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POISSON HOPF STRUCTURE INDUCED BY THE UNIVERSAL ENVELOPING ALGEBRA OF A GRADED LIE ALGEBRA

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Abstract. Let G be an abelian group, α an antisymmetric bicharacter on G and $\mathfrak{g}$ a (G, α)-Lie algebra. Here we give a complete proof for that the associated graded algebra determined by a natural filtration in the universal enveloping algebra $U(\mathfrak{g})$ is a (G, α)-Poisson Hopf algebra.

1. Introduction

Let $\mathfrak{g}$ be a (classical) Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. Then $U(\mathfrak{g})$ has a filtration $\{U_n\}_{n=0}^{\infty}$, where U_n is the subspace spanned by monomials with length less than or equal to n. Hence there exists an associated graded algebra

$$\mathrm{gr}(U) = \bigoplus_{n=0}^{\infty}(U_n/U_{n-1}), \quad U_{-1} = 0.$$

By [2, 2.8.7], $\mathrm{gr}(U)$ is a Poisson algebra with Poisson bracket

$$\{\overline{x}, \overline{y}\} = \overline{xy} - \overline{yx} = \overline{[x, y]}$$

for all $x, y \in \mathfrak{g}$, where $\overline{x}$ and $\overline{y}$ are the canonical images of x and y, respectively. This arises the question that the associated graded algebra determined by a natural filtration in the universal enveloping algebra of a graded Lie algebra is a graded Poisson Hopf algebra. Here we establishes that this is true.

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Let G be an abelian group, α an anti-symmetric bicharacter on G and $\mathfrak{g}$ a (G,α)-Lie algebra. There is known that the universal enveloping algebra of $\mathbb{Z}$-graded Lie algebra has a certain Hopf algebra structure. But we do not find a complete proof for this fact. Here we give a complete proof that the universal enveloping algebra $U(\mathfrak{g})$ of (G,α)-Lie algebra $\mathfrak{g}$ is a (G,α)-Hopf algebra. Next we find that the universal enveloping algebra $U(\mathfrak{g})$ has a natural filtration determined by length of monomials and give a complete proof for that its associated graded algebra is a (G,α)-Poisson Hopf algebra.

We assume throughout the paper that $\mathbf{k}$ denotes a field of characteristic zero and all vector spaces are over $\mathbf{k}$.

2. Universal enveloping algebras of graded Lie algebras

2.1.

Let G be an abelian group and let α be an antisymmetric bicharacter on G, that is, $\alpha : G \times G \longrightarrow \mathbf{k}^{\times} = \mathbf{k} \setminus \{0\}$ is a map satisfying the conditions

$$\alpha(a,b) = \alpha(b,a)^{-1}, \quad \alpha(ab,c) = \alpha(a,c)\alpha(b,c)$$

for all $a,b,c \in G$. Recall the definition of (G,α)-Lie algebra. A G-graded vector space $\mathfrak{g} = \underset{a\in G}{\oplus} \mathfrak{g}_a$ is said to be a (G,α)-Lie algebra if there exists a bilinear map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (x,y) \mapsto [x,y]$$

satisfying the following conditions

(i) $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{ab}$. ($G$-gradation)
(ii) $[x,y] = -\alpha(a,b)[y,x]$ for $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$. ($\alpha$-skew symmetry)
(iii) $\alpha(c,a)[x,[y,z]] + \alpha(a,b)[y,[z,x]] + \alpha(b,c)[z,[x,y]]=0$ for all $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b, z \in \mathfrak{g}_c$. ($\alpha$-Jacobi identity)

For each $a \in G$, an element $x \in \mathfrak{g}_a$ is said to be homogeneous of degree a and we will write $|x| = a$ for convenience.

Set

$$U(\mathfrak{g}) = T(\mathfrak{g})/I,$$

where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ and I is the two-sided ideal of $T(\mathfrak{g})$ generated by

$$x \otimes y - \alpha(|x|,|y|)y \otimes x - [x,y]$$

for all homogeneous elements $x,y \in \mathfrak{g}$. Then the algebra $U(\mathfrak{g})$, called the universal enveloping algebra of $\mathfrak{g}$, is spanned by monomials which are products of finite homogeneous elements in $\mathfrak{g}$, called homogeneous monomials of $U(\mathfrak{g})$. (See [1, 2.4].) Since each homogeneous monomial has a degree induced from the degrees of homogeneous elements in $\mathfrak{g}$, $U(\mathfrak{g})$ is a G-graded algebra.

Moreover each homogeneous monomial of $U(\mathfrak{g})$ has the length. Denote by $\ell(X) = n$ the length of the homogeneous monomial $X = x_1x_2\cdots x_n$, where all x_i are homogeneous elements of $\mathfrak{g}$.

2.2.

Recall the semidirect product $A \rtimes_\alpha B$ for G-graded algebras A and B. The semidirect product $A \rtimes_\alpha B$ is the vector space $A \otimes B$ with multiplication

$$(a \otimes b)(c \otimes d) = \alpha(|b|, |c|)ac \otimes bd$$

for homogeneous elements $a \in A$ and $b \in B$. Refer to [3, 1.7] for the definition of (G, α)-Hopf algebra.

THEOREM 2.1. *Let $\mathfrak{g}$ be a (G, α)-Lie algebra. Then its universal enveloping algebra $U(\mathfrak{g})$ is a (G, α)-Hopf algebra. That is, $U(\mathfrak{g})$ satisfies the following three conditions:*

(i) There exists a G-graded algebra homomorphism

$$\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \rtimes_\alpha U(\mathfrak{g}), \quad x \mapsto x \otimes 1 + 1 \otimes x \quad (x \in \mathfrak{g}) \tag{2.1}$$

such that

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta,$$

where $U(\mathfrak{g}) \rtimes_\alpha [U(\mathfrak{g}) \rtimes_\alpha U(\mathfrak{g})]$ is identified with $[U(\mathfrak{g}) \rtimes_\alpha U(\mathfrak{g})] \rtimes_\alpha U(\mathfrak{g})$.

(ii) There exists a G-graded algebra homomorphism

$$\epsilon : U(\mathfrak{g}) \longrightarrow \mathbf{k}, \quad x \mapsto 0 \quad (x \in \mathfrak{g}) \tag{2.2}$$

such that

$$\sum \epsilon(z')z'' = z, \quad \sum z'\epsilon(z'') = z$$

for all $z \in U(\mathfrak{g})$, where $\Delta(z) = \sum z' \otimes z''$ and $\mathbf{k}$ has the trivial grading

$$\mathbf{k}_e = \mathbf{k}, \quad \mathbf{k}_a = 0$$

for all $e \neq a \in G$.

(iii) Denote by $U(\mathfrak{g})^{\mathrm{op}}_\alpha$ the algebra $U(\mathfrak{g})$ with a new multiplication

$$x \cdot y = \alpha(|x|, |y|)yx$$

for any homogeneous monomials $x, y \in U(\mathfrak{g})$. Then there exists a G-graded algebra homomorphism

$$\sigma : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\mathrm{op}}_\alpha, \quad x \mapsto -x \quad (x \in \mathfrak{g}) \tag{2.3}$$

such that

$$\sum \sigma(z')z'' = \epsilon(z)1, \quad \sum z'\sigma(z'') = \epsilon(z)1$$

for all $z \in U(\mathfrak{g})$, *where* $\Delta(z) = \sum z' \otimes z''$.

Proof. (i) Since

$$\Delta(x)\Delta(y) - \alpha(|x|, |y|)\Delta(y)\Delta(x) - \Delta([x, y]) = 0$$

for all homogeneous elements $x, y \in \mathfrak{g}$, there exists a G-graded algebra homomorphism Δ given in (1). Note that $\Delta \otimes 1$ and $1 \otimes \Delta$ are algebra homomorphisms. Moreover

$$(\Delta \otimes 1)\Delta(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x = (1 \otimes \Delta)\Delta(x)$$

for all $x \in \mathfrak{g}$. Thus $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$ since $\mathfrak{g}$ generates $U(\mathfrak{g})$.

(ii) There exists an algebra homomorphism ϵ given in (2) since

$$\epsilon(x)\epsilon(y) - \alpha(|x|, |y|)\epsilon(y)\epsilon(x) - \epsilon([x, y]) = 0$$

for all homogeneous elements $x, y \in \mathfrak{g}$. For each homogeneous monomial $X = x_1x_2 \cdots x_n$ in $U(\mathfrak{g})$, let $\Delta(X) = \sum X' \otimes X''$. If $\ell(X) = n = 1$ then $X = x_1$ and $\sum \epsilon(X')X'' = x_1 = X$. Suppose that $n > 1$ and that $\sum \epsilon(Y')Y'' = Y$ for all homogeneous monomials Y with length less than n. Since

$$\begin{aligned}\Delta(X) = \Delta(Y)\Delta(x_n) &= \sum (Y' \otimes Y'')(x_n \otimes 1 + 1 \otimes x_n)\\ &= \sum \alpha(|Y''|, |x_n|)(Y'x_n) \otimes Y'' + \sum Y' \otimes (Y''x_n),\end{aligned}$$

where $Y = x_1x_2 \cdots x_{n-1}$, we have

$$\sum \epsilon(X')X'' = \sum \epsilon(Y')(Y''x_n) = Yx_n = X$$

by the induction hypothesis. The other equation $\sum z'\epsilon(z'') = z$ is proved similarly.

(iii) For homogeneous elements $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}\sigma(x) \cdot \sigma(y) - \alpha(|x|, |y|)\sigma(y) \cdot \sigma(x) - \sigma([x, y])&\\ = \alpha(|x|, |y|)yx - xy + [x, y]&\\ = -(xy - \alpha(|x|, |y|)yx - [x, y]) = 0.&\end{aligned}$$

Thus there exists a G-graded algebra homomorphism σ given in (3).

Continue the notation in the proof of (ii). If $n = 1$ then $X = x_1$ and $\sum \sigma(X')X'' = \sigma(x_1) + x_1 = 0 = \epsilon(X)1$. Suppose that $n > 1$ and that

$\sum \sigma(Y')Y'' = \epsilon(Y)1$ for all homogeneous monomials Y with length less than n. Then

$$\begin{aligned}\sum \sigma(X')X'' &= \sum \alpha(|Y''|,|x_n|)\sigma(Y'x_n)Y'' + \sum \sigma(Y')(Y''x_n)\\ &= \sum \alpha(|Y''|,|x_n|)[\sigma(Y')\cdot\sigma(x_n)]Y'' + \sum(\sigma(Y')Y'')x_n\\ &= -\alpha(|Y|,|x_n|)x_n \sum \sigma(Y')Y'' + \sum(\sigma(Y')Y'')x_n\\ &= -\alpha(|Y|,|x_n|)x_n\epsilon(Y) + \epsilon(Y)x_n = 0 = \epsilon(X)1\end{aligned}$$

by the induction hypothesis. The other equation $\sum z'\sigma(z'') = \epsilon(z)1$ is proved similarly. □

2.3.

Let $\mathfrak{g}$ be a (G,α)-Lie algebra. Then its universal enveloping algebra $U(\mathfrak{g})$ has a filtration $\mathfrak{U} = \{U_i \mid i = 0,1,\cdots\}$ such that

$$U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots,$$

where U_n is the subspace spanned by all homogeneous monomials X with length less than or equal to n. Suppose that the length of a homogeneous monomial X is n. In the associated graded algebra

$$\mathrm{gr}\mathfrak{U} = \bigoplus_{n=0}^{\infty}(U_n/U_{n-1}), \quad U_{-1} = 0,$$

the canonical image $\overline{X} = X + U_{n-1} \in \mathrm{gr}\mathfrak{U}$ has a G-grading $|X|$ induced by that of $\mathfrak{g}$. Thus the algebra $\mathrm{gr}\mathfrak{U}$ is a G-graded algebra.

2.4.

Theorem 2.2. *The associated graded algebra* $\mathrm{gr}\mathfrak{U}$ *is a* (G,α)*-Poisson Hopf algebra with Poisson bracket*

$$\{\overline{X},\overline{Y}\} = \overline{XY} - \alpha(|X|,|Y|)\overline{YX} \tag{2.4}$$

for all homogeneous monomials $X, Y \in U(\mathfrak{g})$*. That is,* $\mathrm{gr}\mathfrak{U}$ *is a* (G,α)*-Hopf algebra such that there exists a bilinear map* $\{\cdot,\cdot\} : \mathrm{gr}\mathfrak{U} \times \mathrm{gr}\mathfrak{U} \longrightarrow \mathrm{gr}\mathfrak{U}$*, called the Poisson bracket, satisfying*

(i) $\mathrm{gr}\mathfrak{U}$ *is a* (G,α)*-Lie algebra under the Poisson bracket* $\{\cdot,\cdot\}$.

(ii) $\{\overline{X},\overline{YZ}\} = \{\overline{X},\overline{Y}\}\overline{Z} + \alpha(|X|,|Y|)\overline{Y}\{\overline{X},\overline{Z}\}$ *for all homogeneous elements* $\overline{X},\overline{Y},\overline{Z} \in \mathrm{gr}\mathfrak{U}$*. (*$\alpha$*-Leibniz rule)*

(iii) $\Delta(\{\overline{X},\overline{Y}\}) = \{\Delta(\overline{X}),\Delta(\overline{Y})\}$ *for all* $\overline{X},\overline{Y} \in \mathrm{gr}\mathfrak{U}$*, where the bracket* $\{\cdot,\cdot\}$ *in* $\mathrm{gr}\mathfrak{U} \rtimes_\alpha \mathrm{gr}\mathfrak{U}$ *is given by*

$$\{\overline{X}\otimes\overline{Y},\overline{Z}\otimes\overline{W}\} = \alpha(|Y|,|Z|)\overline{XZ}\otimes\{\overline{Y},\overline{W}\} + \alpha(|Y|,|Z|)\{\overline{X},\overline{Z}\}\otimes\overline{YW}$$

for homogeneous elements $\overline{X}, \overline{Y}, \overline{Z}, \overline{W} \in gr\mathfrak{U}$.

Proof. It is proved by using the length of homogeneous monomials in $U(\mathfrak{g})$ that gr$\mathfrak{U}$ is a (G, α)-Hopf algebra with structure maps induced by those of $U(\mathfrak{g})$. Since gr$\mathfrak{U}$ is a G-graded algebra, it is (G, α)-Lie algebra with bracket (4). Moreover it is easy to see that (4) satisfies the α-Leibniz rule

$$\{\overline{X}, \overline{YZ}\} = \{\overline{X}, \overline{Y}\}\overline{Z} + \alpha(|X|, |Y|)\overline{Y}\{\overline{X}, \overline{Z}\}$$

for all homogeneous monomials $X, Y, Z \in U(\mathfrak{g})$.

It remains to prove that

$$\Delta(\{\overline{X}, \overline{Y}\}) = \{\Delta(\overline{X}), \Delta(\overline{Y})\} \tag{2.5}$$

for all homogeneous monomials $X, Y \in U(\mathfrak{g})$. We proceed by induction on $\ell(X) + \ell(Y)$ and simply write X for $\overline{X}$ for convenience. If $Y = 1$ then

$$\Delta(\{X, 1\}) = 0 = \{\Delta(X), 1 \otimes 1\} = \{\Delta(X), \Delta(1)\}.$$

If $X = 1$ then

$$\Delta(\{1, Y\}) = -\Delta(\{Y, 1\}) = 0 = \{1 \otimes 1, \Delta(Y)\} = \{\Delta(1), \Delta(Y)\}.$$

If $X = x$ and $Y = y$ for some homogeneous elements $x, y \in \mathfrak{g}$ then

$$\begin{gathered}\Delta(\{x, y\}) = [x, y] \otimes 1 + 1 \otimes [x, y] \\ \{\Delta(x), \Delta(y)\} = \{x \otimes 1 + 1 \otimes x, y \otimes 1 + 1 \otimes y\} = [x, y] \otimes 1 + 1 \otimes [x, y],\end{gathered}$$

thus $\Delta(\{x, y\}) = \{\Delta(x), \Delta(y)\}$. Suppose that $\ell(X) + \ell(Y) > 2$ and that (5) is true for all homogeneous monomials Z, W such that $\ell(Z) + \ell(W) < \ell(X) + \ell(Y)$. Thus $X = X_1X_2$ for some homogeneous monomials X_1, X_2 with $\ell(X_1) < \ell(X)$ and $\ell(X_2) < \ell(X)$, or $Y = Y_1Y_2$ for some homogeneous monomials Y_1, Y_2 with $\ell(Y_1) < \ell(Y)$ and $\ell(Y_2) < \ell(Y)$. Let $Y = Y_1Y_2$. We use the notation $\Delta(Z) = \sum Z' \otimes Z''$ for a homogeneous monomial $Z \in U(\mathfrak{g})$. Then

$$
\begin{aligned}
&\Delta(\{X, Y\}) = \Delta(\{X, Y_1Y_2\})\\
&= \Delta(\{X, Y_1\})\Delta(Y_2) + \alpha(|X|, |Y_1|)\Delta(Y_1)\Delta(\{X, Y_2\})\\
&= \{\Delta(X), \Delta(Y_1)\}\Delta(Y_2) + \alpha(|X|, |Y_1|)\Delta(Y_1)\{\Delta(X), \Delta(Y_2)\}\\
&= \sum \alpha(|X''|, |Y_1'|)(X'Y_1' \otimes \{X'', Y_1''\})(Y_2' \otimes Y_2'')\\
&\quad + \sum \alpha(|X''|, |Y_1'|)(\{X', Y_1'\} \otimes X''Y_1'')(Y_2' \otimes Y_2'')\\
&\quad + \sum \alpha(|X|, |Y_1|)\alpha(|X''|, |Y_2'|)(Y_1' \otimes Y_1'')(X'Y_2' \otimes \{X'', Y_2''\})\\
&\quad + \sum \alpha(|X|, |Y_1|)\alpha(|X''|, |Y_2'|)(Y_1' \otimes Y_1'')(\{X', Y_2'\} \otimes X''Y_2'')\\
&= \sum \alpha(|X''|, |Y_1'|)\alpha(|X''||Y_1''|, |Y_2'|)X'Y_1'Y_2' \otimes \{X'', Y_1''\}Y_2''\\
&\quad + \sum \alpha(|X''|, |Y_1'|)\alpha(|X''||Y_1''|, |Y_2'|)\{X', Y_1'\}Y_2' \otimes X''Y_1''Y_2''\\
&\quad + \sum \alpha(|X|, |Y_1|)\alpha(|X''|, |Y_2'|)\alpha(|Y_1''|, |X'||Y_2'|)Y_1'X'Y_2' \otimes Y_1''\{X'', Y_2''\}\\
&\quad + \sum \alpha(|X|, |Y_1|)\alpha(|X''|, |Y_2'|)\alpha(|Y_1''|, |X'||Y_2'|)Y_1'\{X', Y_2'\} \otimes Y_1''X''Y_2''
\end{aligned}
$$

by the induction hypothesis and

$$
\begin{aligned}
&\{\Delta(X), \Delta(Y)\} = \{\Delta(X), \Delta(Y_1)\Delta(Y_2)\}\\
&= \{\sum(X' \otimes X''), \sum(Y_1' \otimes Y_1'')\sum(Y_2' \otimes Y_2'')\}\\
&= \sum \alpha(|X''|, |Y_1'||Y_2'|)\alpha(|Y_1''|, |Y_2'|)X'Y_1'Y_2' \otimes \{X'', Y_1''\}Y_2''\\
&\quad + \sum \alpha(|X''|, |Y_1'||Y_2'|)\alpha(|Y_1''|, |Y_2'|)\{X', Y_1'\}Y_2' \otimes X''Y_1''Y_2''\\
&\quad + \sum \alpha(|X''|, |Y_1'||Y_2'|)\alpha(|Y_1''|, |Y_2'|)\alpha(|X_2''|, |Y_1''|)X'Y_1'Y_2' \otimes Y_1''\{X'', Y_2''\}\\
&\quad + \sum \alpha(|X''|, |Y_1'||Y_2'|)\alpha(|Y_1''|, |Y_2'|)\alpha(|X'|, |Y_1'|)Y_1'\{X', Y_2'\} \otimes X''Y_1''Y_2''.
\end{aligned}
$$

Thus $\Delta(\{X, Y\}) = \{\Delta(X), \Delta(Y)\}$ for $Y = Y_1Y_2$. If $X = X_1X_2$ then

$$
\begin{aligned}
\Delta(\{X, Y\}) &= -\alpha(|X|, |Y|)\Delta(\{Y, X\}) = -\alpha(|X|, |Y|)\{\Delta(Y), \Delta(X)\}\\
&= -\alpha(|X|, |Y|)\sum \alpha(|Y''|, |X'|)(Y'X' \otimes \{Y'', X''\})\\
&\quad - \alpha(|X|, |Y|)\sum \alpha(|Y''|, |X'|)(\{Y', X'\} \otimes Y''X'')\\
&= \{\Delta(X), \Delta(Y)\}
\end{aligned}
$$

by the case $Y = Y_1Y_2$. Hence (5) holds. This completes the proof. $\square$

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