JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **23**, No. 1, March 2010

EXISTENCE OF NASH EQUILIBRIUM IN A COMPACT ACYCLIC STRATEGIC GAME

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Abstract. In this note, we will prove a new equilibrium existence theorem for a compact acyclic strategic game by using Begle's fixed point theorem.

1. Introduction

The classical results of Debreu [3], Nash [6], and Nikaido-Isoda[7] have served as basic references for the existence of Nash equilibrium for non-cooperative strategic games. In all of them, convexity of strategy spaces, continuity and convexity of payoff functions were assumed. Till now, there have been a number of generalizations and applications of those theorems have been found in several areas, e.g., see [1,4,5,8,9] and references therein. However, in some economic models, neither convexity nor quasiconvexity assumptions can be guaranteed, e.g., the best response correspondences in the pure strategy spaces of auctions, political contests, models of imperfect competition are not convex-valued (e.g., see [1]). Hence we shall need general concepts for removing the convexity assumptions of strategy spaces and payoff functions. For these purposes, some homological conditions are suitable in general. Indeed, the contractible and acyclic conditions can be useful for many existence results in equilibrium theories by applying Eilenberg-Montgomery's fixed point theorem instead of Kakutani's fixed point theorem as in [3]. In this direction, Lu [5] recently investigated some existence of pure-strategy

Received September 08, 2009; Accepted January 18, 2010.
2000 Mathematics Subject Classification: Primary 90A13, 52A07.
Key words and phrases: Nash equilibrium, acyclic, lc space, Begle's fixed point theorem.
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* This work was supported by the research grant of the Chungbuk National University in 2009.

Nash equilibria with acyclic values by using Gorniewicz's fixed point theorem.

In this paper, using Begle's fixed point theorem, we will prove a new existence theorem of pure-strategy Nash equilibrium in a compact acyclic strategic game which is comparable with equilibrium existence theorems due to Nash, Debreu and Lu in several aspects.

2. Preliminaries

Let I be a (possibly infinite) set of players. For each $i \in I$, let X_i be a nonempty topological space as an action space, and denote $X_{\hat{i}} := \Pi_{j \in I \setminus \{i\}} X_j$. For an action profile $x = (x_i)_{i \in I} \in X$, we shall write $x = (x_{\hat{i}}, x_i) \in X = X_{\hat{i}} \times X_i$. Recall some terminologies which generalize the convex condition. A nonempty space X is *contractible* if the identity map is homotopic to a constant map, and a nonempty space X is *acyclic* if it is connected and its Čech homology $H_n(X; F)$ is zero for each $n \geq 1$, where F is a fixed field. Then it is easy to see that every convex set is contractible, and a contractible space is acyclic. However, an acyclic space need not be contractible (e.g., see [8,9]). In [2], Begle introduced a common notion of an lc space which contains both a convex subset of a locally convex space, and an absolute neighborhood retract(ANR), and for the definition and properties of lc space, see [2].

The following Begle fixed point theorem, which is a generalization of the Eilenberg-Montgomery fixed point theorem into an lc space, is essential in the acyclic settings:

LEMMA 2.1. ([2]) *Let X be a nonempty compact acyclic lc space and $T : X \to 2^X$ have closed graph in $X \times X$ with nonempty acyclic values. Then T has a fixed point.*

If X is a compact convex subset of a locally convex topological vector space, then X is compact acyclic so that Fan-Glicksberg's fixed point theorem is a consequence of Lemma 2.1. For general minimax theorems and applications, there have been many generalization of Lemma 2.1 in general spaces by several authors, e.g., see [8,9]. For the other standard notations and terminologies, we shall refer to [4,8,9].

3. Existence of Nash equilibrium in a compact acyclic setting

We now recall some notions and terminologies in Nash equilibrium for a pure strategic game. Let I be a (possibly infinite) set of players. Then, a *non-cooperative strategic game of normal form* is an ordered tuples $\Gamma = (X_i; f_i)_{i \in I}$ where for each player $i \in I$, the nonempty set X_i is a player's pure strategy space, and $f_i : X = \Pi_{i\in I} X_i \to \mathbb{R}$ is a player's payoff function. The set X, *joint strategy space*, is the Cartesian product of the individual strategy spaces, and the element of X_i is called a *strategy.* A strategy profile $(\bar{x}_i)_{i\in I} \in X$ is called the *Nash equilibrium* for the game Γ if the following system of inequalities hold: for each $i \in I$,

$$f_i(\bar{x}_{\hat{i}}, \bar{x}_i) \ \geq \ f_i(\bar{x}_{\hat{i}}, x_i) \quad \text{for all } x_i \in X_i.$$

In [7], Nikaido and Isoda first introduced the total sum of all payoffs of players for the existence of Nash equilibrium in the case of an n-person game, and we may generalize the total sum of payoff functions for the infinite generalized strategic game. For this, as in [4], we shall need the following condition: Let I be a countably infinite set of players, and let $\Gamma = (X_i; f_i)_{i\in I}$ be a non-cooperative strategic game of normal form. Then $\{f_i : X_{\hat{i}} \times X_i \to \mathbb{R} \mid i \in I\}$ is said to be *unconditionally summable* if for each $z_i \in X_i$ and $y_{\hat{i}} \in X_{\hat{i}}$, any rearrangement $\sum_{j\in I} f_j(y_{\hat{j}}, z_j)$ of the infinite sum $\sum_{i\in I} f_i(y_{\hat{i}}, z_i)$ converges to the same real value. Indeed, the unconditionally summable condition should be needed in the infinite strategic game, and it should be noted that $\{f_i \mid i \in I\}$ is unconditionally summable if and only if the sum $\sum_{i\in I} f_i(y_{\hat{i}}, z_i)$ converges absolutely for each $z_i \in X_i$ and $y_{\hat{i}} \in X_{\hat{i}}$.

From now on, the unconditional summablity of the payoff functions is always assumed, and let us define the total sum of payoff function $H : X \times X \to \mathbb{R}$ associated with the infinite strategic game $\Gamma = (X_i; f_i)_{i\in I}$ as follows:

$$H(y, x) := \sum_{i\in I} f_i(x_{\hat{i}}, y_i), \quad \text{for each } \ x, y \in X = \Pi_{i\in I} X_i.$$

Then we can obtain the following equivalences which generalizes the total sum of payoff functions due to Nikaido-Isoda [7] and Lu [5]:

Lemma 3.1. *Let $\Gamma = (X_i; f_i)_{i\in I}$ be a non-cooperative strategic game of normal form where I be a countably infinite set of players.*

Then the followings are equivalent:

(1) $\bar{x} \in X = \Pi_{i\in I} X_i$ is a Nash equilibrium for Γ;

(2) $f_i(\bar{x}_1, \ldots, \bar{x}_i, \ldots, \bar{x}_n, \ldots) \geq f_i(\bar{x}_1, \ldots, x_i, \ldots, \bar{x}_n, \ldots)$ *for all* $x_i \in X_i$ *and* $i \in I$;

(3) $H(\bar{x}, \bar{x}) \geq H(y, \bar{x})$ *for all* $y \in X$.

Proof. The equivalence of (1) and (2) follows immediately from the definition of a Nash equilibrium in case of the infinite set of players. The implication (2) $\Rightarrow$ (3) is obtained by adding both sides in the inequalities of all $i \in I$,

$$f_i(\bar{x}_1, \ldots, \bar{x}_i, \ldots, \bar{x}_n, \ldots) \geq f_i(\bar{x}_1, \ldots, y_i, \ldots, \bar{x}_n, \ldots).$$

To prove (3) implies (2), we fix i and take $y = (\bar{x}_{\hat{i}}, y_i)$ where $y_i \in X_i$. Then the inequality $H(\bar{x}, \bar{x}) \geq H(y, \bar{x})$ may be written as

$$f_i(\bar{x}_{\hat{i}}, \bar{x}_i) - f_i(\bar{x}_{\hat{i}}, y_i) + \sum_{j \neq i} \big(f_j(\bar{x}_{\hat{j}}, \bar{x}_j) - f_j(\bar{x}_{\hat{j}}, y_j)\big) \geq 0.$$

Since $y_j = \bar{x}_j$ whenever $j \neq i$, we have

$$f_i(\bar{x}_{\hat{i}}, \bar{x}_i) - f_i(\bar{x}_{\hat{i}}, y_i) \geq 0 \qquad \text{for all } \ i \in I \text{ and } \ y_i \in X_i,$$

which proves (2). □

Using Lemma 2.1, we now prove the following new existence theorem of Nash equilibrium for a compact acyclic strategic game:

THEOREM 3.2. *Let* $\Gamma = (X_i\,; f_i)_{i \in I}$ *be a non-cooperative strategic game of normal form, where* I *be a countably infinite set of players, with the strategy set* X_i *being nonempty subset in a topological vector space* E, $X = \Pi_{i \in I} X_i$, *and* $f_i : X \to \mathbb{R}$ *being player's payoff function. Assume that* X *is a compact acyclic lc space, and the total sum of payoff function* $H : X \times X \to \mathbb{R}$ *satisfy the following conditions:*

(1) $(x, y) \mapsto H(y, x) - H(x, x)$ *is lower semicontinuous in* X;

(2) for each $x \in X$, $\{y \in X \mid H(x, x) < H(y, x)\}$ *is convex in* X;

(3) for each $y \in X$, $\{x \in X \mid H(y, x) \leq H(x, x)\}$ *is acyclic.*

Then the strategic game Γ *has a Nash equilibrium.*

Proof. Suppose the contrary, then by Lemma 3.1, for all $x \in X$, there exists $y \in X$ such that $H(x, x) < H(y, x)$. For each $y \in X$, we let

$$N(y) := \{x \in X \mid H(x, x) < H(y, x)\}.$$

By the assumption (1), each $N(y)$ is open in X and we have $\bigcup_{y \in X} N(y) = X$. Since X is compact, there exists a finite number of nonempty open sets $N(y_1), \ldots, N(y_n)$ such that $\bigcup_{i=1}^{n} N(y_i) = X$. Let $\{\alpha_i \mid i = 1, \ldots, n\}$ be the continuous partition of unity subordinated to the open covering $\{N(y_i) \mid i = 1, \ldots, n\}$ of X.

We now define a mapping $\phi : X \to E$ by

$$\phi(x) := \sum_{i=1}^{n} \alpha_i(x)\, y_i \qquad \text{for each} \quad x \in X.$$

Then, ϕ is a continuous mapping since each α_i is continuous. By the assumption (2), for fixed $x \in X$, the set $\{y \in X \mid H(x,x) < H(y,x)\}$ is convex in X so that we have

$$\phi(x) \in co\,\{y_i \in X \mid \alpha_i(x) \neq 0\} \subseteq \{y \in X \mid H(x,x) < H(y,x)\};$$

and we know that ϕ maps X into X.

Next, we define a multimap $T : X \to 2^X$ by

$$T(y) := \{x \in X \mid H(y,x) \leq H(x,x)\} \qquad \text{for each} \quad y \in X.$$

By the assumption (1) again, the set $\{(x,y) \in X \times X \mid H(y,x) - H(x,x) \leq 0\}$ is nonempty closed in $X \times X$ and hence it is compact. Therefore, for each $y \in X$, $T(y)$ is compact. By the assumption (3), each $T(y)$ is a nonempty acyclic subset of X. Next, it is easy to see that T has a closed graph in $X \times X$. In fact, for any nets $(x_\alpha) \to x_o$, $(y_\alpha) \to y_o$, $x_\alpha \in T(y_\alpha)$, we have $H(y_\alpha, x_\alpha) \leq H(x_\alpha, x_\alpha)$. Since the mapping $(x,y) \mapsto H(y,x) - H(x,x)$ is lower semicontinuous, we have $H(y_o, x_o) \leq H(x_o, x_o)$. Hence $x_o \in T(y_o)$ and T has a closed graph in $X \times X$.

Finally, we define a multimap $S : X \to 2^X$ by

$$S(x) = (T \circ \phi)(x) \qquad \text{for each} \quad x \in X.$$

Since T has a closed graph and ϕ is continuous, S has a closed graph in $X \times X$ and each $S(x)$ is nonempty compact acyclic. Therefore, by Lemma 2.1, there exists a point $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x}) = T(\phi(\bar{x}))$. Let $x^* = \phi(\bar{x}) \in X$; then

$$\bar{x} \in T(x^*) = \{x \in X \mid H(x^*, x) \leq H(x,x)\}$$

so that $H(x^*, \bar{x}) \leq H(\bar{x}, \bar{x})$. On the other hand, since $x^* = \phi(\bar{x})$, by $(*)$, we have

$$x^* = \phi(\bar{x}) \in \{y \in X \mid H(\bar{x}, \bar{x}) < H(y, \bar{x})\}$$

so that $H(x^*, \bar{x}) > H(\bar{x}, \bar{x})$ which is a contradiction. This completes the proof. □

Remark 3.3. (1) Theorem 3.2 is a new equilibrium existence theorem which is comparable with the previous existence theorems of Nash equilibrium due to Nash [6], Nikaido-Isoda [7], and Lu [5] in the following aspects:

(a) the set I of players need not be a finite set;

(b) every strategy set X_i need not be convex, and it is sufficient that X is an acyclic lc space;

(c) every payoff function f_i need not be (quasi)concave nor continuous on X; however, we shall need the weaker convex assumption (2) and acyclic assumption (3).

(2) Theorem 3.2 is different from the existence theorems of Nash equilibrium due to Park [8,9] using admissible compact convex assumption. Indeed, in Theorem 3.2, X need not be admissible.

Finally, it should be noted that our theorem can be considered as a basic result on non-convex general topological spaces so that it is possible to generalize numerous existence results on equilibria in the game theory into general settings by using suitable fixed point theorems as in Theorem 3.2.

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