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ANALYTIC OPERATOR-VALUED GENERALIZED FEYNMAN INTEGRALS ON FUNCTION SPACE

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Abstract. In this paper we use a generalized Brownian motion process to defined an analytic operator-valued generalized Feynman integral. We then obtain explicit formulas for the analytic operator-valued generalized Feynman integrals for functionals of the form

$$F(x) = f\bigg(\int_0^T \alpha_1(t)dx(t), \cdots, \int_0^T \alpha_n(t)dx(t)\bigg),$$

where x is a continuous function on $[0,T]$ and $\{\alpha_1, \cdots, \alpha_n\}$ is an orthonormal set of functions from $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$.

1. Introduction

Since the Feynman integral was introduced by R. P. Feynman in 1948, there has been considerable progress on the Feynman integration theory. In [1], Cameron and Storvick introduced the analytic operator-valued function space "Feynman integral", $J_q^{\mathrm{an}}(F)$, which mapped an $L^2(\mathbb{R})$ function ψ into an $L^2(\mathbb{R})$ function $(J_q^{\mathrm{an}}(F)\psi)$ on the classical Wiener space $C_0[0,T]$.

The function space $C_{a,b}[0,T]$ induced by generalized Brownian motion process was introduced by J. Yeh in [6] and was used extensively by Chang and Chung in [3]. In [2, 4], the authors defined a generalized analytic Feynman integral and a generalized Fourier-Feynman transform on $C_{a,b}[0,T]$ and studied their properties and related topics.

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In this paper we use a generalized Brownian motion process to define the analytic operator-valued generalized Feynman integral. We then obtain explicit formulas for the generalized analytic operator-valued Feynman integrals of functionals of the form

$$F(x) = f\Big(\int_0^T \alpha_1(t)dx(t), \cdots, \int_0^T \alpha_n(t)dx(t) \Big)$$

where x is a continuous function on $[0,T]$ and $\{\alpha_1, \cdots, \alpha_n\}$ is an orthonormal set of functions from $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$. We also give several examples for our results. The Wiener process used in [1] is free of drift and is stationary in time while the stochastic process used in this paper as well as in [2-4, 6] is nonstationary in time, is subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [5].

2. Definitions and preliminaries

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega) = 0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \cdots, Y(t_n,\omega))$ is normally distributed with the density function

$$\begin{aligned} K(\vec{t},\vec{\eta}) &= \Big((2\pi)^n \prod_{j=1}^n (b(t_j) - b(t_{j-1}))\Big)^{-1/2} \\ &\quad \cdot \exp\Big\{ -\frac{1}{2} \sum_{j=1}^n \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})} \Big\}, \end{aligned} \tag{2.1}$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [7, pp.18-20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real-valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [7, p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable (s.i.m.) if ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set if $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., then we write $F \approx G$.

Let $L^2_{a,b}[0,T]$ be the set of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$\begin{aligned} &L^2_{a,b}[0,T] \\ &= \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\} \end{aligned} \tag{2.2}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)]. \tag{2.3}$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore, $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space. Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j, \phi_k)_{a,b} = \begin{cases} 0 & , \ j \neq k \\ 1 & , \ j = k \end{cases}.$$

Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund(PWZ) stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (v, \phi_j)_{a,b}\phi_j(t)dx(t) \tag{2.4}$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists; one can show that for each $v \in L^2_{a,b}[0,T]$, the PWZ stochastic integral $\langle v, x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$ and is a Gaussian random variable on $C_{a,b}[0,T]$ with mean $\int_0^T v(s)da(s)$ and variance $\int_0^T v^2(s)db(s)$. If v is of bounded variation on $[0,T]$, then the PWZ integral $\langle v, x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)dx(t)$ for μ-a.e. $x \in C_{a,b}[0,T]$. For more details, see [4]. We next give the definition of the analytic operator-valued function space integral as an element of $\mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$.

Throughout this paper, let $\mathbb{C}$ be the set of complex numbers, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$. Let $C[0,T]$ denote the space of real-valued continuous functions x on $[0,T]$. Let F be a functional from $C[0,T]$ to $\mathbb{C}$. For each $\lambda > 0$, $\psi \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$, assume that $F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(T) + \xi)$ is μ-integrable with respect to x on $C_{a,b}[0,T]$, and let

$$(I_\lambda(F)\psi)(\xi) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(T) + \xi)d\mu(x). \tag{2.5}$$

If $I_\lambda(F)\psi$ is in $L^2(\mathbb{R})$ as a function of ξ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$, the space of continuous linear operators from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next, suppose there exists an $\mathcal{L}$-valued function which is analytic in $\mathbb{C}_+$ and agrees with $I_\lambda(F)$ on $(0,\infty)$; then this $\mathcal{L}$-valued function is denoted by $I^{\mathrm{an}}_\lambda(F)$ and is called the analytic operator-valued function space integral of F associated with λ.

The following notations are used throughout this paper:

$$A_h \equiv \int_0^T h(t)da(t) \quad \text{and} \quad B_h \equiv \int_0^T h^2(t)db(t) \tag{2.6}$$

for $h \in L^2_{a,b}[0,T]$. Furthermore, for all $\mathbb{C}_+$, $\sqrt{\lambda} = \lambda^{1/2}$ is always chosen to have positive real part.

Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal set of functions from $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ and for $j \in \{1, \cdots, n\}$, let $A_j \equiv A_{\alpha_j}$ and $B_j \equiv B_{\alpha_j}$. Note that for any $h \in L^2_{a,b}[0,T]$, B_h is always positive, while A_h may be positive, negative or zero.

Next we state a fundamental integration formula for the function space $C_{a,b}[0,T]$.

THEOREM 2.1. *Let* $\{\alpha_1, \cdots, \alpha_n\}$ *be an orthonormal set of functions from* $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$. *Let* $f : \mathbb{R}^n \to \mathbb{C}$ *be Lebesgue measurable, and let*

$$F(x) = f(\langle \alpha_1, x\rangle, \cdots, \langle \alpha_n, x\rangle) \equiv f(\langle \vec{\alpha}, x\rangle).$$

Then

$$\begin{aligned} E[F] &\equiv \int_{C_{a,b}[0,T]} f(\langle \vec{\alpha}, x\rangle) d\mu(x) \\ &= \Big(\prod_{j=1}^n \frac{1}{2\pi B_j}\Big)^{\frac{1}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{ -\sum_{j=1}^n \frac{[u_j - A_j]^2}{2B_j}\Big\} d\vec{u} \end{aligned} \tag{2.7}$$

in the sense that if either side exists, both sides exist and equality holds.

3. Analytic operator-valued function space integral of finite dimensional functionals

Let $\{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal set of $L^2_{a,b}[0,T]$ where α_j is of bounded variation on $[0,T]$. Let $F : C[0,T] \to \mathbb{C}$ be given by

$$F(x) = f\Big(\int_0^T \alpha_1(t) dx(t), \cdots, \int_0^T \alpha_n(t) dx(t)\Big), \tag{3.1}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function. In this case, we see that for $x \in C_{a,b}[0,T]$, $\rho > 0$ and $\xi \in \mathbb{R}$,

$$F(\rho x + \xi) = f(\rho\langle \alpha_1, x\rangle, \cdots, \rho\langle \alpha_n, x\rangle). \tag{3.2}$$

We also note that for each $x \in C_{a,b}[0,T]$,

$$x(T) = \langle 1, x\rangle = \sum_{j=1}^n (1, \alpha_j)_{a,b} \langle \alpha_j, x\rangle + \langle p, x\rangle \tag{3.3}$$

where

$$p = 1 - \sum_{j=1}^n (1, \alpha_j)_{a,b} \alpha_j \tag{3.4}$$

is an element of $(\text{span}\{\alpha_1, \cdots, \alpha_n\})^{\perp}$.

Throughout this section we will use the following notation for convenience; for $\lambda \in \tilde{\mathbb{C}}_+$ and $p \in L^2_{a,b}[0,T]$, let

$$K(\lambda;p) = \left(\frac{\lambda}{2\pi B_p}\prod_{j=1}^{n}\frac{\lambda}{2\pi B_j}\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}H(\lambda;\vec{u},u_{n+1}) &\equiv H(\lambda;\vec{u})\times H(\lambda;u_{n+1})\\ &= \exp\bigg\{-\sum_{j=1}^{n}\frac{[\sqrt{\lambda}u_j - A_j]^2}{2B_j}\bigg\}\exp\bigg\{-\frac{[\sqrt{\lambda}u_{n+1}-A_p]^2}{2B_p}\bigg\}.\end{aligned}$$

THEOREM 3.1. *Let F be given by equation* (3.1) *with*

$$\int_{\mathbb{R}^n}|f(\vec{u})|^2\exp\bigg\{-\sum_{j=1}^{n}\frac{[\sqrt{\lambda}u_j - A_j]^2}{2B_j}\bigg\}d\vec{u} < \infty. \tag{3.5}$$

Then the analytic operator-valued function space integral of F, $I^{\mathrm{an}}_\lambda(F)$ exists for all $\lambda \in \mathbb{C}_+$ and is given by the formula

$$\begin{aligned}(I^{\mathrm{an}}_\lambda(F)\psi)(\xi) &= K(\lambda;p)\int_{\mathbb{R}}\int_{\mathbb{R}^n} f(\vec{u})\\ &\cdot\psi\bigg(\sum_{j=1}^{n}(1,\alpha_j)_{a,b}u_j + u_{n+1}+\xi\bigg)H(\lambda;\vec{u},u_{n+1})d\vec{u}du_{n+1}\end{aligned} \tag{3.6}$$

where p is given by equation (3.4).

Proof. First note that $\{\alpha_1,\cdots,\alpha_n,p/\|p\|_{a,b}\}$ is an orthonormal set in $L^2_{a,b}[0,T]$. Using equations (2.5), (3.2), (3.3) and (2.7), we obtain that for all $\lambda > 0$ and $\xi \in \mathbb{R}$

$$\begin{aligned}&(I_\lambda(F)\psi)(\xi)\\ &= \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(T)+\xi)d\mu(x)\\ &= \int_{C_{a,b}[0,T]} f(\lambda^{-1/2}\langle\alpha_1,x\rangle,\cdots,\lambda^{-1/2}\langle\alpha_n,x\rangle)\\ &\quad\cdot\psi\bigg(\lambda^{-1/2}\sum_{j=1}^{n}(1,\alpha_j)_{a,b}\langle\alpha_j,x\rangle + \lambda^{-1/2}\|p\|_{a,b}\langle p/\|p\|_{a,b},x\rangle+\xi\bigg)d\mu(x)\end{aligned} \tag{3.7}$$

$$= K(\lambda; p/\|p\|_{a,b}) \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(\vec{u}) \psi\bigg(\sum_{j=1}^n (1,\alpha_j)_{a,b} u_j + \|p\|_{a,b} u_{n+1} + \xi \bigg)$$

$$\cdot \exp\bigg\{ - \sum_{j=1}^n \frac{[\sqrt{\lambda} u_j - A_j]^2}{2B_j} - \frac{[\sqrt{\lambda} u_{n+1} - A_{p/\|p\|_{a,b}}]^2}{2B_{p/\|p\|_{a,b}}} \bigg\} d\vec{u} du_{n+1}$$

$$= K(\lambda; p) \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(\vec{u}) \psi\bigg(\sum_{j=1}^n (1,\alpha_j)_{a,b} u_j + u_{n+1} + \xi \bigg)$$

$$\cdot \exp\bigg\{ - \sum_{j=1}^n \frac{[\sqrt{\lambda} u_j - A_j]^2}{2B_j} - \frac{[\sqrt{\lambda} u_{n+1} - A_p]^2}{2B_p} \bigg\} d\vec{u} du_{n+1}.$$

Note that for $\lambda \in \mathbb{C}_+$, $\sqrt{\lambda} = c + di$ with $c^2 - d^2 > 0$. Then for each f with $\int_{\mathbb{R}^n} |f(\vec{u})|^2 \exp\{-\sum_{j=1}^n \frac{[\sqrt{\lambda} u_j - A_j]^2}{2B_j}\} d\vec{u} < \infty$ and $\psi \in L^2(\mathbb{R})$, we obtain that

(3.8)

$$\bigg| \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(\vec{u}) \psi\bigg(\sum_{j=1}^n (1,\alpha_j)_{a,b} u_j + u_{n+1} + \xi \bigg) H(\lambda; \vec{u}, u_{n+1}) d\vec{u} du_{n+1} \bigg|$$

$$\le \int_{\mathbb{R}^n} |f(\vec{u})| H(\lambda : \vec{u})$$

$$\int_{\mathbb{R}} \bigg| \psi\bigg(\sum_{j=1}^n (1,\alpha_j)_{a,b} u_j + u_{n+1} + \xi \bigg) \bigg| H(\lambda; u_{n+1}) du_{n+1} d\vec{u}$$

$$\le \int_{\mathbb{R}^n} |f(\vec{u})| H(\lambda : \vec{u}) \bigg(\int_{\mathbb{R}} H(\lambda; u_{n+1}) du_{n+1} \bigg)^{\frac{1}{2}}$$

$$\bigg(\int_{\mathbb{R}} \bigg| \psi\bigg(\sum_{j=1}^n (1,\alpha_j)_{a,b} u_j + u_{n+1} + \xi \bigg) \bigg|^2 H(\lambda; u_{n+1}) du_{n+1} \bigg)^{\frac{1}{2}} d\vec{u}$$

$$\le \bigg[\int_{\mathbb{R}^n} |f(\vec{u})|^2 H(\lambda; \vec{u})$$

$$\bigg(\int_{\mathbb{R}} \bigg| \psi\bigg(\sum_{j=1}^n (1,\alpha_j)_{a,b} u_j + u_{n+1} + \xi \bigg) \bigg|^2 H(\lambda; u_{n+1}) du_{n+1} \bigg) d\vec{u} \bigg]^{\frac{1}{2}}$$

$$\bigg(\int_{\mathbb{R}^n} H(\lambda; \vec{u}) d\vec{u} \bigg)^{\frac{1}{2}} \bigg(\int_{\mathbb{R}} H(\lambda; u_{n+1}) du_{n+1} \bigg)^{\frac{1}{2}}.$$

Therefore, we have

(3.9) $$\|I_\lambda^{\mathrm{an}}(F)\psi\|_2^2$$

$$\le M_1M_2\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}}|f(\vec{u})|^2H(\lambda;\vec{u})$$
$$\left|\psi\Big(\sum_{j=1}^{n}(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\Big)\right|^2H(\lambda;u_{n+1})du_{n+1}d\vec{u}d\xi$$
$$\le M_1^2M_2\|\psi\|_2^2\int_{\mathbb{R}^n}|f(\vec{u})|^2H(\lambda;\vec{u})d\vec{u}$$

where

$$M_1=\int_{\mathbb{R}}H(\lambda;u_{n+1})du_{n+1}\quad\text{and}\quad M_2=\int_{\mathbb{R}^n}H(\lambda;\vec{u})d\vec{u}. \tag{3.10}$$

Therefore, for all $\lambda\in\mathbb{C}_+$, we have

$$\|I_\lambda^{\text{an}}(F)\|\le M_1M_2^{1/2}\Big(\int_{\mathbb{R}^n}|f(\vec{u})|^2H(\lambda;\vec{u})d\vec{u}\Big)^{1/2}.$$

Hence we complete the proof as desired. □

4. Analytic operator-valued generalized Feynman integral

In this section we study the analytic operator-valued generalized Feynman integral of functional F given by equation (3.1). We first give the existence of the analytic operator-valued generalized Feynman integral.

Let $I_\lambda^{\text{an}}(F)$ be the analytic operator-valued function space integral. Suppose there exists an operator $J_q^{\text{an}}(F)$ in $\mathcal{L}(L^2(\mathbb{R},\nu_\delta),L^2(\mathbb{R},\nu_{-\sigma}))$ for some $\delta,\sigma>0$ such that

$$\|I_\lambda^{\text{an}}(F)\psi-J_q^{\text{an}}(F)\psi\|_{L^2(\mathbb{R},\nu_{-\sigma})}\to 0 \tag{4.1}$$

as $\lambda\to -iq$ through $\mathbb{C}_+$where ν is a measure on $\mathcal{B}(\mathbb{R})$ with $d\nu_\delta=\exp\{\delta\eta^2\}d\eta$; then $J_q^{\text{an}}(F)$ is called the analytic operator-valued generalized Feynman integral of F with parameter q.

Fix $q\in\mathbb{R}-\{0\}$. Then as $\lambda\to -iq$ through values in $\mathbb{C}_+$, $c=\text{Re}(\sqrt{\lambda})\to\sqrt{|q|/2}$ and $d=\text{Im}(\sqrt{\lambda})\to\sqrt{|q|/2}$. Then for all $\lambda\in\tilde{\mathbb{C}}_+$, $\sqrt{\lambda}=c+di$ with $c^2-d^2>0$

$$\begin{aligned}&|H(\lambda;u_1,\cdots,u_n,u_{n+1})|\\&=\exp\Big\{-\sum_{j=1}^{n}\frac{[(c^2-d^2)u_j^2-2cA_ju_j+A_j^2]}{2B_j}\\&\qquad-\frac{[(c^2-d^2)u_{n+1}^2-2cA_pu_{n+1}+A_p^2]}{2B_p}\Big\}\end{aligned} \tag{4.2}$$

$$\le \exp\left\{\sum_{j=1}^{n}\frac{cA_ju_j}{B_j}+\frac{cA_pu_{n+1}}{B_p}\right\}$$
$$\le \exp\left\{\sum_{j=1}^{n}\frac{\sqrt{|q|}|A_ju_j|}{B_j}+\frac{\sqrt{|q|}|A_pu_{n+1}|}{B_p}\right\}.$$

Theorem 4.1. *Let F be given by equation* (3.1) *with*

$$\int_{\mathbb{R}^n}|f(\vec{u})|\exp\left\{\sum_{j=1}^{n}\frac{\sqrt{|q|}|A_ju_j|}{B_j}\right\}d\vec{u}<\infty \tag{4.3}$$

for $q\in\mathbb{R}-\{0\}$. Then the analytic operator-valued generalized Feynman integral of F, $J_q^{an}(F)$ exists and is given by the formula

(4.4)
$$(J_q^{an}(F)\psi)(\xi)=K(-iq;p)\int_{\mathbb{R}}\int_{\mathbb{R}^n}f(\vec{u})$$
$$\cdot\,\psi\bigg(\sum_{j=1}^{n}(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\bigg)H(-iq;\vec{u},u_{n+1})d\vec{u}du_{n+1}$$

where p is given by equation (3.4).

Proof. By using equations (4.2) and (4.3), we have for all $\psi\in L^2(\mathbb{R},\nu_\delta)$

(4.5)
$$\bigg|\int_{\mathbb{R}}\int_{\mathbb{R}^n}f(\vec{u})\psi\bigg(\sum_{j=1}^{n}(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\bigg)H(-iq;\vec{u},u_{n+1})d\vec{u}du_{n+1}\bigg|$$
$$\le\int_{\mathbb{R}^n}|f(\vec{u})|\exp\left\{\sum_{j=1}^{n}\frac{\sqrt{|q|}|A_ju_j|}{B_j}\right\}$$
$$\int_{\mathbb{R}}\bigg|\psi\bigg(\sum_{j=1}^{n}(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\bigg)\bigg|\exp\left\{\frac{\sqrt{|q|}|A_pu_{n+1}|}{B_p}\right\}du_{n+1}d\vec{u}$$
$$\le\int_{\mathbb{R}^n}|f(\vec{u})|\exp\left\{\sum_{j=1}^{n}\frac{\sqrt{|q|}|A_ju_j|}{B_j}\right\}$$
$$\bigg(\int_{\mathbb{R}}\exp\left\{\frac{2\sqrt{|q|}|A_pu_{n+1}|}{B_p}-\frac{\delta u_{n+1}^2}{2}\right\}du_{n+1}\bigg)^{\frac{1}{2}}$$

$$\left(\int_{\mathbb{R}}\left|\psi\left(\sum_{j=1}^n(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\right)\right|^2\exp\left\{\frac{\delta u_{n+1}^2}{2}\right\}du_{n+1}\right)^{\frac{1}{2}}d\vec{u}$$
$$=M_1^{1/2}\|\psi\|_{L^2(\mathbb{R},\nu_\delta)}\int_{\mathbb{R}^n}|f(\vec{u})|\exp\left\{\sum_{j=1}^n\frac{\sqrt{|q|}|A_ju_j|}{B_j}\right\}d\vec{u}$$

where

$$M_1=\int_{\mathbb{R}}\exp\left\{\frac{2\sqrt{|q|}|A_pu_{n+1}|}{B_p}-\frac{\delta u_{n+1}^2}{2}\right\}du_{n+1}.$$

Therefore we have

(4.6)
$$\|J_q^{\mathrm{an}}(F)\psi\|^2_{L^2(\mathbb{R},\nu_{-\sigma})}$$
$$\le M_1\|\psi\|^2_{L^2(\mathbb{R},\nu_\delta)}\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|f(\vec{u})|\exp\left\{\sum_{j=1}^n\frac{\sqrt{|q|}|A_ju_j|}{B_j}\right\}d\vec{u}\right)^2d\nu_{-\sigma}(\xi)$$

and

$$\|J_q^{\mathrm{an}}(F)\|\le M_1^{1/2}\left(\int_{\mathbb{R}^n}|f(\vec{u})|\exp\left\{\sum_{j=1}^n\frac{\sqrt{|q|}|A_ju_j|}{B_j}\right\}d\vec{u}\right).$$

To show that $J_q^{\mathrm{an}}(F)$ exists and is given by equation (4.4) it suffices to show that

$$\lim_{\lambda\to -iq}\int_{\mathbb{R}}\left|(I_\lambda^{\mathrm{an}}(F)\psi)(\xi)-(J_q^{\mathrm{an}}(F)\psi)(\xi)\right|^2d\nu_{-\sigma}(\xi)=0. \tag{4.7}$$

But for all $\lambda\in\mathbb{C}_+$, $\sqrt{\lambda}=c+di$ with $c^2-d^2>0$,

(4.8)
$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}^n}f(\vec{u})\psi\left(\sum_{j=1}^n(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\right)H(\lambda;\vec{u},u_{n+1})d\vec{u}du_{n+1}\right.$$
$$\left.-\int_{\mathbb{R}}\int_{\mathbb{R}^n}f(\vec{u})\psi\left(\sum_{j=1}^n(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\right)H(-iq;\vec{u},u_{n+1})d\vec{u}du_{n+1}\right|^2$$
$$\le 2\left|\int_{\mathbb{R}}\int_{\mathbb{R}^n}f(\vec{u})\psi\left(\sum_{j=1}^n(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\right)H(\lambda;\vec{u},u_{n+1})d\vec{u}du_{n+1}\right|^2$$

$$+2\bigg|\int_{\mathbb{R}}\int_{\mathbb{R}^n} f(\vec{u})\psi\bigg(\sum_{j=1}^{n}(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\bigg)H(-iq;\vec{u},u_{n+1})d\vec{u}du_{n+1}\bigg|^2$$

$$\leq 2\bigg(\int_{\mathbb{R}^n}|f(\vec{u})|\exp\bigg\{\sum_{j=1}^{n}\frac{cA_ju_j}{B_j}\bigg\}$$

$$\bigg(\int_{\mathbb{R}}\exp\bigg\{-\frac{\delta u_{n+1}^2}{2B_p}+\frac{2cA_pu_{n+1}}{B_p}\bigg\}du_{n+1}\bigg)^{\frac{1}{2}}$$

$$\bigg(\int_{\mathbb{R}}\bigg|\psi\bigg(\sum_{j=1}^{n}(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\bigg)\bigg|^2\exp\bigg\{\frac{\delta u_{n+1}^2}{2B_p}\bigg\}du_{n+1}\bigg)^{\frac{1}{2}}d\vec{u}\bigg)^2$$

$$+2\bigg(\int_{\mathbb{R}^2}|f(\vec{u})|\exp\bigg\{\sum_{j=1}^{n}\frac{\sqrt{-iq}A_ju_j}{B_j}\bigg\}$$

$$\bigg(\int_{\mathbb{R}}\exp\bigg\{-\frac{\delta u_{n+1}^2}{2B_p}+\frac{2\sqrt{-iq}A_pu_{n+1}}{B_p}\bigg\}du_{n+1}\bigg)^{\frac{1}{2}}$$

$$\bigg(\int_{\mathbb{R}}\bigg|\psi\bigg(\sum_{j=1}^{n}(1,\alpha_j)_{a,b}u_j+u_{n+1}+\xi\bigg)\bigg|^2\exp\bigg\{\frac{\delta u_{n+1}^2}{2B_p}\bigg\}du_{n+1}\bigg)^{\frac{1}{2}}d\vec{u}\bigg)^2$$

$$=4\|\psi\|_{L^2(\mathbb{R},\nu_\delta)}M_1\bigg(\int_{\mathbb{R}^n}|f(\vec{u})|\exp\bigg\{\sum_{j=1}^{n}\frac{cA_ju_j}{B_j}\bigg\}d\vec{u}\bigg)^2.$$

Hence by using equations (3.8), (4.5) and the Dominated Convergence Theorem, we have the desired result. □

In the following example, we exhibit the analytic operator-valued generalized Feynman integral $J_q^{\text{an}}(F)$.

Example 4.2. *Let* $f(u_1,u_2)=\exp\{-(u_1^2+u_2^2)\}$ *and let*

$$F(x)=f\bigg(\int_0^T\alpha_1(t)dx(t),\int_0^T\alpha_2(t)dx(t)\bigg)$$

where $\alpha\equiv\alpha_1(t)=(b(T)+|a|(T))^{-1/2}$ *and* $\{\alpha_1,\alpha_2\}$ *are orthonormal. Then the analytic operator-valued generalized Feynman integral of* F, $J_q^{\text{an}}(F)$ *exists and is given by the formula*

$$(J_q^{\text{an}}(F)\psi)(\xi)$$
$$=\bigg(\prod_{j=1}^{2}\frac{-iq}{2\pi B_j}\bigg)^{\frac{1}{2}}\int_{\mathbb{R}^2}\exp\{-(u_1^2+u_2^2)\}\psi(\alpha^{-1}u_1+\xi)$$

$$\cdot \exp\bigg\{-\sum_{j=1}^{2}\frac{[\sqrt{-iq}u_j - A_j]^2}{2B_j}\bigg\}du_1du_2$$

$$= \bigg(\frac{-iq}{2B_2 - iq}\bigg)^{\frac{1}{2}}\exp\bigg\{\frac{(-iq)A_2^2(2B_2+iq)}{2B_2(4B_2^2+q^2)}\bigg\}$$

$$\cdot \bigg(\frac{-iq}{2\pi B_1}\bigg)^{\frac{1}{2}}\int_{\mathbb{R}}\psi(\alpha^{-1}u_1+\xi)\exp\bigg\{-u_1^2 - \frac{[\sqrt{-iq}u_1 - A_1]^2}{2B_1}\bigg\}du_1.$$

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