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FIXED POINT AND FUNCTIONAL EQUATION STEMMING FROM GENERALIZED JORDAN TRIPLE DERIVATION

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Abstract. We adopt the idea of Cădariu and Radu to prove the generalized Hyers-Ulam stability of generalized Jordan triple derivation in Banach algebra. In addition, we take account of problems for generalized Jordan triple linear derivation in Banach algebra.

1. Introduction

The stability problem of functional equations has originally been formulated by Ulam [24] in 1940: *Under what condition does there exists a homomorphism near an approximate homomorphism*? As an answer to the problem of Ulam, Hyers has proved the stability of the additive functional equation [13] in 1941, which states that *if* $\varepsilon > 0$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is mapping with* $\mathcal{X}, \mathcal{Y}$ *Banach spaces, such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \tag{1.1}$$

for all $x, y \in \mathcal{X}$, *then there exists a unique additive mapping* $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - \mathcal{L}(x)\| \leq \varepsilon$$

for all $x \in \mathcal{X}$. This stability phenomenon is called the *Hyers-Ulam stability* of the additive functional equation $f(x+y) = f(x) + f(y)$.

A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [1] in 1950 (cf. [5]) and for approximately linear mappings was presented by Rassias [20] in 1978 by

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considering the case when the inequality (1.1) is not bounded. Due to the fact, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have the *generalized Hyers-Ulam stability* property. Since then, a great deal of work has been done by a number of mathematicians and the problems concerned with the generalizations and the applications of the stability to functional equations have been developed as well (for instance, [10, 11, 19]). In particular, the stability result concerning derivations between operator algebras was first obtained by Šemrl [21]. Badora [2] gave a generalization of the Bourgin's result [4]. He also dealt with the Hyers-Ulam stability and the Bourgin-type superstability of derivations in [3].

Cădariu and Radu [9] applied the fixed point method to the investigation of the Cauchy additive functional equation. Using such an elegant idea, they could present a short and simple proof for the stability of that equation [8, 18]. In the present paper, we adopt the idea of Cădariu and Radu and establish the generalized Hyers-Ulam stability of generalized Jordan triple derivation in Banach algebra. Moreover, we investigate problems for generalized Jordan triple linear derivation in Banach algebra.

2. Preliminaries

We introduce the concept needed in this paper.

Definition 2.1. [25] Let $\mathcal{A}$ be an algebra over the real or complex field $\mathbb{F}$.

(1) An additive mapping $\mu : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if $\mu(xy) = \mu(x)y + x\mu(y)$ holds for all $x, y \in \mathcal{A}$.
(2) An additive mapping $\mu : \mathcal{A} \to \mathcal{A}$ is said to be a *Jordan derivation* if $\mu(x^2) = \mu(x)x + x\mu(x)$ is fulfilled for all $x \in \mathcal{A}$.
(3) An additive mapping $\mu : \mathcal{A} \to \mathcal{A}$ is said to be a *generalized derivation* if $\mu(xy) = \mu(x)y + xD(y)$ is valid for all $x, y \in \mathcal{A}$, where $D : \mathcal{A} \to \mathcal{A}$ is a derivation.
(4) An additive mapping $\mu : \mathcal{A} \to \mathcal{A}$ is called a *Jordan triple derivation* if $\mu(xyx) = \mu(x)yx + x\mu(y)x + xy\mu(x)$ is valid for all $x, y \in \mathcal{A}$.
(5) An additive mapping $\mu : \mathcal{A} \to \mathcal{A}$ μ is said to be a *generalized Jordan triple derivation* if $\mu(xyx) = \mu(x)yx + x\delta(y)x + xy\delta(x)$ is fulfilled for all $x, y \in \mathcal{A}$, where $\delta : \mathcal{A} \to \mathcal{A}$ is a Jordan triple derivation.

Note that if $\mu(\lambda x) = \lambda\mu(x)$ for all $\lambda \in \mathbb{F}$ and all $x \in \mathcal{A}$ in the definition, then we say that μ is *linear*.

The following fundamental result in fixed point theory plays an important role in proving the stability problem.

Theorem 2.2. (The alternative of fixed point) [16] *Suppose that we are given a complete generalized metric space $(\mathcal{X}, d)$, i.e., one for which d may assume infinite values, and a strictly contractive mapping $T : \mathcal{X} \to \mathcal{X}$ with the Lipschitz constant $L < 1$. Then, for each given $x \in \mathcal{X}$, either*

(1) *$d(T^n x, T^{n+1}x) = \infty$ for all $n \geq 0$, or*

(2) *there exists a nonnegative integer n_0 such that $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$.*

Actually, if (2) holds, then the followings are true:

- *the sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
- *y^* is the unique fixed point of T in the set $\Delta = \{y \in \mathcal{X} : d(T^{n_0}x, y) < \infty\}$;*
- *$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.*

The reader is referred to the book of Hyers, Isac and Rassias [14] for extensive theory of fixed points with a large variety of applications.

3. Auxiliary lemma

In the whole this paper, the element e of an algebra $\mathcal{A}$ will denote a unit. Now we construct the functional equation stemming from generalized Jordan triple derivation.

Lemma 3.1. *Let $\mathcal{A}$ be an algebra with unit. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping. If there exists a mapping $S : \mathcal{A} \to \mathcal{A}$ such that*

$$f(x + y + zwz) - f(x) - f(y) - f(z)wz - zS(w)z - zwS(z) = 0 \tag{3.1}$$

for all $x, y, z, w \in \mathcal{A}$, then $f : \mathcal{A} \to \mathcal{A}$ is a generalized Jordan triple derivation.

Proof. If we set $z = w = 0$ in (3.1), we see that f is additive. It is clear that $f(0) = 0$.

Letting $x = y = 0$ in (3.1), we obtain

$$f(zwz) = f(z)wz + zS(w)z + zwS(z) \tag{3.2}$$

for all $z, w \in \mathcal{A}$. In particular, put $z = w = e$ in (3.2) and then $S(e) = 0$. Considering $z = e$ in (3.2), we get

$$f(w) = f(e)w + S(w). \tag{3.3}$$

By using (3.3) and the additivity of f, S is additive. Combining (3.2) and (3.3), we have

$$S(zwz) = S(z)wz + zS(w)z + zwS(z)$$

for all $z, w \in \mathcal{A}$. So S is a Jordan triple derivation.

Therefore f is a generalized Jordan triple derivation. This completes the proof. □

4. The results

Using the fixed point method, we deal with the generalized Hyers-Ulam stability of (3.1).

THEOREM 4.1. *Let $\mathcal{A}$ be a Banach algebra with unit. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exist a mapping $S : \mathcal{A} \to \mathcal{A}$ and a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z, 2^n w)}{2^n} = 0, \ \lim_{n\to\infty} \frac{\varphi(0, 0, 2^n z, w)}{2^n} = 0 \tag{4.1}$$

for all $x, y, z, w \in \mathcal{A}$ and

$$\begin{aligned} &\|f(x + y + zwz) - f(x) - f(y) - f(z)wz - zS(w)z - zwS(z)\| \\ &\le \varphi(x, y, z, w) \end{aligned} \tag{4.2}$$

for all $x, y, z, w \in \mathcal{A}$. If there exists a positive constant $L < 1$ such that

$$\varphi(2x, 2x, 0, 0) \le 2L\varphi(x, x, 0, 0) \tag{4.3}$$

for all $x \in \mathcal{A}$, then there exists a unique generalized Jordan triple derivation $\mu : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \mu(x)\| \le \frac{1}{2(1 - L)}\varphi(x, x, 0, 0) \tag{4.4}$$

for all $x \in \mathcal{A}$. In this case, $S : \mathcal{A} \to \mathcal{A}$ is a Jordan triple derivation.

Proof. We consider the set $\mathcal{X} := \{g \,|\, g : \mathcal{A} \to \mathcal{A},\ g(0) = 0\}$ and the generalized metric on $\mathcal{X}$,

$$d(g, h) = \inf\{K \in [0, \infty] : \|g(x) - h(x)\| \le K\varphi(x, x, 0, 0), \text{ for all } x \in \mathcal{A}\}.$$

One can easily check that $(\mathcal{X}, d)$ is complete.

Next, let $T : \mathcal{X} \to \mathcal{X}$ be a mapping defined by $Tg(x) := \frac{g(2x)}{2}$ for all $x \in \mathcal{A}$.

We first verify that T is strictly contractive on $\mathcal{X}$: Observe that for all $g, h \in \mathcal{X}$,

$$\begin{aligned} d(g,h) \le K \quad &\Longrightarrow \|g(x) - h(x)\| \le K\varphi(x,x,0,0),\ x \in \mathcal{A} \\ &\Longrightarrow \left\| \frac{g(2x)}{2} - \frac{h(2x)}{2} \right\| \le LK\varphi(x,x,0,0),\ x \in \mathcal{A} \\ &\Longrightarrow \left\| Tg(x) - Th(x) \right\| \le LK\varphi(x,x,0,0),\ x \in \mathcal{A} \\ &\Longrightarrow d(Tg, Th) \le LK. \end{aligned}$$

Hence we see that $d(Tg, Th) \le Ld(g,h)$ for all $g, h \in \mathcal{X}$.

We now assert that $d(Tf, f) < \infty$: If we put $y = x, z = w = 0$ in (4.2) and we divide both sides by 2, then we arrive at

$$\|Tf(x) - f(x)\| \le \frac{\varphi(x,x,0,0)}{2}$$

for all $x \in \mathcal{A}$, that is, $d(Tf, f) \le \frac{1}{2} < \infty$.

Therefore, by the alternative of fixed point, we can prove that there is a unique generalized Jordan triple derivation $\mu : \mathcal{A} \to \mathcal{A}$ satisfying the inequality (4.4): Now, from the alternative of fixed point, it follows that there exists a fixed point μ of T such that $\lim_{n\to\infty} d(T^n f, \mu) = 0$, that is, $\mu(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ for all $x \in \mathcal{A}$.

Again, by use of the alternative of fixed point, we lead to the inequality

$$d(f, \mu) \le \frac{1}{1-L} d(Tf, f) \le \frac{1}{2(1-L)},$$

which yields the inequality (4.4).

In order to claim that the mapping $\mu : \mathcal{A} \to \mathcal{A}$ is a generalized Jordan triple derivation, let us take $z = w = 0$ in (4.2). Then it becomes

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y,0,0) \tag{4.5}$$

for all $x, y \in \mathcal{A}$. Replacing $2^n x$ and $2^n y$ instead of x and y in (4.5) and dividing by 2^n, we have by (4.1)

$$\lim_{n\to\infty} \left\| \frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} \right\| \le \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 0, 0)}{2^n} = 0.$$

This means that $\mu(x+y) = \mu(x) + \mu(y)$ for all $x, y \in \mathcal{A}$ and so we can conclude that μ is additive.

Let us now take $x = y = 0$ in (4.2). Then it follows that

$$\|f(zwz) - f(z)wz - zS(w)z - zwS(z)\| \le \varphi(0,0,z,w) \tag{4.6}$$

for all $z, w \in \mathcal{A}$. Replacing $2^n z$ and $2^n w$ instead of z and w in (4.6) and dividing by 8^n, we have by (4.1)

$$\lim_{n\to\infty}\left\|\frac{f(8^n zwz)}{8^n}-\frac{f(2^n z)}{2^n}wz-z\frac{S(2^n w)}{2^n}z-zw\frac{S(2^n z)}{2^n}\right\| \le \lim_{n\to\infty}\frac{\varphi(0,0,2^n z,2^n w)}{2^n}=0. \tag{4.7}$$

So we get

$$\mu(zwz)-\mu(z)wz=\lim_{n\to\infty}\left[z\frac{S(2^n w)}{2^n}z+zw\frac{S(2^n z)}{2^n}\right] \tag{4.8}$$

for all $z, w \in \mathcal{A}$. Putting $z = w = e$ in (4.8), one obtains $\lim_{n\to\infty}\frac{S(2^n e)}{2^n} = 0$. Again, take $z = e$ in (4.8). Then we find that

$$\mu(w)-\mu(e)w=\lim_{n\to\infty}\frac{S(2^n w)}{2^n}.$$

Thus if $\delta(w) = \mu(w) - \mu(e)w$, then, by the additivity of μ, we have

$$\delta(x+y)=(\mu(x)-\mu(e)x)+(\mu(y)-\mu(e)y)=\delta(x)+\delta(y)$$

for all $x, y \in \mathcal{A}$. Hence we show that δ is additive. In (4.8) set $w = e$ and use the definition of δ to yield $\delta(z^2) = \delta(z)z+z\delta(z)$, that is, δ is a Jordan derivation. Since any Jordan derivation is a Jordan triple derivation [6], δ is a Jordan triple derivation. In view of (4.7), we conclude that

$$\mu(zwz)=\mu(z)wz+z\delta(w)z+zw\delta(z)$$

for all $z, w \in \mathcal{A}$. Thus μ is a generalized Jordan triple derivation.

Assume that there exists another generalized Jordan triple derivation $\mu_1 : \mathcal{A} \to \mathcal{A}$ satisfying the inequality (4.4). Since μ_1 is additive, we get

$$\mu_1(x)=\frac{\mu_1(2x)}{2}=(T\mu_1)(x)$$

and so μ_1 is a fixed point of T. In view of (4.4) and the definition of d, we know that

$$d(f,\mu_1)\le\frac{1}{2(1-L)}<\infty,$$

that is, $\mu_1 \in \Delta = \{g \in \mathcal{X} : d(f, g) < \infty\}$. Due to the alternative of fixed point, we find that $\mu = \mu_1$, which proves that μ is unique.

Finally, we prove that S is a Jordan triple derivation: Replacing z by $2^n z$ in (4.6) and dividing by 4^n, we have by (4.1)

$$\lim_{n\to\infty}\left\|\frac{f(4^n zwz)}{4^n}-\frac{f(2^n z)}{2^n}wz - zS(w)z - zw\frac{S(2^n z)}{2^n}\right\| \le \lim_{n\to\infty}\frac{\varphi(0,0,2^n z,w)}{4^n}=0,$$

which means that

$$\mu(zwz)=\mu(z)wz+zS(w)z+zw\delta(z)$$

for all $z, w \in \mathcal{A}$. Using the additivity of μ and δ, this equation now can be rewritten as

$$\mu(2^n z\cdot w\cdot 2^n z)=4^n\mu(z)wz+4^n zS(w)z+4^n zw\delta(z),$$
$$\mu(z\cdot 4^n w\cdot z)=4^n\mu(z)wz+zS(4^n w)z+4^n zw\delta(z).$$

Hence $zS(w)z = z\frac{S(4^n w)}{4^n}z$, and then we obtain $zS(w)z = z\delta(w)z$ as $n\to\infty$. If $z = e$, we arrive at $S = \delta$. Since δ is a Jordan triple derivation, we see that S is a Jordan triple derivation. This ends the proof of the theorem. □

Similarly, as we did in the proof of Theorem 4.1, we also apply the fixed point method and verify the following theorem.

THEOREM 4.2. *Let $\mathcal{A}$ be a Banach algebra with unit. Suppose that $f : \mathcal{A}\to\mathcal{A}$ is a mapping for which there exist a mapping $S : \mathcal{A}\to\mathcal{A}$ and a function $\varphi : \mathcal{A}^4\to[0,\infty)$ satisfying*

$$\lim_{n\to\infty}8^n\varphi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}\Big)=0,\ \lim_{n\to\infty}8^n\varphi\Big(0,0,\frac{z}{2^n},w\Big)=0 \tag{4.9}$$

and the inequality (4.2). *If there exists a positive constant $L < 1$ such that*

$$\varphi(x,x,0,0)\le\frac{L}{2}\varphi(2x,2x,0,0) \tag{4.10}$$

for all $x\in\mathcal{A}$, then there exists a unique generalized Jordan triple derivation $\mu : \mathcal{A}\to\mathcal{A}$ such that

$$\|f(x)-\mu(x)\|\le\frac{L}{2(1-L)}\varphi(x,x,0,0) \tag{4.11}$$

for all $x\in\mathcal{A}$. In this case, $S : \mathcal{A}\to\mathcal{A}$ is a Jordan triple derivation.

Proof. First of all, if we take $x = 0$ in (4.10), then we see that $\varphi(0,0,0,0) = 0$, because of $0 < L < 1$. So letting $x = y = z = w = 0$ in (4.2), one obtains $f(0) = 0$.

We now use the definitions for $\mathcal{X}$ and d, the generalized metric on $\mathcal{X}$, as in the proof of Theorem 4.1. Then $(\mathcal{X}, d)$ is complete.

We define a mapping $T : \mathcal{X} \to \mathcal{X}$ by $Tg(x) := 2g(\frac{x}{2})$ for all $x \in \mathcal{A}$. Using the same argument as in the proof of Theorem 4.1, T is strictly contractive on $\mathcal{X}$ with the Lipschitz constant L. In addition, we prove that $d(Tf, f) \leq \frac{L}{2} < \infty$.

Now it follows from the alternative of fixed point that there exists a fixed point μ of T such that $\mu(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ for all $x \in \mathcal{A}$, because $\lim_{n\to\infty} d(T^n f, \mu) = 0$. In addition, we have the inequality

$$d(f, \mu) \leq \frac{1}{1-L} d(Tf, f) \leq \frac{L}{2(1-L)},$$

that is, the inequality (4.11) is true.

Let us replace $\frac{x}{2^n}$ and $\frac{y}{2^n}$ instead of x and y in (4.5) and multiply by 2^n. Then, by virtue of (4.9), we find that

$$\lim_{n\to\infty} \left\| 2^n f\Big(\frac{x+y}{2^n}\Big) - 2^n f\Big(\frac{x}{2^n}\Big) - 2^n f\Big(\frac{y}{2^n}\Big) \right\| \leq \lim_{n\to\infty} 2^n \varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}, 0, 0\Big) = 0.$$

Thus we see that $\mu(x + y) = \mu(x) + \mu(y)$ for all $x, y \in \mathcal{A}$, that is, μ is additive.

Replacing z and w by $\frac{z}{2^n}$ and $\frac{w}{2^n}$ in (4.6) and multiplying by 8^n, we obtain by (4.9)

$$\lim_{n\to\infty} \left\| 8^n f\Big(\frac{zwz}{8^n}\Big) - 2^n f\Big(\frac{z}{2^n}\Big) wz - 2^n z S\Big(\frac{w}{2^n}\Big) z - 2^n zwS\Big(\frac{z}{2^n}\Big) \right\| \leq \lim_{n\to\infty} 8^n \varphi\Big(0, 0, \frac{z}{2^n}, \frac{w}{2^n}\Big) = 0.$$

Following the same fashion as the proof of Theorem 4.1, we can show that μ is a generalized Jordan triple derivation. Of course, as in the proof of Theorem 4.1, we see that μ is unique.

On the other hand, replacing $\frac{z}{2^n}$ instead of z in (4.6) and multiplying by 4^n, we obtain by (4.9)

$$\lim_{n\to\infty} \left\| 4^n f\Big(\frac{zwz}{4^n}\Big) - 2^n f\Big(\frac{z}{2^n}\Big) wz - zS(w)z - 2^n zwS\Big(\frac{z}{2^n}\Big) \right\| \leq \lim_{n\to\infty} 4^n \varphi\Big(0, 0, \frac{z}{2^n}, w\Big) = 0.$$

So that

$$\mu(zwz) = \mu(z)wz + zS(w)z + zw\delta(z)$$

for all $z, w \in \mathcal{A}$. Similar to the proof of Theorem 4.1, S is a Jordan triple derivation. The proof of the theorem is complete. □

From Theorem 4.1 and Theorem 4.2, we obtain the following corollaries concerning the generalized Hyers-Ulam stability.

Corollary 4.3. *Let $\mathcal{A}$ be a Banach algebra with unit. Assume that p is given with $0 < p < 1$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping for which there exists a mapping $S : \mathcal{A} \to \mathcal{A}$ such that*

$$\begin{aligned} \text{(4.12)}\quad & \|f(x+y+zwz) - f(x) - f(y) - f(z)wz - zS(w)z - zwS(z)\| \\ & \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^{p/2}\|w\|^{p/2}) \end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$ and for some $\varepsilon > 0$. Then there exists a unique a generalized Jordan triple derivation $\mu : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \mu(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p$$

for all $x \in \mathcal{A}$. In this case, $S : \mathcal{A} \to \mathcal{A}$ is a Jordan triple derivation.

Proof. Put $x = y = z = w = 0$ in (4.12) to get $f(0) = 0$. Consider a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ defined by

$$\varphi(x, y, z, w) := \varepsilon(\|x\|^p + \|y\|^p + \|z\|^{p/2}\|w\|^{p/2})$$

for all $x, y, z, w \in \mathcal{A}$, where $L = 2^{p-1}$. Then it follows that

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z, 2^n w)}{2^n} = \lim_{n\to\infty} \frac{\varepsilon(\|x\|^p + \|y\|^p + \|z\|^{p/2}\|w\|^{p/2})}{2^{n(1-p)}} = 0,$$

$$\lim_{n\to\infty} \frac{\varphi(0, 0, 2^n z, w)}{2^n} = \lim_{n\to\infty} \frac{\varepsilon(\|z\|^{p/2}\|w\|^{p/2})}{2^{n(1-(p/2))}} = 0.$$

Since $\varphi(2x, 2x, 0, 0) = 2 \cdot 2^p \cdot \varepsilon\|x\|^p = 2L\varphi(x, x, 0, 0)$, the inequality (4.4) yields the desired property, which completes the proof. □

Corollary 4.4. *Let $\mathcal{A}$ be a Banach algebra with unit. Assume that p is given with $p > 6$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping for which there exists a mapping $S : \mathcal{A} \to \mathcal{A}$ satisfying the inequality* (4.12)*. Then there exist a unique generalized Jordan triple derivation $\mu : \mathcal{A} \to \mathcal{A}$ such that*

$$\|f(x) - \mu(x)\| \leq \frac{2\varepsilon}{|2^p - 2|}\|x\|^p$$

for all $x \in \mathcal{A}$. In this case, $S : \mathcal{A} \to \mathcal{A}$ is a Jordan triple derivation.

Proof. The proof follows from Theorem 4.2 by taking

$$\varphi(x, y, z, w) := \varepsilon(\|x\|^p + \|y\|^p + \|z\|^{p/2}\|w\|^{p/2})$$

for all $x, y, z, w \in \mathcal{A}$, where $L = \frac{1}{2^{p-1}}$. Then we get the desired result. □

5. The applications

Singer and Wermer [22] in 1955 obtained a fundamental result which started investigation into the ranges of linear derivations on Banach algebras. The result, which is called the Singer-Wermer theorem, states that any continuous linear derivation on a commutative Banach algebra maps into the Jacobson radical. They also made a very insightful conjecture, namely that the assumption of continuity is unnecessary. This was known as the Singer-Wermer conjecture and was proved in 1988 by Thomas [23]. The Singer-Wermer conjecture implies that any linear derivation on a commutative semisimple Banach algebra is identically zero which is the result of Johnson [15]. On the other hand, Hatori and Wada [12] showed that a zero operator is the only derivation on a commutative semisimple Banach algebra with the maximal ideal space without isolated points. Note that this differs from the above result of Johnson. Based on these facts and a private communication with Watanabe [17], Miura *et al.* proved the generalized Hyers-Ulam stability and Bourgin-type superstability of derivations on Banach algebras in [17].

Note that any generalized Jordan triple derivation (resp., Jordan triple derivation) on 2-torsion semiprime free ring is a generalized derivation (resp., derivation) [7].

THEOREM 5.1. *Let $\mathcal{A}$ be a commutative semiprime Banach algebra with unit. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exist a mapping $S : \mathcal{A} \to \mathcal{A}$ and a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying* (4.1) *and*

(5.1)
$$\|f(\alpha x + \beta y + zwz) - \alpha f(x) - \beta f(y) - f(z)wz - zS(w)z - zwS(z)\| \leq \varphi(x, y, z, w)$$

for all $x, y, z, w \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$. If there exists a positive constant $L < 1$ satisfying (4.3)*, then there is a unique generalized linear derivation $\mu : \mathcal{A} \to \mathcal{A}$ satisfying* (4.4)*. In this case, $S : \mathcal{A} \to \mathcal{A}$ is a linear derivation which maps $\mathcal{A}$ into its Jacobson radical.*

Proof. We consider $\alpha = \beta = 1 \in \mathbb{U}$ in (5.1) and then f satisfies the inequality (4.2). It follows from Theorem 4.1 that there exists a unique generalized Jordan triple derivation $\mu : \mathcal{A} \to \mathcal{A}$ satisfying (4.4), where

$$\mu(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad S(x) := \mu(x) - \mu(e)x$$

for all $x \in \mathcal{A}$. In particular, the mapping $S : \mathcal{A} \to \mathcal{A}$ is a Jordan triple derivation.

Letting $z = w = 0$ in (5.1), we have

$$\|f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)\| \le \varphi(x, y, 0, 0) \tag{5.2}$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$. If we also replace x and y with $2^n x$ and $2^n y$ in (5.2), respectively, and then divide both sides by 2^n, we see that

$$\begin{aligned}
&\lim_{n\to\infty} \left\| \frac{f(2^n(\alpha x + \beta y))}{2^n} - \alpha \frac{f(2^n x)}{2^n} - \beta \frac{f(2^n y)}{2^n} \right\| \\
&\le \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 0, 0)}{2^n} = 0.
\end{aligned}$$

So we get

$$\mu(\alpha x + \beta y) = \alpha\mu(x) + \beta\mu(y)$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$. Let us now assume that λ is a nonzero complex number and that N is a positive integer greater than $|\lambda|$. Then by applying a geometric argument, there exist $\lambda_1, \lambda_2 \in \mathbb{U}$ such that $2\frac{\lambda}{N} = \lambda_1 + \lambda_2$. In particular, due to the additivity of μ, we obtain $\mu(\frac{1}{2}x) = \frac{1}{2}\mu(x)$ for all $x \in \mathcal{A}$. Thus we have that

$$\begin{aligned}
\mu(\lambda x) &= \mu\left(\frac{N}{2} \cdot 2 \cdot \frac{\lambda}{N} x\right) = N\mu\left(\frac{1}{2} \cdot 2 \cdot \frac{\lambda}{N} x\right) = \frac{N}{2}\mu((\lambda_1 + \lambda_2)x) \\
&= \frac{N}{2}(\lambda_1 + \lambda_2)\mu(x) = \frac{N}{2} \cdot 2 \cdot \frac{\lambda}{N}\mu(x) = \lambda\mu(x)
\end{aligned}$$

for all $x \in \mathcal{A}$. Also, it is obvious that $\mu(0x) = 0 = 0\mu(x)$ for all $x \in \mathcal{A}$, that is, μ is linear. Therefore μ is a generalized Jordan triple linear derivation and S is also a Jordan triple linear derivation. Since $\mathcal{A}$ is a commutative semiprime Banach algebra, μ is a generalized linear derivation and S is a linear derivation which maps $\mathcal{A}$ into its Jacobson radical. The proof of the theorem is ended. □

Using the same argument as in the proof of Theorem 5.1, we can obtain the following result.

Theorem 5.2. *Let A be a commutative semiprime Banach algebra with unit. Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping for which there exist a mapping $S : \mathcal{A} \to \mathcal{A}$ and a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying* (4.9) *and the inequality* (5.1)*. If there exists a positive constant $L < 1$ satisfying* (4.10)*, then there is a unique generalized linear derivation $\mu : \mathcal{A} \to \mathcal{A}$ satisfying* (4.11)*. In this case, $S : \mathcal{A} \to \mathcal{A}$ is a linear derivation which maps $\mathcal{A}$ into its Jacobson radical.*

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