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LINEAR $*$-DERIVATIONS ON C^*-ALGEBRAS

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Abstract. It is shown that for a derivation

$$f(x_1 \cdots x_{j-1} x_j x_{j+1} \cdots x_k) = \sum_{j=1}^{k} x_1 \cdots x_{j-1} x_{j+1} \cdots x_k f(x_j)$$

on a unital C^*-algebra $\mathcal{B}$, there exists a unique $\mathbb{C}$-linear $*$-derivation $D : \mathcal{B} \to \mathcal{B}$ near the derivation, by using the Hyers-Ulam-Rassias stability of functional equations. The concept of Hyers-Ulam-Rassias stability originated from the Th.M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

1. Introduction

Borelli [1] proved the Hyers-Ulam-Rassias stability problem of the functional equation

$$D(xy) = xD(y) + yD(x)$$

on the interval $(0, 1]$, which is called a *derivation*, and Páles [5] proved the stability of the functional equation for real-valued functions on $[1, \infty)$. Tabor [9] investigated the Hyers-Ulam-Rassias stability problem of the functional equation for Banach space-valued functions and obtained the following result: Let X be a Banach space with norm $\| \cdot \|$ and let $f : (0, 1] \to X$ be a mapping and $\theta > 0$. Suppose that

$$\|f(xy) - xf(y) - yf(x)\| \le \theta$$

for all $x, y \in (0, 1]$. Then there exists a derivation $D : (0, 1] \to X$ such that

$$\|f(x) - D(x)\| \le 4e\theta$$

for all $x \in (0, 1]$.

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Let E_1 and E_2 be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias [8] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in E_1$. Găvruta [2] generalized the Th.M. Rassias' result, and Park [6] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra. See [4, 7] for details.

Throughout this paper, let $\mathbb{R}^+$ be the set of nonnegative real numbers and k an integer greater than 1. Let $\mathcal{B}$ be a unital C^*-algebra with norm $\| \cdot \|$ and unitary group $\mathcal{U}(\mathcal{B})$.

In this paper, we prove that for a derivation

$$f(x_1 \cdots x_{j-1}x_jx_{j+1} \cdots x_k) = \sum_{j=1}^{k} x_1 \cdots x_{j-1}x_{j+1} \cdots x_k f(x_j)$$

on a unital C^*-algebra $\mathcal{B}$, there exists a unique $\mathbb{C}$-linear $*$-derivation $D : \mathcal{B} \to \mathcal{B}$ near the derivation.

2. Stability of linear $*$-derivations on C^*-algebras

We prove the Hyers-Ulam-Rassias stability of linear $*$-derivations on C^*-algebras.

THEOREM 2.1. *Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying*

$$f(x_1 \cdots x_{j-1}x_jx_{j+1} \cdots x_k) = \sum_{j=1}^{k} x_1 \cdots x_{j-1}x_{j+1} \cdots x_k f(x_j)$$

for which there exists a function $\varphi : \mathcal{B}^k \to [0, \infty)$ *such that*

(i) $$\widetilde{\varphi}(x_1, \cdots, x_k) = \sum_{j=0}^{\infty} k^{-j} \varphi(k^j x_1, \cdots, k^j x_k) < \infty,$$

(ii)

$$\|T_\mu f(x_1, \cdots, x_k)\| := \|f(\sum_{j=1}^{k} \mu x_j) - \mu \sum_{j=1}^{k} f(x_j)\| \le \varphi(x_1, \cdots, x_k),$$

(iii) $$\|f(k^n u^*) - f(k^n u)^*\| \le \varphi(\underbrace{k^n u, \cdots, k^n u}_{k \text{ times}})$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} | \ |\lambda| = 1\}$*, all* $u \in \mathcal{U}(\mathcal{B})$*,* $n = 0, 1, \cdots,$ *and all* $x_1, \cdots, x_k \in \mathcal{B}$*. Then there exists a unique* $\mathbb{C}$*-linear* $*$*-derivation* $D : \mathcal{B} \to \mathcal{B}$ *such that*

(iv) $$\|f(x) - D(x)\| \le \frac{1}{k} \widetilde{\varphi}(\underbrace{x, \cdots, x}_{k \text{ times}})$$

for all $x \in \mathcal{B}$*.*

Proof. Put $\mu = 1 \in \mathbb{T}^1$. Replacing x_j by x in (ii) for all $j = 1, \cdots, k$, we get

$$\|f(kx) - kf(x)\| \le \varphi(\underbrace{x, \cdots, x}_{k \text{ times}})$$

for all $x \in \mathcal{B}$. So one can obtain that

$$\|f(x) - \frac{1}{k} f(kx)\| \le \frac{1}{k} \varphi(\underbrace{x, \cdots, x}_{k \text{ times}}),$$

and hence

$$\|\frac{1}{k^n} f(k^n x) - \frac{1}{k^{n+1}} f(k^{n+1} x)\| \le \frac{1}{k^{n+1}} \varphi(\underbrace{k^n x, \cdots, k^n x}_{k \text{ times}})$$

for all $x \in \mathcal{B}$. So we get

(1) $$\|f(x) - \frac{1}{k^n} f(k^n x)\| \le \frac{1}{k} \sum_{l=0}^{n-1} \frac{1}{k^l} \varphi(\underbrace{k^l x, \cdots, k^l x}_{k \text{ times}})$$

for all $x \in \mathcal{B}$.

Let x be an element in $\mathcal{B}$. For positive integers n and m with $n > m$,

$$\|\frac{1}{k^n} f(k^n x) - \frac{1}{k^m} f(k^m x)\| \le \frac{1}{k} \sum_{l=m}^{n-1} \frac{1}{k^l} \varphi(\underbrace{k^l x, \cdots, k^l x}_{k \text{ times}}),$$

which tends to zero as $m \to \infty$ by (i). So $\{\frac{1}{k^n} f(k^n x)\}$ is a Cauchy sequence for all $x \in \mathcal{B}$. Since $\mathcal{B}$ is complete, the sequence $\{\frac{1}{k^n} f(k^n x)\}$ converges for all $x \in \mathcal{B}$. We can define a mapping $D : \mathcal{B} \to \mathcal{B}$ by

$$D(x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n x) \tag{2}$$

for all $x \in \mathcal{B}$.

By (i) and (2), we get

$$\begin{aligned}\|T_1 D(x_1, \cdots, x_k)\| &= \lim_{n\to\infty} \frac{1}{k^n} \|T_1 f(k^n x_1, \cdots, k^n x_k)\| \\ &\le \lim_{n\to\infty} \frac{1}{k^n} \varphi(k^n x_1, \cdots, k^n x_k) = 0\end{aligned}$$

for all $x_1, \cdots, x_k \in \mathcal{B}$. Hence $T_1 D(x_1, \cdots, x_k) = 0$ for all $x_1, \cdots, x_k \in \mathcal{B}$. But by the assumption $f(0) = 0$. Putting $x_3 = \cdots = x_k = 0$, one can obtain that D is additive. Moreover, by passing to the limit in (1) as $n \to \infty$, we get the inequality (iv).

Now let $S : \mathcal{B} \to \mathcal{B}$ be another additive mapping satisfying

$$\|f(x) - S(x)\| \le \frac{1}{k} \widetilde{\varphi}(\underbrace{x, \cdots, x}_{k \text{ times}})$$

for all $x \in \mathcal{B}$.

$$\begin{aligned}\|D(x) - S(x)\| &= \frac{1}{k^l} \|D(k^l x) - S(k^l x)\| \\ &\le \frac{1}{k^l} \|D(k^l x) - f(k^l x)\| + \frac{1}{k^l} \|f(k^l x) - S(k^l x)\| \\ &\le \frac{2}{k} \frac{1}{k^l} \widetilde{\varphi}(\underbrace{k^l x, \cdots, k^l x}_{k \text{ times}}),\end{aligned}$$

which tends to zero as $l \to \infty$ by (i). Thus $D(x) = S(x)$ for all $x \in \mathcal{B}$. This proves the uniqueness of D.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\|f(k^n \mu x) - k\mu f(k^{n-1} x)\| \le \varphi(\underbrace{k^{n-1} x, \cdots, k^{n-1} x}_{k \text{ times}})$$

for all $x \in \mathcal{B}$. And one can show that

$$\begin{aligned}\|\mu f(k^n x) - k\mu f(k^{n-1} x)\| &\le |\mu| \cdot \|f(k^n x) - k f(k^{n-1} x)\| \\ &\le \varphi(\underbrace{k^{n-1} x, \cdots, k^{n-1} x}_{k \text{ times}})\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. So

$$\begin{aligned}\|f(k^n\mu x) - \mu f(k^n x)\| \leq & \|f(k^n\mu x) - k\mu f(k^{n-1}x)\| \\ & + \|k\mu f(k^{n-1}x) - \mu f(k^n x)\| \\ \leq & \varphi(\underbrace{k^{n-1}x, \cdots, k^{n-1}x}_{k \text{ times}}) + \varphi(\underbrace{k^{n-1}x, \cdots, k^{n-1}x}_{k \text{ times}})\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. Thus $k^{-n}\|f(k^n\mu x) - \mu f(k^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$. Hence

$$D(\mu x) = \lim_{n\to\infty} \frac{f(k^n\mu x)}{k^n} = \lim_{n\to\infty} \frac{\mu f(k^n x)}{k^n} = \mu D(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{B}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [3, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. And $D(x) = D(3 \cdot \frac{1}{3}x) = 3D(\frac{1}{3}x)$ for all $x \in \mathcal{B}$. So $D(\frac{1}{3}x) = \frac{1}{3}D(x)$ for all $x \in \mathcal{B}$. Thus

$$\begin{aligned}D(\lambda x) &= D(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot D(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}D(3\frac{\lambda}{M}x) \\ &= \frac{M}{3}D(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(D(\mu_1 x) + D(\mu_2 x) + D(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)D(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}D(x) \\ &= \lambda D(x)\end{aligned}$$

for all $x \in \mathcal{B}$. Hence

$$D(\alpha x + \beta y) = D(\alpha x) + D(\beta y) = \alpha D(x) + \beta D(y)$$

for all $\alpha, \beta \in \mathbb{C}(\alpha, \beta \neq 0)$ and all $x, y \in \mathcal{B}$. And $D(0x) = 0 = 0D(x)$ for all $x \in \mathcal{B}$. So the unique additive mapping $D : \mathcal{B} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

It follows from (2) that

$$D(x) = \lim_{n\to\infty} \frac{f(k^{kn}x)}{k^{kn}}$$

for all $x \in \mathcal{B}$. Since $f(x_1 \cdots x_j \cdots x_k) = \sum_{j=1}^{k} x_1 \cdots x_{j-1} x_{j+1} \cdots x_k f(x_j)$,

$$\begin{aligned} D(x_1 \cdots x_{j-1} x_j x_{j+1} \cdots x_k) &= \lim_{n\to\infty} \frac{f(k^{kn} x_1 \cdots x_k)}{k^{kn}} \\ &= \lim_{n\to\infty} \frac{f(k^n x_1 \cdots k^n x_k)}{k^{(k-1)n} \cdot k^n} \\ &= \lim_{n\to\infty} \frac{k^{(k-1)n} \sum_{j=1}^{k} x_1 \cdots x_{j-1} x_{j+1} \cdots x_k f(k^n x_j)}{k^{(k-1)n} \cdot k^n} \\ &= \sum_{j=1}^{k} x_1 \cdots x_{j-1} x_{j+1} \cdots x_k D(x_j) \end{aligned}$$

for all $x_1, \cdots, x_k \in \mathcal{B}$.

By (i) and (iii), we get

$$\begin{aligned} D(u^*) &= \lim_{n\to\infty} \frac{f(k^n u^*)}{k^n} = \lim_{n\to\infty} \frac{(f(k^n u))^*}{k^n} = (\lim_{n\to\infty} \frac{f(k^n u)}{k^n})^* \\ &= D(u)^* \end{aligned} \tag{3}$$

for all $u \in \mathcal{U}(\mathcal{B})$.

Now let $x \in \mathcal{B}$ $(x \neq 0)$ and M an integer greater than $4|x|$. Then

$$|\frac{x}{M}| = \frac{1}{M}|x| < \frac{|x|}{4|x|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [3, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(\mathcal{B})$ such that $3\frac{x}{M} = u_1 + u_2 + u_3$. Thus by (3)

$$\begin{aligned} D(x^*) &= D(\frac{M}{3} \cdot 3\frac{x^*}{M}) = \frac{M}{3} D((3\frac{x}{M})^*) \\ &= \frac{M}{3} D(u_1^* + u_2^* + u_3^*) = \frac{M}{3}(D(u_1^*) + D(u_2^*) + D(u_3^*)) \\ &= \frac{M}{3}(D(u_1)^* + D(u_2)^* + D(u_3)^*) \\ &= \frac{M}{3}(D(u_1) + D(u_2) + D(u_3))^* \\ &= \frac{M}{3} D(u_1 + u_2 + u_3)^* = D(\frac{M}{3} 3\frac{x}{M})^* \\ &= D(x)^* \end{aligned}$$

for all $x \in \mathcal{B}$. Hence the additive mapping $D : \mathcal{B} \to \mathcal{B}$ is a $\mathbb{C}$-linear $*$-derivation satisfying the inequality (iv), as desired. □

Corollary 2.2. *Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying*

$$f(x_1 \cdots x_{j-1} x_j x_{j+1} \cdots x_k) = \sum_{j=1}^{k} x_1 \cdots x_{j-1} x_{j+1} \cdots x_k f(x_j)$$

for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(\sum_{j=1}^{k} \mu x_j) - \mu \sum_{j=1}^{k} f(x_j)\| \leq \theta \sum_{j=1}^{k} \|x_j\|^p,$$
$$\|f(k^n u^*) - f(k^n u)^*\| \leq k \cdot k^{np} \theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{B})$, $n = 0, 1, \cdots$, and all $x_1, \cdots, x_k \in \mathcal{B}$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : \mathcal{B} \to \mathcal{B}$ such that

$$\|f(x) - D(x)\| \leq \frac{k\theta}{k - k^p} \|x\|^p$$

for all $x \in \mathcal{B}$.

Proof. Define $\varphi(x_1, \cdots, x_k) = \theta \sum_{j=1}^{k} \|x_j\|^p$, and apply Theorem 2.1. □

Theorem 2.3. *Let $f : \mathcal{B} \to \mathcal{B}$ be a mapping satisfying*

$$f(x_1 \cdots x_{j-1} x_j x_{j+1} \cdots x_k) = \sum_{j=1}^{k} x_1 \cdots x_{j-1} x_{j+1} \cdots x_k f(x_j)$$

for which there exists a function $\varphi : \mathcal{B}^k \to [0, \infty)$ satisfying (i) *and* (iii) *such that*

$$\text{(v)} \qquad \|f(\sum_{j=1}^{k} \mu x_j) - \mu \sum_{j=1}^{k} f(x_j)\| \leq \varphi(x_1, \cdots, x_k)$$

for $\mu = 1, i$, and all $x_1, \cdots, x_k \in \mathcal{B}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{B}$, then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : \mathcal{B} \to \mathcal{B}$ satisfying the inequality (iv).

Proof. Put $\mu = 1$ in (v). By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $D : \mathcal{B} \to \mathcal{B}$ satisfying the inequality (iv). By the same reasoning as in the proof of [8, Theorem], the additive mapping $D : \mathcal{B} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (v). By the same method as the proof of Theorem 2.1, one can obtain that

$$D(ix) = \lim_{n\to\infty} \frac{f(k^n ix)}{k^n} = \lim_{n\to\infty} \frac{if(k^n x)}{k^n} = iD(x)$$

for all $x \in \mathcal{B}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = \eta + i\nu$, where $\eta, \nu \in \mathbb{R}$. So

$$\begin{aligned} D(\lambda x) &= D(\eta x + i\nu x) = \eta D(x) + \nu D(ix) = \eta D(x) + i\nu D(x) \\ &= \lambda D(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{B}$. So

$$D(\alpha x + \beta y) = D(\alpha x) + D(\beta y) = \alpha D(x) + \beta D(y)$$

for all $\alpha, \beta \in \mathbb{C}$, and all $x, y \in \mathcal{B}$. Hence the additive mapping $D : \mathcal{B} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as the proof of Theorem 2.1. □

REMARK 2.4. Suppose that $\mathcal{B}$ is not unital. If the inequalities (iii) in Theorems 2.1 and 2.3 are replaced by

$$\|f(x^*) - f(x)^*\| \le \varphi(\underbrace{x, \cdots, x}_{k \text{ times}})$$

for all $x \in \mathcal{B}$, then the results do also hold. The proofs are similar to the proofs of Theorems 2.1 and 2.3.

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