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ON THE PARITY OF THE CLASS NUMBER OF SOME REAL BIQUADRATIC FUNCTION FIELD

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Abstract. Let $k = \mathbb{F}_q(T)$ and $\mathbb{A} = \mathbb{F}_q[T]$. In this paper, we obtain the the criterion for the parity of the ideal class number $h(\mathcal{O}_K)$ of the real biquadratic function field $K = k(\sqrt{P_1}, \sqrt{P_2})$, where $P_1, P_2 \in \mathbb{A}$ be two distinct monic primes of even degree.

1. Introduction and statement of main result

In the paper [2], Kucera has determined the parity of the class number of any real biquadratic field $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ in terms of quadratic residue symbols, where p and q are different primes with $p \equiv q \equiv 1 \bmod 4$. In this paper we give a similar result for some real biquadratic function fields.

Let $k = \mathbb{F}_q(T)$ and $\mathbb{A} = \mathbb{F}_q[T]$. Assume that q is odd. Let $P \in \mathbb{A}$ be a monic prime of even degree. Let $\mathbb{M}_P := \{A \in \mathbb{A} : \deg A < \deg P \text{ and } \gcd(P, A) = 1\}$ and $\mathbb{M}_P^+ := \{A \in \mathbb{M}_P : A \text{ is monic}\}$. Then we have ([4, Lemma 16.13]) that

$$P = \prod_{A \in \mathbb{M}_P} \lambda_P^A = \left(\prod_{A \in \mathbb{M}_P^+} \lambda_P^A \right)^{q-1},$$

where λ_P is a primitive P-torsion point of the Carlitz module. We set

$$\sqrt{P} := (-1)^{\frac{q^{\deg P}-1}{2(q-1)}} \left(\prod_{A \in \mathbb{M}_P^+} \lambda_P^A \right)^{\frac{q-1}{2}}.$$

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Then $k(\sqrt{P})$ is the unique real quadratic subfield of K_P^+, where K_P^+ is the maximal real subfield of the P-th cyclotomic function field K_P.

Now let $P_1, P_2 \in \mathbb{A}$ be two distinct monic primes of even degree. We let $K := k(\sqrt{P_1}, \sqrt{P_2})$ and $F := k(\sqrt{P_1P_2})$. Fix a primitive root $Q_i \in \mathbb{M}_{P_i}$ of P_i for $i = 1, 2$. For $P_i \nmid A$, let $\delta_i(A)$ be the index of A relative to Q_i. We set

$$\mathcal{M}_i := \{A \in \mathbb{M}_{P_i} : 0 \le \delta_i(A) < (q^{\deg P_i} - 1)/(q-1)\},$$
$$\mathcal{M}_i^+ := \{A \in \mathcal{M}_i : (A/P_i) = 1\}, \quad \mathcal{M}_i^- := \{A \in \mathcal{M}_i : (A/P_i) = -1\}$$

for $i = 1, 2$. Then $\mathbb{M}_{P_i}$ is the disjoint union of the $c\mathcal{M}_i$, where $c \in \mathbb{F}_q^*$. Let $\mathcal{X}$ be the set of all $A \in \mathbb{M}_{P_1P_2}$ such that $(A/P_2) = 1$ and $A \equiv \tilde{A} \bmod P_1$ for some $\tilde{A} \in \mathcal{M}_1^+$. We define

$$\beta := \prod_{A \in \mathcal{X}} \lambda_{P_1P_2}^A.$$

Let $\mathcal{O}_K$ and $\mathcal{O}_F$ be the integral closure of $\mathbb{A}$ in K and F, respectively. Denote by $h(\mathcal{O}_K)$ and $h(\mathcal{O}_F)$ the class numbers of $\mathcal{O}_K$ and $\mathcal{O}_F$, respectively. We also let σ_i be the generator of $\mathrm{Gal}(K/k(\sqrt{P_i})$ for $i = 1, 2$. Then the main result of this paper is

THEOREM 1.1. *Let $P_1, P_2 \in \mathbb{A}$ be two distinct monic primes of even degree and $K = k(\sqrt{P_1}, \sqrt{P_2}), F = k(\sqrt{P_1P_2})$. Let μ_F be a fundamental unit of F.*

(I) *If $(P_2/P_1) = -1$, then $h(\mathcal{O}_K)$ is odd and $h(\mathcal{O}_F) \equiv 2 \bmod 4$ with $N_{F/k}(\mu_F) \in \mathbb{F}_q^* \backslash (\mathbb{F}_q^*)^2$.*

(II) *When $(P_2/P_1) = 1$, fix $U, V \in \mathbb{A}$ satisfying $U^2 \equiv P_1 \bmod P_2$ and $V^2 \equiv P_2 \bmod P_1$.*

  (i) *If $(U/P_2) \neq (V/P_1)$, then $h(\mathcal{O}_K)$ is odd and $h(\mathcal{O}_F) \equiv 2 \bmod 4$ with $N_{F/k}(\mu_F) \in (\mathbb{F}_q^*)^2$.*

  (ii) *Assume that $(U/P_2) = (V/P_1) = -1$.*
  - *If $\beta^{1+\sigma_1} = \beta^{1+\sigma_2}$, then $h(\mathcal{O}_K)$ is even and $h(\mathcal{O}_F) \equiv 4 \bmod 8$ with $N_{F/k}(\mu_F) \in \mathbb{F}_q^* \backslash (\mathbb{F}_q^*)^2$.*
  - *If $\beta^{1+\sigma_1} \neq \beta^{1+\sigma_2}$, then $h(\mathcal{O}_K)$ is odd and $h(\mathcal{O}_F) \equiv 2 \bmod 4$ with $N_{F/k}(\mu_F) \in (\mathbb{F}_q^*)^2$.*

  (iii) *Assume that $(U/P_2) = (V/P_1) = 1$.*
  - *If $\beta^{1+\sigma_1} = \beta^{1+\sigma_2}$, then $h(\mathcal{O}_K)$ is even and $4 | h(\mathcal{O}_F)$ (resp. $8 | h(\mathcal{O}_F)$ whenever $N_{F/k}(\mu_F) \in \mathbb{F}_q^* \backslash (\mathbb{F}_q^*)^2$).*
  - *If $\beta^{1+\sigma_1} \neq \beta^{1+\sigma_2}$, then $h(\mathcal{O}_K)$ is odd and $h(\mathcal{O}_F) \equiv 2 \bmod 4$ with $N_{F/k}(\mu_F) \in (\mathbb{F}_q^*)^2$.*

2. Cyclotomic units

Fix two distinct primes $P_1, P_2 \in \mathbb{A}^+$ of even degree. We let $K := k(\sqrt{P_1}, \sqrt{P_2}), F := k(\sqrt{P_1 P_2})$ and $K_i := k(\sqrt{P_i})$ for $i = 1, 2$. We also let

$$\varepsilon := N_{K_{P_1P_2}/K}(\lambda_{P_1P_2}) \in \mathcal{O}_K^*, \quad \varepsilon_i := \frac{1}{\sqrt{P_i}} N_{K_{P_i}/K_i}(\lambda_{P_i}) \in \mathcal{O}_{K_i}^*.$$

For any $A \in \mathbb{A}$ with $\gcd(A, P_i) = 1$, let $sgn_{P_i}(A)$ be the leading coefficient of $\bar{A}$, where $\bar{A}$ is the unique element of $\mathbb{M}_{P_i}$ such that $A \equiv \bar{A} \bmod P_i$. Let $\mathcal{R}_i$ be any complete set of representatives of $(\mathbb{A}/P_i\mathbb{A})^*/\mathbb{F}_q^*$. Define

$$\beta_i := \prod_{A \in \mathcal{R}_i} \Big(\frac{\lambda_{P_i}^A}{sgn_{P_i}(A)\lambda_{P_i}} \Big)^{(A/P_i)},$$

where (A/P_i) denotes the Legendre symbol. It is shown in [4, Lemma 16.14] that β_i is independent of the choice of $\mathcal{R}_i$ and $\beta_i \in \mathcal{O}_{K_i}^*$. Moreover, we have

Lemma 2.1. $\beta_i^{\frac{q-1}{2}} = \varepsilon_i$.

Proof. It follows from that

$$\beta_i = \prod_{\substack{A \in \mathbb{M}_{P_i}^+ \\ (A/P_i)=1}} \frac{\lambda_{P_i}^A}{\lambda_{P_i}} \prod_{\substack{A \in \mathbb{M}_{P_i}^+ \\ (A/P_i)=-1}} \frac{\lambda_{P_i}}{\lambda_{P_i}^A} = \frac{\big(\prod_{A \in \mathbb{M}_{P_i}^+, (A/P_i)=1} \lambda_{P_i}^A\big)^2}{\prod_{A \in \mathbb{M}_{P_i}^+} \lambda_{P_i}^A}.$$

by taking $\mathcal{R}_i = \mathbb{M}_{P_i}^+$. □

Lemma 2.2. *$\varepsilon = \beta^{q-1}$ with $\beta \in \mathcal{O}_K^*$.*

Proof. At first, we note that $(c/P_1) = (c/P_2) = 1$ for any $c \in \mathbb{F}_q^*$. Let $\mathcal{Y}$ be the set of all $A \in \mathbb{M}_{P_1P_2}$ such that $(A/P_1) = (A/P_2) = 1$. Then $\mathcal{Y}$ is the disjoint union of the $c\mathcal{X}$, where $c \in \mathbb{F}_q^*$, and $|\mathcal{X}|$ is a multiple of $q-1$. Thus we have

$$\varepsilon = \prod_{A \in \mathcal{Y}} \lambda_{P_1P_2}^A = \prod_{A \in \mathcal{X}} \prod_{c \in \mathbb{F}_q^*} \lambda_{P_1P_2}^{cA} = (-1)^{|\mathcal{X}|} \big(\prod_{A \in \mathcal{X}} \lambda_{P_1P_2}^A \big)^{q-1} = \beta^{q-1}.$$

It remains to show that $\beta \in K^*$. For any $A \in \mathcal{Y}$, let $c(A)$ be the unique element of $\mathbb{F}_q^*$ such that $c(A)A \in \mathcal{X}$. For any $A \in \mathbb{A}$ with $\gcd(A, P_1P_2) = 1$, let $\bar{A}$ be the unique element of $\mathbb{M}_{P_1P_2}$ such that $A \equiv \bar{A} \bmod P_1P_2$.

For any $B \in \mathbb{M}_{P_1P_2}$, we have $\{c(\overline{AB})\overline{AB} : A \in \mathcal{X}\} = \mathcal{X}$. Thus, for any $B \in \mathcal{Y}$, we have

$$\beta^{\sigma_B} = \prod_{A\in\mathcal{X}} \lambda_{P_1P_2}^{AB} = \prod_{A\in\mathcal{X}} c(\overline{AB})^{-1}\lambda_{P_1P_2}^{c(\overline{AB})\overline{AB}} = \beta \prod_{A\in\mathcal{X}} c(\overline{AB})^{-1}.$$

For any $C \in \mathcal{M}_1^+$, we have $|\{A \in \mathcal{X} : A \equiv C \bmod P_1\}| = (q^{\deg P_2} - 1)/2$. Thus

$$\begin{aligned}\prod_{A\in\mathcal{X}} c(\overline{AB})^{-1} &= \prod_{C\in\mathcal{M}_1^+} \prod_{\substack{A\in\mathcal{X}\\ A\equiv C \bmod P_1}} c(\overline{AB})^{-1} \\ &= \prod_{C\in\mathcal{M}_1^+} (c(\overline{CB})^{-1})^{\frac{q^{\deg P_2 -1}}{2}} = 1.\end{aligned}$$

Therefore $\beta \in K^*$. $\square$

Let $c_i \in \mathbb{F}_q^*$ be the unique element such that $Q_i^{\frac{q^{\deg P_i -1}}{q-1}} \equiv c_i \bmod P_i$. We also let σ_i be the generator of $\mathrm{Gal}(K/K_i)$.

LEMMA 2.3.

$$\beta^{1+\sigma_1} = \begin{cases} c_1^{\frac{\delta_1(P_2)}{2}} & \text{if } (P_2/P_1) = 1, \\ c_1^{\frac{\delta_1(P_2)+1}{2}} \Big(\frac{\prod_{A\in\mathcal{M}_1^+} sgn_{P_1}(A)}{\prod_{A\in\mathcal{M}_1^-} sgn_{P_1}(A)}\Big)\beta_1 & \text{if } (P_2/P_1) = -1. \end{cases}$$

Proof. Choose $A' \in \mathbb{M}_{P_1P_2}$ such that $A' \equiv 1 \bmod P_1$ and $(A'/P_2) = -1$. Then $\sigma_{A'}|_K = \sigma_1$, where $\sigma_{A'} \in Gal(K_{P_1P_2}/k)$. Thus we have

$$\beta^{1+\sigma_1} = \prod_{B\in\mathcal{M}_1^+} (\lambda_{P_1}^B)^{1-\mathrm{Frob}^{-1}(P_2,K_{P_1})} = \frac{\prod_{B\in\mathcal{M}_1^+} \lambda_{P_1}^B}{\prod_{B\in\mathcal{M}_1^+} \lambda_{P_1}^{BD}},$$

where $D \in \mathbb{A}$ with $DP_2 \equiv 1 \bmod P_1$. We only prove the case that $(P_1/P_2) = -1$. In this case, we have $(D/P_1) = -1$, and so $\delta_1(D)$ is odd. Write $\frac{\delta_1(D)-1}{2} = n_1 + \frac{(q^{d_1}-1)}{2(q-1)} n_2$ with $0 \le n_1 < \frac{(q^{d_1}-1)}{2(q-1)}$ and $0 \le n_2 < q-1$. Then

$$\prod_{B\in\mathcal{M}_1^+} \lambda_{P_1}^{BD} = c_1^{\frac{\delta_1(D)-1}{2}} \prod_{0\le j<\frac{q^{d_1}-1}{2(q-1)}} \lambda_{P_1}^{Q_1^{2j+1}} = c_1^{\frac{\delta_1(D)-1}{2}} \prod_{B\in\mathcal{M}_1^-} \lambda_{P_1}^B.$$

Thus, by taking $\mathcal{R}_1 = \mathcal{M}_1$ in definition of β_1, we have

$$\beta^{1+\sigma_1} = c_1^{\frac{\delta_1(P_2)+1}{2}} \Big(\frac{\prod_{A\in\mathcal{M}_1^+} sgn_{P_1}(A)}{\prod_{A\in\mathcal{M}_1^-} sgn_{P_1}(A)}\Big)\beta_1.$$

This completes the proof. □

Corollary 2.4.

$$\beta^{1+\sigma_2} = \begin{cases} c_0^{-2} c_2^{\frac{\delta_2(P_1)}{2}} & \text{if } (P_2/P_1) = 1, \\ c_0^{-2} c_2^{\frac{\delta_2(P_1)+1}{2}} \Big(\frac{\prod_{A \in \mathcal{M}_2^+} \mathrm{sgn}_{P_2}(A)}{\prod_{A \in \mathcal{M}_2^-} \mathrm{sgn}_{P_2}(A)} \Big) \beta_2 & \text{if } (P_2/P_1) = -1. \end{cases}$$

Lemma 2.5. *For any $B \in \mathbb{A}$ with $P_i \nmid B$, we have*

$$\prod_{A \in \mathcal{M}_i} \Big(\frac{sgn_{P_i}(AB)}{sgn_{P_i}(A) sgn_{P_i}(B)} \Big)^{(A/P_i)} = \begin{cases} 1 & \text{if } (B/P_i) = 1, \\ \in \mathbb{F}_q^* \backslash (\mathbb{F}_q^*)^2 & \text{if } (B/P_i) = -1. \end{cases}$$

Proof. We only prove the case that $(B/P_i) = -1$. We may assume that $B \in \mathcal{M}_i^-$, so that $B \equiv Q_i^{2n+1} \bmod P_i$ for some $0 \leq n < (q^{\deg P_i} - 1)/2(q-1)$. Then we have

$$\prod_{A \in \mathcal{M}_i} \Big(\frac{sgn_{P_i}(AB)}{sgn_{P_i}(A) sgn_{P_i}(B)} \Big)^{(A/P_i)} = c_i^n \frac{\prod_{A \in \mathcal{M}_i^-} sgn_{P_i}(A)}{\prod_{A \in \mathcal{M}_i^+} sgn_{P_i}(A)} \cdot c_i^{-n-1} \frac{\prod_{A \in \mathcal{M}_i^-} sgn_{P_i}(A)}{\prod_{A \in \mathcal{M}_i^+} sgn_{P_i}(A)}.$$

Hence

$$\prod_{A \in \mathcal{M}_i} \Big(\frac{sgn_{P_i}(AB)}{sgn_{P_i}(A) sgn_{P_i}(B)} \Big)^{(A/P_i)} = c_i^{-1} \Big(\frac{\prod_{A \in \mathcal{M}_i^-} sgn_{P_i}(A)}{\prod_{A \in \mathcal{M}_i^+} sgn_{P_i}(A)} \Big)^2 \in \mathbb{F}_q^* \backslash (\mathbb{F}_q^*)^2.$$

□

3. Proof of Theorem 1.1

Let $\mathcal{C}'_K$ be the subgroup of $\mathcal{O}_K^*$ generated by $\mathbb{F}_q^*$ and $\{\beta, \beta_1, \beta_2\}$. Then, by Lemma 2.1 and 2.2, we have $[\mathcal{O}_K^* : \mathcal{C}'_K] = h(\mathcal{O}_K)$. Similarly the subgroup $\mathcal{C}'_F$ of $\mathcal{O}_F^*$ generated by $\mathbb{F}_q^*$ and $N_{K/F}(\beta) = \beta^{1+\sigma_1\sigma_2}$ has index $[\mathcal{O}_F^* : C'_F] = \frac{1}{2} h(\mathcal{O}_F)$ by Lemma 2.2. Thus $h(\mathcal{O}_F)$ is always even.

Lemma 3.1. $(c\beta_1)^{1+\sigma_2}$ *and* $(c\beta_2)^{1+\sigma_1}$ *are contained in* $\mathbb{F}_q^* \backslash (\mathbb{F}_q^*)^2$ *for any* $c \in \mathbb{F}_q^*$.

Proof. We only prove that $(c\beta_1)^{1+\sigma_2} \in \mathbb{F}_q^*\backslash(\mathbb{F}_q^*)^2$ for any $c \in \mathbb{F}_q^*$. Choose $B \in \mathbb{M}_{P_1P_2}$ such that $(B/P_1) = -1$ and $(B/P_2) = 1$. Then $\sigma_B|_K = \sigma_2$, where $\sigma_B \in \mathrm{Gal}(K_{P_1P_2}/k)$. By taking $\mathcal{R}_1 = \mathcal{M}_1$, we have

$$\beta_1^{\sigma_2} = \beta_1^{-1} \prod_{A\in\mathcal{M}_1} \Big(\frac{sgn_{P_1}(AB)}{sgn_{P_1}(A)sgn_{P_1}(B)}\Big)^{(A/P_1)}.$$

Thus, by Lemma 2.5, $\beta_1^{1+\sigma_2} \in \mathbb{F}_q^*\backslash(\mathbb{F}_q^*)^2$. Therefore $(c\beta_1)^{1+\sigma_2} \in \mathbb{F}_q^*\backslash(\mathbb{F}_q^*)^2$ for any $c \in \mathbb{F}_q^*$. □

COROLLARY 3.2. *None of $c\beta_1, c\beta_2$ and $c\beta_1\beta_2$ is a square in K for any $c \in \mathbb{F}_q^*$.*

Proof. By Lemma 3.1, $c\beta_1$ and $c\beta_2$ are not square in K for any $c \in \mathbb{F}_q^*$. Since $(c\beta_1\beta_2)^{1+\sigma_1} = c^2\beta_1^2\beta_2^{1+\sigma_1}$ and $(c\beta_1\beta_2)^{1+\sigma_2} = c^2\beta_2^2\beta_1^{1+\sigma_2}$, $c\beta_1\beta_2$ is not a square in K. □

LEMMA 3.3. *$h(\mathcal{O}_K)$ is even if and only if $\delta = c\beta\beta_1^x\beta_2^y$ is a square in K for some $x, y \in \{0,1\}$ and $c \in \mathbb{F}_q^*$.*

Proof. It is an immediate consequence of Lemma 3.1, Corollary 3.2 and the fact that $h(\mathcal{O}_K) = [\mathcal{O}_K^* : \mathcal{C}_K']$. □

Set $G_K := \mathrm{Gal}(K/k)$ for simplicity. A function $f : G_K \to K$ is called a *crossed homomorphism* if $f(\sigma\tau) = f(\sigma)f(\tau)^\sigma$ for any $\sigma, \tau \in G_K$. The following Lemma is taken from [3, Proposition 2].

LEMMA 3.4. *Let $\mu \in \mathcal{O}_K^*$ be such that there exists a function $f : G_K \to K$ satisfying $\mu^{1-\sigma} = f(\sigma)^2$ for any $\sigma \in G_K$. If there is a function $g : G_K \to \{\pm 1\}$ such that fg is a crossed homomorphism, then $c\mu$ is a square in K for some $c \in \mathbb{F}_q^*$.*

Now we give the proof of Theorem 1.1. At first, let us consider the case that $(P_1/P_2) = -1$. By Lemma 2.3 and Corollary 2.4, we have

$$\beta^{1+\sigma_1\sigma_2} = (\beta^{1+\sigma_1})^{\sigma_2}(\beta^{1+\sigma_2})^{-1}\beta^2 = c\beta_1^{\sigma_2}\beta_2^{-1}\beta^2$$

for some $c \in \mathbb{F}_q^*$. Thus $(\beta^{1+\sigma_1\sigma_2})^{1+\sigma_1} \in \mathbb{F}_q^*\backslash(\mathbb{F}_q^*)^2$, and so $c\beta^{1+\sigma_1\sigma_2}$ is not a square in F for any $c \in \mathbb{F}_q^*$. Hence $h(\mathcal{O}_F) \equiv 2 \bmod 4$ and $N_{F/k}(\mu_F) \in \mathbb{F}_q^*\backslash(\mathbb{F}_q^*)^2$. Consider $\delta = c\beta\beta_1^x\beta_2^y$ with $c \in \mathbb{F}_q^*$ and $x, y \in \{0,1\}$. Then, by Lemma 2.3, we have

$$\delta^{1-\sigma_2} = \beta^{1-\sigma_2}\beta_1^{x(1-\sigma_2)} = \beta^2(\beta^{1+\sigma_2})^{-1}\beta_1^{2x}(\beta_1^{1+\sigma_2})^{-x} = c\beta_1^{-1}(\beta\beta_1^x)^2$$

for some $c \in \mathbb{F}_q^*$. Thus, by Corollary 3.2, $\delta^{1-\sigma_2}$ is not a square in K and so is δ. Hence, by Lemma 3.3, $h(\mathcal{O}_K)$ is odd.

Now let us suppose that $(P_1/P_2) = 1$. Fix $U, V \in \mathbb{A}$ such that $U^2 \equiv P_1 \bmod P_2$ and $V^2 \equiv P_2 \bmod P_1$. Clearly $(V/P_1) = (-1)^{\frac{\delta_1(P_2)}{2}}$ and $(U/P_2) = (-1)^{\frac{\delta_2(P_1)}{2}}$. By Lemma 2.3 and Corollary 2.4, we have

$$\beta^{1+\sigma_1} = c_1^{\frac{\delta_1(P_2)}{2}} \in \begin{cases} (\mathbb{F}_q^*)^2 & \text{if } (V/P_1) = 1, \\ \mathbb{F}_q^* \backslash (\mathbb{F}_q^*)^2 & \text{if } (V/P_1) = -1, \end{cases} \tag{3.1}$$

and

$$\beta^{1+\sigma_2} = c_0^{-2} c_2^{\frac{\delta_2(P_1)}{2}} \in \begin{cases} (\mathbb{F}_q^*)^2 & \text{if } (U/P_2) = 1, \\ \mathbb{F}_q^* \backslash (\mathbb{F}_q^*)^2 & \text{if } (U/P_2) = -1. \end{cases} \tag{3.2}$$

Consider $\delta = c\beta\beta_1^x\beta_2^y$ with $c \in \mathbb{F}_q^*$ and $x, y \in \{0, 1\}$. We have

$$\begin{aligned} \delta^{1-\sigma_1} &= \beta^2(\beta^{1+\sigma_1})^{-1}(\beta_2^{1+\sigma_1})^{-y}\beta_2^{2y}, \\ \delta^{1-\sigma_2} &= \beta^2(\beta^{1+\sigma_2})^{-1}(\beta_1^{1+\sigma_2})^{-x}\beta_1^{2x}, \\ \delta^{1-\sigma_1\sigma_2} &= \beta^2(\beta^{1+\sigma_1\sigma_2})^{-1}(\beta_1^{1+\sigma_2})^{-x}\beta_1^{2x}(\beta_2^{1+\sigma_1})^{-y}\beta_2^{2y}. \end{aligned} \tag{3.3}$$

In the case that $(U/P_2) \neq (V/P_1)$, by (3.1) and (3.2), we have

$$\beta^{1+\sigma_1\sigma_2} = (\beta^{1+\sigma_1})^{\sigma_2}(\beta^{1+\sigma_2})^{-1}\beta^2 = c\beta^2 \tag{3.4}$$

for some $c \in \mathbb{F}_q^*/(\mathbb{F}_q^*)^2$. Thus $\beta \notin F$ and $c\beta^{1+\sigma_1\sigma_2}$ cannot be a square in F for any $c \in \mathbb{F}_q^*$. Hence $h(\mathcal{O}_F) \equiv 2 \bmod 4$ and $c\beta^{1+\sigma_1\sigma_2}$ is an odd power of μ_F for some $c \in \mathbb{F}_q^*$. By (3.2) and (3.4), we have $N_{F/k}(c\beta^{1+\sigma_1\sigma_2}) = c^2(\beta^{1+\sigma_1\sigma_2})^{1+\sigma_1} \in (\mathbb{F}_q^*)^2$. Thus $N_{F/k}(\mu_F) \in (\mathbb{F}_q^*)^2$. If δ is a square in K, then $(-1)^y = (U/P_2)$ and $(-1)^x = (V/P_1)$. Define

$$f(\sigma_1) = \begin{cases} \beta(\beta^{1+\sigma_1})^{-\frac{1}{2}} & \text{if } (U/P_2) = 1, \\ \beta\beta_2(\beta^{1+\sigma_1}\beta_2^{1+\sigma_1})^{-\frac{1}{2}} & \text{if } (U/P_2) = -1, \end{cases}$$

and

$$f(\sigma_2) = \begin{cases} \beta\beta_1(\beta^{1+\sigma_2}\beta_1^{1+\sigma_2})^{-\frac{1}{2}} & \text{if } (U/P_2) = 1, \\ \beta(\beta^{1+\sigma_2})^{-\frac{1}{2}} & \text{if } (U/P_2) = -1, \end{cases}$$

where $(\beta^{1+\sigma_1})^{-\frac{1}{2}}$, $(\beta^{1+\sigma_1}\beta_2^{1+\sigma_1})^{-\frac{1}{2}}$, $(\beta^{1+\sigma_2}\beta_1^{1+\sigma_2})^{-\frac{1}{2}}$ and $(\beta^{1+\sigma_2})^{-\frac{1}{2}}$ are uniquely determined up to $\{\pm 1\}$. Then $\delta^{1-\sigma_1} = f(\sigma_1)^2$ and $\delta^{1-\sigma_2} = f(\sigma_2)^2$. Moreover

$$f(\sigma_1)g(\sigma_1)(f(\sigma_2)g(\sigma_2))^{\sigma_1} \neq f(\sigma_2)g(\sigma_2)(f(\sigma_1)g(\sigma_1))^{\sigma_2}$$

for any $g(\sigma_1), g(\sigma_2) \in \{-1, 1\}$. Thus δ is not square in K. Hence $h(\mathcal{O}_K)$ is odd, by Lemma 3.3. This complete the proof of case (i). Similar arguments will give the proof for the rest cases (ii) and (iii). We leave it to the readers.

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