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OPERATORS WITH RANK ONE SELFCOMMUTATORS

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Abstract. In this paper it is shown that if $[T^*, T]$ is of rank one and $\ker [T^*, T]$ is invariant for T, then T is quasinormal. Thus, we can know that the hyponormal condition is superfluous in the Morrel's theorem.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *quasinormal* if $T^*T^2 = TT^*T$, *hyponormal* if $T^*T \geq TT^*$, *paranormal* if $||T^2x|| \geq ||Tx||^2$ for all unit vector $x \in \mathcal{H}$, and *subnormal* if it has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$. It is well known that

$$normal \Rightarrow quasinormal \Rightarrow subnormal \Rightarrow hyponormal \Rightarrow paranormal.$$

The selfcommutator of an operator plays an important role in the study of subnormality. Subnormal operators with finite rank selfcommutators have been extensively studied ([5], [7], [12], [13], [14], [16], [17]). Particular attention has been paid to hyponormal operators with rank one or rank two selfcommutators ([4], [6], [8], [9], [10], [12], [15], [18]). In particular, B. Morrel [6] showed that a pure subnormal operator with rank one selfcommutator (pure means having no normal summand) is unitarily equivalent to a linear function of the unilateral shift. Morrel's theorem can be essentially stated (also see [3, p.162]) that if

$$(1.1) \qquad \begin{cases} \text{(i) } T \text{ is hyponormal;} \\ \text{(ii) } [T^*, T] \text{ is of rank one; and} \\ \text{(iii) } \ker [T^*, T] \text{ is invariant for } T, \end{cases}$$

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then $T \cong \alpha U + \beta$, where U is the unilateral shift on ℓ^2 and $\alpha, \beta \in \mathbb{C}$. Hence T is quasinormal. It would be interesting (in the sense of giving a simple sufficiency for the subnormality) to note that Morrel's theorem gives that

$$\text{if } T \text{ satisfies the condition (1.1) then } T \text{ is subnormal.} \tag{1.2}$$

On the other hand, M. Putinar [11] gave a matricial model for the hyponormal operator $T \in \mathcal{L}(\mathcal{H})$ with finite rank selfcommutator, in the cases where

$$\mathcal{H}_0 := \bigvee_{k=0}^{\infty} T^{*k}\big(\text{ran}\,[T^*, T]\big) \text{ has finite dimension } d \quad \text{and} \quad \mathcal{H} = \bigvee_{n=0}^{\infty} T^n \mathcal{H}_0.$$

In this case T has the following two-diagonal structure relative to the decomposition $H = H_0 \oplus H_1 \oplus \cdots$:

$$T = \begin{pmatrix} B_0 & 0 & 0 & 0 & \cdots \\ A_0 & B_1 & 0 & 0 & \cdots \\ 0 & A_1 & B_2 & 0 & \cdots \\ 0 & 0 & A_2 & B_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.3}$$

where

$$\begin{cases} \mathcal{H}_n := G_n \ominus G_{n-1} \quad (n \geq 1) \\ \qquad\qquad\qquad \text{and} \quad G_n := \bigvee_{k=0}^{n} T^k \mathcal{H}_0 \quad (n \geq 0); \\ \dim(\mathcal{H}_n) = \dim(\mathcal{H}_{n+1}) = d \ \ (n \geq 0); \\ [T^*, T] = ([B_0^*, B_0] + A_0^* A_0) \oplus 0_\infty; \\ [B_{n+1}^*, B_{n+1}] + A_{n+1}^* A_{n+1} = A_n A_n^* \ \ (n \geq 0); \\ A_n^* B_{n+1} = B_n A_n^* \ \ (n \geq 0). \end{cases} \tag{1.4}$$

We will refer the operator (1.3) to the *Putinar's matricial model of rank* d. This model was also introduced in [8], [15], [16], and etc.

Recently, in [4], it was shown that if $T \in \mathcal{L}(\mathcal{H})$ satisfies

(i) T is a pure hyponormal operator;
(ii) $[T^*, T]$ is of rank-two; and
(iii) $\ker[T^*, T]$ is invariant for T,

then T is either a subnormal operator or the Putinar's matricial model of rank two. More precisely, if $T|_{\ker[T^*,T]}$ has the rank one selfcommutator then T is subnormal and if instead $T|_{\ker[T^*,T]}$ has the rank two selfcommutator then T is either a subnormal operator or the k-th minimal partially normal extension, $\widehat{T_k}^{(k)}$, of a $(k+1)$-hyponormal operator T_k which has rank-two selfcommutator for any $k \in \mathbb{Z}_+$.

In the sense of Morrel's theorem, it seems to be interesting to consider the following problem:

Problem 1.1. *If*

$$\begin{cases} \text{(i) } T \text{ is paranormal;} \\ \text{(ii) } [T^*, T] \text{ is of rank one; and} \\ \text{(iii) } \ker [T^*, T] \text{ is invariant for } T, \end{cases} \tag{1.5}$$

does T is subnormal?

If the answer to Problem1.1 were negative, then we can have an paranormal operator with rank one selfcommutator which is not hyponormal. However, the purpose of the present paper is to give an affirmative answer to the Problem1.1. In fact, we show that the hyponormal condition is superfluous in the Morrel's theorem.

2. The main result

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *paranormal* if $||T^2x|| \geq ||Tx||^2$ for every unit vector $x \in \mathcal{H}$. It is well known that

$$\text{hyponormal} \Longrightarrow \text{p-hyponormal} \Longrightarrow \text{paranormal} \tag{2.1}$$

The following result provide an abundancy of the examples of non hyponormal paranormal.

Lemma 2.1. ([1, Corollary 1]). *If $T \in \mathcal{L}(\mathcal{H})$ is p-hyponormal and n is a positive integer, then T^n is $\frac{p}{n}$-hyponormal for every $0 < p < 1$.*

In the conditions (1.1), there is no relation codition (ii) and condition (iii) even though T is hyponormal. For example, let $A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $B := \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ and let

$$T := \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & A^{\frac{1}{2}} & (0) & 0 & \cdots \\ \cdots & 0 & A^{\frac{1}{2}} & 0 & \cdots \\ \cdots & 0 & 0 & B^{\frac{1}{2}} & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where (0) denote the $(0,0)$ position in the matrix representation. Observe that $B \geq A$ but not $B^2 \geq A^2$. By a direct calculation, we can see that

$$[T^*, T] = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & (0) & 0 & 0 & \cdots \\ \cdots & 0 & B-A & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$[T^{*2}, T^2] = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & (A^{\frac{1}{2}}(B-A)A^{\frac{1}{2}}) & 0 & 0 & \cdots \\ \cdots & 0 & B^2 - A^2 & 0 & \cdots \\ \cdots & 0 & 0 & B^{\frac{1}{2}}(B-A)B^{\frac{1}{2}} & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence T is a pure hyponormal operator with rank one selfcommutator because $B - A$ is of rank one. However, T^2 is not hyponormal. Thus, T is not subnormal. Therefore, by Morrel's theorem, $\ker[T^*, T]$ is not invariant for T. Note that T^2 is paranormal by Lemma 2.1.

However, we have:

Lemma 2.2. *If $\mathcal{M}$ is a subspace of $\ker[T^*, T]$ which is invariant for $T \in \mathcal{L}(\mathcal{H})$, then $T|_{\mathcal{M}}$ is hyponormal.*

Proof. Write T as the following form relative to the direct sum $\mathcal{M} \oplus \mathcal{M}^{\perp}$:

$$T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix}.$$

Then

$$[T^*, T] = \begin{pmatrix} [S^*, S] - AA^* & S^*A - AB^* \\ A^*S - BA^* & A^*A + [B^*, B] \end{pmatrix}.$$

Since $\mathcal{M} \subseteq \ker[T^*, T]$, we have $[S^*, S] = AA^* \geq 0$ on $\mathcal{M}$. Thus $S = T|_{\mathcal{M}}$ is hyponormal. □

Corollary 2.3. *If* $\ker[T^*,T]$ *is invariant for* $T \in \mathcal{L}(\mathcal{H})$*, then* $T|_{\ker[T^*,T]}$ *is hyponormal.*

In general, $T \in \mathcal{L}(\mathcal{H})$ may not hyponormal, albeit $[T^*,T]$ is of rank two and $\ker[T^*,T]$ is invariant for T. For example, let $T = W_\alpha$ with weight sequence $\alpha \equiv (a,b,b,\cdots)$, $(a > b > 0)$. Then T is not hyponormal. But, $[T^*,T] = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 - a^2 \end{pmatrix} \bigoplus O$ is of rank two and $\ker[T^*,T]$ is invariant for T.

For weighted shift W_α, observe that

$$[W_\alpha^*, W_\alpha] = \begin{pmatrix} \alpha_0^2 & 0 & 0 & \cdots \\ 0 & \alpha_1^2 - \alpha_0^2 & 0 & \cdots \\ 0 & 0 & \alpha_2^2 - \alpha_1^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence if $\operatorname{rank}[W_\alpha^*, W_\alpha] = 1$ then $\alpha_n = \alpha_{n+1}$ for all $n \geq 0$. Thus W_α is quasinormal (i.e., all weights are equal).

Now, we are ready for our main theorem:

Theorem 2.4. *If* $[T^*,T]$ *is of rank one and* $\ker[T^*,T]$ *is invariant for* T*, then* T *is hyponormal.*

Proof. Let $\operatorname{ran}[T^*,T] = \bigvee\{e\}$ with $||e|| = 1$. Write T as the following form relative to the direct sum $\ker[T^*,T] \oplus \bigvee\{e\}$:

$$T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix}.$$

Then

$$[T^*,T] = \begin{pmatrix} [S^*,S] - AA^* & S^*A - AB^* \\ A^*S - BA^* & A^*A + [B^*,B] \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A^*A + [B^*,B] \end{pmatrix}.$$

Since $B : \bigvee\{e\} \longrightarrow \bigvee\{e\}$, we have $[B^*,B] = 0$. Therefore T is hyponormal. □

Corollary 2.5. *If* $[T^*,T]$ *is of rank one and* $\ker[T^*,T]$ *is invariant for* T*, then* T *is quasinormal.*

Proof. It follows from Theorem 2.4 and Morrel's theorem. □

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