha\psi$-*derived* set and the $\alpha\psi$-*shell* of a set A of a topological space (X, τ).

Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned. Let A be a subset of a space X. The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

2. Preliminaries

Before entering to our work, we recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a space (X, τ) is called

1. a *semi-open* set [4] if $A \subseteq cl(int(A))$ and a *semi-closed* set if $int(cl(A)) \subseteq A$ and

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2. an *α-open* set [5] if $A \subseteq int(cl(int(A)))$ and an *α-closed* set if $cl(int(cl(A))) \subseteq A$.

The *semi*-closure (resp. α-closure) of a subset A of a space (X, τ) is the intersection of all *semi-closed* (resp. *α-closed*) sets that contain A and is denoted by $scl(A)$ (resp. $\alpha cl(A)$).

Definition 2.2. A subset A of a topological space (X, τ) is called a

1. a *semi-generalized closed* (briefly *sg-closed*) set [1] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is *semi-open* in (X, τ). The complement of *sg-closed* set is called *sg-open* set,
2. a *ψ-closed* set [6] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is *sg-open* in (X, τ). The complement of *ψ-closed* set is called *ψ-open* set and
3. an *$\alpha\psi$-closed* set [2] if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and U is *α-open* in (X, τ). The complement of *$\alpha\psi$-closed* set is called *$\alpha\psi$-open* set.

The $\alpha\psi$-closure of a subset A of a space (X, τ) is the intersection of all *$\alpha\psi$-closed* sets that contain A and is denoted by $\alpha\psi cl(A)$. The $\alpha\psi$-interior of a subset A of a space (X, τ) is the union of all *$\alpha\psi$-open* sets that are contained in A and is denoted by $\alpha\psi int(A)$. By $\alpha\psi O(X, \tau)$ or $\alpha\psi O(X)$, we denote the family of all *$\alpha\psi$-open* sets of (X, τ).

3. $\alpha\psi$-kernel and $\alpha\psi$-closure

Definition 3.1. The intersection of all *$\alpha\psi$-open* subsets of (X, τ) containing A is called the *$\alpha\psi$-kernel* of A (briefly, *$\alpha\psi$-ker*(A)).

$$\text{i.e } \alpha\psi\text{-}ker(A) = \cap\{G \in \alpha\psi O(X) : A \subseteq G\}$$

Definition 3.2. Let $x \in X$. Then *$\alpha\psi$-kernel* of x is denoted by $\alpha\psi\text{-}ker(\{x\}) = \cap\{G \in \alpha\psi O(X) : x \in G\}$.

Definition 3.3. Let X be a topological space and $x \in X$, then a subset N_x of X is called an $\alpha\psi$-neighbourhood (briefly, *$\alpha\psi$-nbd*) of X if there exists an *$\alpha\psi$-open* set G such that $x \in G \subseteq N_x$.

Theorem 3.4. *Let X be a toplogical space. Then for any nonempty subset A of X, $\alpha\psi\text{-}ker(A) = \{x \in X : \alpha\psi cl(\{x\}) \cap A \neq \phi\}$.*

Proof. Let $x \in \alpha\psi\text{-}ker(A)$. Suppose that $\alpha\psi cl(\{x\}) \cap A = \phi$. Then $A \subseteq X - \alpha\psi cl(\{x\})$ and $X - \alpha\psi cl(\{x\})$ is *$\alpha\psi$-open* set containing A but not x, which is a contradiction.

Conversely, let us assume that $x \notin \alpha\psi\text{-}ker(A)$ and $\alpha\psi cl(\{x\}) \cap A \neq \phi$. Then there exist an *$\alpha\psi$-open* set D containing A but not x and a $y \in \alpha\psi cl(\{x\}) \cap A$.

Hence an $\alpha\psi$-closed set $X - D$ contains x, and $\{x\} \subset X - D, y \notin X - D$. This is a contradiction to $y \in \alpha\psi cl(\{x\}) \cap A$. Therefore $x \in \alpha\psi\text{-}ker(A)$. □

DEFINITION 3.5. In a space X, a set A is said to be *weakly ultra-$\alpha\psi$-separated* from a set B if there exists an *$\alpha\psi$-open* set G such that $A \subseteq G$ and $G \cap B = \phi$ or $A \cap \alpha\psi cl(B) = \phi$.

By the definition 3.6 and the theorem 3.4, we have the following for $x, y \in X$ of a topological space,

(i) $\alpha\psi\text{-}cl(\{x\}) = \{y : \{x\}$ *is not weakly ultra-$\alpha\psi$-separated from* $\{x\}\}$ and

(ii) $\alpha\psi\text{-}ker(\{x\}) = \{y : \{y\}$ *is not weakly ultra-$\alpha\psi$-separated from* $\{y\}\}$.

DEFINITION 3.6. For any point x of a space X,

(i) the *$\alpha\psi$-derived* (briefly, $\alpha\psi\text{-}d(\{x\})$) *set* of x is defined to be the set
$\alpha\psi\text{-}d(\{x\}) = \alpha\psi\text{-}cl(\{x\}) - \{x\} = \{y : y \neq x$ *and* $\{y\}$*is not weakly-ultra-$\alpha\psi$-separated from* $\{x\}\}$,

(ii) the $\alpha\psi$-shell (briefly, $\alpha\psi\text{-}shl(\{x\})$) of a singleton set $\{x\}$ is defined to be the set
$\alpha\psi\text{-}shl(\{x\}) = \alpha\psi\text{-}ker(\{x\}) - \{x\} = \{y : y \neq x$ *and* $\{x\}$*is not weakly-ultra-$\alpha\psi$-separated from* $\{y\}\}$.

DEFINITION 3.7. Let X be a topological space. Then we define

(i) $\alpha\psi\text{-}N\text{-}D = \{x : x \in X$ *and* $\alpha\psi\text{-}d(\{x\}) = \phi\}$,

(ii) $\alpha\psi\text{-}N\text{-}shl = \{x : x \in X$ *and* $\alpha\psi\text{-}shl(\{x\}) = \phi\}$ and

(iii) $\alpha\psi\text{-}\langle x\rangle = \alpha\psi\text{-}cl(\{x\}) \cap \alpha\psi\text{-}ker(\{x\})$.

THEOREM 3.8. *Let $x, y \in X$. Then the following conditions hold.*

(i) $y \in \alpha\psi\text{-}ker(\{x\})$ *if and only if* $x \in \alpha\psi\text{-}cl(\{y\})$,

(ii) $y \in \alpha\psi\text{-}shl(\{x\})$ *if and only if* $x \in \alpha\psi\text{-}d(\{y\})$,

(iii) $y \in \alpha\psi\text{-}cl(\{x\})$ *implies* $\alpha\psi\text{-}cl(\{y\}) \subseteq \alpha\psi\text{-}cl(\{x\})$ *and*

(iv) $y \in \alpha\psi\text{-}ker(\{x\})$ *implies* $\alpha\psi\text{-}ker(\{y\}) \subseteq \alpha\psi\text{-}ker(\{x\})$.

Proof. The proof of (i) and (ii) are obvious.
(iii) Let $z \in \alpha\psi\text{-}cl(\{y\})$. Then $\{z\}$ is not *weakly ultra-$\alpha\psi$-separated* from $\{y\}$. So there exists an $\alpha\psi$-open set G containing z such that $G \cap \{y\} \neq \phi$. Hence $y \in G$ and by assumption $G \cap \{x\} \neq \phi$. Hence $\{z\}$ is not *weakly ultra-$\alpha\psi$-separated* from $\{x\}$. So $z \in \alpha\psi\text{-}cl(\{x\})$.

Therefore $\alpha\psi$-$cl(\{y\}) \subseteq \alpha\psi$-$cl(\{x\})$.
(iv) Let $z \in \alpha\psi$- $ker(\{y\})$. Then $\{y\}$ is not *weakly ultra-$\alpha\psi$-separated* from $\{z\}$. So $y \in \alpha\psi$-$cl(\{z\})$. Hence $\alpha\psi$-$cl(\{y\}) \subseteq \alpha\psi$-$cl(\{z\})$. By assumption $y \in \alpha\psi$-$ker(\{x\})$ and then $x \in \alpha\psi$-$cl(\{y\})$. So $\alpha\psi$-$cl(\{x\}) \subseteq \alpha\psi$-$cl(\{y\})$. Ultimately $\alpha\psi$-$cl(\{x\}) \subseteq \alpha\psi$-$cl(\{z\})$. Hence $x \in \alpha\psi$-$cl(\{z\})$, that is $z \in \alpha\psi$-$ker(\{x\})$. Therefore $\alpha\psi$-$ker(\{y\}) \subseteq \alpha\psi$-$ker(\{x\})$. □

Let us recall that a subset A of X is called a degenerate set if A is either a null set or a singleton set.

THEOREM 3.9. *Let $x, y \in X$. Then,*

(i) *for every $x \in X$, $\alpha\psi$-$shl(\{x\})$ is degenerate if and only if for all $x, y \in X$, $x \neq y$, $\alpha\psi$-$d(\{x\}) \cap \alpha\psi$-$d(\{y\}) = \phi$,*
(ii) *for every $x \in X$, $\alpha\psi$-$d(\{x\})$ is degenerate if and only if for every $x, y \in X$, $x \neq y$, $\alpha\psi$-$shl(\{x\}) \cap \alpha\psi$-$shl(\{y\}) = \phi$.*

Proof. Let $\alpha\psi$-$d(\{x\}) \cap \alpha\psi$-$d(\{y\}) \neq \phi$. Then there exists a $z \in X$ such that $z \in \alpha\psi$-$d(\{x\})$ and $z \in \alpha\psi$-$d(\{y\})$. Then $z \neq y \neq x$ and $z \in \alpha\psi$-$cl(\{x\})$ and $z \in \alpha\psi$-$cl(\{y\})$, that is $x, y \in \alpha\psi$-$ker(\{z\})$. Hence $\alpha\psi$-$ker(\{z\})$ and so $\alpha\psi$-$shl(\{z\})$ is not a degenerate set.

Conversely, let $x, y \in \alpha\psi$-$shl(\{z\})$. Then we get $x \neq z$, $x \in \alpha\psi$-$ker(\{z\})$ and $y \neq z$ and $y \in \alpha\psi$-$ker(\{z\})$ and hence z is an element of both $\alpha\psi$-$cl(\{x\})$ and $\alpha\psi$-$cl(\{y\})$, which is a contradiction.
The proof of (ii) is simple and hence omitted. □

THEOREM 3.10. *If $y \in \alpha\psi$-$\langle x\rangle$, then $\alpha\psi$-$\langle x\rangle = \alpha\psi$-$\langle y\rangle$.*

Proof. If $y \in \alpha\psi$-$\langle x\rangle$, then $y \in \alpha\psi$-$cl(\{x\}) \cap \alpha\psi$-$ker(\{x\})$. Hence $y \in \alpha\psi$-$cl(\{x\})$ and $y \in \alpha\psi$-$ker(\{x\})$ and so we have $\alpha\psi$-$cl(\{y\}) \subseteq \alpha\psi$-$cl(\{x\})$ and $\alpha\psi$-$ker(\{y\}) \subseteq \alpha\psi$-$ker(\{x\})$. Then $\alpha\psi$-$cl(\{y\}) \cap \alpha\psi$-$ker(\{y\}) \subseteq \alpha\psi$-$cl(\{x\}) \cap \alpha\psi$-$ker(\{x\})$. Hence $\alpha\psi$-$\langle y\rangle \subseteq \alpha\psi$-$\langle x\rangle$. The fact that $y \in \alpha\psi$-$cl(\{x\})$ implies $x \in \alpha\psi$-$ker(\{y\})$ and $y \in \alpha\psi$-$ker(\{x\})$ implies $x \in \alpha\psi$-$cl(\{y\})$. Then we have that $\alpha\psi$-$\langle x\rangle \subseteq \alpha\psi$-$\langle y\rangle$. So $\alpha\psi$-$\langle x\rangle = \alpha\psi$-$\langle y\rangle$. □

THEOREM 3.11. *For all $x, y \in X$, either $\alpha\psi$-$\langle x\rangle \cap \alpha\psi$-$\langle y\rangle = \phi$ or $\alpha\psi$-$\langle x\rangle = \alpha\psi$-$\langle x\rangle$.*

Proof. $\alpha\psi$-$\langle x\rangle \cap \alpha\psi$-$\langle y\rangle \neq \phi$, then there exists $z \in X$ such that $z \in \alpha\psi$-$\langle x\rangle$ and $z \in \alpha\psi$-$\langle y\rangle$. So by Theorem 3.11, $\alpha\psi$-$\langle z\rangle = \alpha\psi$-$\langle x\rangle = \alpha\psi$-$\langle y\rangle$. Hence the result. □

THEOREM 3.12. *For any two points $x, y \in X$, the following statements are equivalent.*

(i) $\alpha\psi$-$ker(\{x\}) \neq \alpha\psi$-$ker(\{y\})$ *and*
(ii) $\alpha\psi$-$cl(\{x\}) \neq \alpha\psi$-$cl(\{y\})$.

Proof. $(i) \Longrightarrow (ii)$ Let us assume $\alpha\psi$-$ker(\{x\}) \neq \alpha\psi$-$ker(\{y\})$. Then there exists a $z \in \alpha\psi$-$ker(\{x\})$ but $z \notin \alpha\psi$-$ker(\{y\})$. As $z \in \alpha\psi$-$ker(\{x\})$, $x \in \alpha\psi$-$cl(\{z\})$ and $\alpha\psi$-$cl(\{x\}) \subseteq \alpha\psi$-$cl(\{z\})$. Also we have taken $z \notin \alpha\psi$-$ker(\{y\})$, by Theorem 3.4, $\alpha\psi$-$cl(\{z\}) \cap \{y\} = \phi$, so $\alpha\psi$-$cl(\{x\}) \cap \{y\} = \phi$ and so $\{y\}$ is *weakly ultra-$\alpha\psi$-separated* from $\{x\}$ and hence we get that $y \notin \alpha\psi$-$cl(\{x\})$. Hence $\alpha\psi$-$cl(\{y\}) \neq \alpha\psi$-$cl(\{x\})$.
$(ii) \Longrightarrow (i)$ Suppose $\alpha\psi$-$cl(\{x\}) \neq \alpha\psi$-$cl(\{y\})$. Then there exists a point $z \in \alpha\psi$-$cl(\{x\})$ but $z \notin \alpha\psi$-$?cl(\{y\})$. So, we get an *$\alpha\psi$-open* set containing z and x but not y. That is $y \notin \alpha\psi$-$ker(\{x\})$. Hence $\alpha\psi$-$ker(\{y\}) \neq \alpha\psi$-$ker(\{x\})$. □

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