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ON A GRONWALL-TYPE INEQUALITY ON TIME SCALES

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Abstract. In this paper we extend a differential inequality presented in Theorem 2.2 [6] to a dynamic inequality on time scales.

1. Introduction

The Gronwall inequalities play a very important role in the study of the qualitative theory of differential and integral equations. Furthermore, they can be widely used to investigate stability properties for solutions of differential and difference equations. See [1, 2, 3, 9, 10, 11] for differential inequalities and difference inequalities.

The theory of time scales (closed subsets of $\mathbb{R}$) was created by Hilger [7] in order to unify the theories of differential equations and of difference equations and in order to extend those theories to other kinds of the so-called "dynamic equations". The two main features of the calculus on time scales are unification and extension of continuous and discrete analysis.

Pachpatte [13, 14] obtained some general versions of Gronwall-Bellman inequality. Oguntuase [12] established some generalizations of the inequalities obtained in [13]. However, there were some defects in the proofs of Theorems 2.1 and 2.7 in [12]. Choi et al. [6] improved the results of [12] and gave an application to boundedness of the solutions of nonlinear integro-differential equations.

In this paper we extend a differential inequality presented in [6, Theorem 2.2] to a dynamic inequality on time scales which unified differential inequality and difference inequality.

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2. Preliminaries on time scales

We mention several foundational definitions about the calculus on time scales in an excellent introductory text by Bohner and Peterson [5].

DEFINITION 2.1. The functions $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} \ : \ s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} \ : \ s < t\}$$

are called the *jump operators*.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ in the following way:

DEFINITION 2.2. A non-maximal element $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$. Also, a non-minimal element $t \in \mathbb{T}$ is called *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$.

DEFINITION 2.3. The function $\mu : \mathbb{T} \to \mathbb{R}_+$ defined by $\mu(t) = \sigma(t) - t$ is called the *graininess* function.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = h\mathbb{Z}$ with a positive constant h, we have $\mu(t) = h$.

DEFINITION 2.4. A function $f : \mathbb{T} \to \mathbb{R}$ is called *differentiable* at $t \in \mathbb{T}^\kappa$, with *(delta) derivative* $f^\Delta(t) \in \mathbb{R}$ if given $\varepsilon > 0$ there exists a neighborhood U of t such that, for all $s \in U$,

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|.$$

If $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable for every $t \in \mathbb{T}^\kappa$, then f is called *delta differentiable* on $\mathbb{T}^\kappa$.

If $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t} = f'(t),$$

and if $\mathbb{T} = h\mathbb{Z}$, then

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+h) - f(t)}{h}.$$

DEFINITION 2.5. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{\rm rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.6. Let $f \in C_{\rm rd}(\mathbb{T},\mathbb{R})$. Then $g : \mathbb{T} \to \mathbb{R}$ is called the *antiderivative* of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}^\kappa$ and satisfies $g^\Delta(t) = f(t)$ for $t \in \mathbb{T}^\kappa$. In this case, we define

$$\int_a^t f(s)\Delta s = g(t) - g(a),\ t \in \mathbb{T}.$$

Remark 2.7. Let $a, b \in \mathbb{T}$ and $f \in C_{\rm rd}(\mathbb{T},\mathbb{R})$, and $t \in \mathbb{T}^\kappa$.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) If $\mathbb{T} = h\mathbb{Z}$, where $h > 0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \text{if a} < \text{b} \\ 0 & \text{if a} = \text{b} \\ -\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \text{if a} > \text{b.} \end{cases}$$

Definition 2.8. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^\kappa$$

holds. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R}(\mathbb{T},\mathbb{R})$.

We use the cylinder transformation to define a generalized exponential function for an arbitrary time scale $\mathbb{T}$.

Definition 2.9. If $p \in \mathcal{R}(\mathbb{T},\mathbb{R})$, then we define the *generalized exponential function* $e_p(t,s)$ by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for all } s,t \in \mathbb{T},$$

where $\xi_h(z)$ is the cylinder transformation given by

$$\xi_h(z) = \begin{cases} \frac{1}{h}\mathrm{Log}(1+zh) & \text{if } h > 0 \\ z & \text{if } h = 0. \end{cases}$$

Here Log is the principal logarithm function.

3. Main results

We give a linear version of the comparison theorem on time scales.

LEMMA 3.1. [5] *Let* $y, f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ *and* $p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$, *i.e., p satisfies* $1 + \mu(t)p(t) > 0$ *for* $t \in \mathbb{T}$. *Suppose that*

$$y^\Delta(t) \leq p(t)y(t) + f(t),\ t \in \mathbb{T}.$$

Then

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau,\ t \in \mathbb{T}.$$

Now, the basic inequality for the unified Gronwall's inequality is the following:

LEMMA 3.2. [5] *Let* $y \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}), p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R}), p \geq 0$ *and* $\alpha \in \mathbb{R}$. *Then*

$$y(t) \leq \alpha + \int_{t_0}^{t} y(\tau)p(\tau)\Delta\tau \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leq \alpha e_p(t, t_0) \text{ for all } t \in \mathbb{T}.$$

The continuous version of Lemma 3.2 was first proved by Bellman [3], while the corresponding discrete version was due to Sugiyama [15, Theorem 1.2.2].

We need the following Lemma to prove Theorem 3.4.

LEMMA 3.3. [4, Theorem 2.5] *Let* $t_0 \in \mathbb{T}^\kappa$ *and assume* $k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ *is continuous at* (t, t), *where* $t \in \mathbb{T}^\kappa$ *with* $t > t_0$. *Also assume that* $k(t, \cdot)$ *is rd-continuous on* $[t_0, \sigma(t)]$. *Suppose that for each* $\varepsilon > 0$ *there exists a neighborhood U of t, independent of* $\tau \in [t_0, \sigma(t)]$, *such that*

$$|k(\sigma(t), \tau) - k(s, \tau) - k^{\Delta_t}(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U,$$

where k^{Δ_t} *denotes the derivative of k with respect to the first variable. Then*

$$g^{\Delta_t}(t) = \int_{t_0}^{t} k^{\Delta_t}(t, \tau)\Delta\tau + k(\sigma(t), t),$$

where $g(t) = \int_{t_0}^{t} k(t, \tau)\Delta\tau$.

The following is our main theorem unified Theorem 2.2 in [6] and Theorem 2.5 in [8].

THEOREM 3.4. *Suppose that $u, f \in C_{\rm rd}(\mathbb{T}, \mathbb{R})$ are nonnegative functions, and c is a nonnegative constant. Let $k(t,s)$ be defined as in Lemma 3.3 such that $k(\sigma(t),t)$ and $k^{\Delta_t}(t,s)$ are nonnegative and rd-continuous functions for $s, t \in \mathbb{T}$ with $s \le t$. Then*

$$u(t) \le c + \int_{t_0}^{t} f(s)[u(s) + \int_{t_0}^{s} k(s,\tau)u(\tau)\Delta\tau]\Delta s \text{ for all } t \in \mathbb{T}_0 \qquad (3.1)$$

implies

$$u(t) \le c[1 + \int_{t_0}^{t} f(s)e_p(s,t_0)\Delta s] \text{ for all } t \in \mathbb{T}_0,$$

where $p(t,t_0) = f(t) + k(\sigma(t),t) + \int_{t_0}^{t} k^{\Delta_t}(t,s)\Delta s$ and $\mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}$.

Proof. Put $v(t)$ by the right hand side of (3.1). Then for all $t \in \mathbb{T}_0$

$$\begin{aligned} v^{\Delta}(t) &= f(t)u(t) + f(t)\int_{t_0}^{t} k(t,\tau)u(\tau)\Delta\tau, \ v(t_0) = c \\ &\le f(t)(v(t) + \int_{t_0}^{t} k(t,\tau)v(\tau)\Delta\tau). \end{aligned} \qquad (3.2)$$

Let

$$m(t) = v(t) + \int_{t_0}^{t} k(t,\tau)v(\tau)\Delta\tau, \ m(t_0) = v(t_0) = c.$$

From Lemma 3.3, we obtain

$$\begin{aligned} m^{\Delta}(t) &= v^{\Delta}(t) + k(\sigma(t),t)v(t) + \int_{t_0}^{t} k^{\Delta_t}(t,\tau)v(\tau)\Delta\tau \\ &\le f(t)m(t) + k(\sigma(t),t)m(t) + \int_{t_0}^{t} k^{\Delta_t}(t,\tau)m(\tau)\Delta\tau \\ &\le (f(t) + k(\sigma(t),t) + \int_{t_0}^{t} k^{\Delta_t}(t,\tau)\Delta\tau)m(t). \end{aligned} \qquad (3.3)$$

By Lemma 3.1, we get

$$m(t) \le ce_p(t,t_0), \ t \ge t_0, \qquad (3.4)$$

where $p(t,t_0) = f(t) + k(\sigma(t),t) + \int_{t_0}^{t} k^{\Delta_t}(t,s)\Delta s$. By substituting (3.4) into (3.2) and then integrating it from t_0 to t, we have

$$\begin{aligned} v(t) &\le v(t_0) + c\int_{t_0}^{t} f(s)e_p(s,t_0)\Delta s \\ &= c[1 + \int_{t_0}^{t} f(s)e_p(s,t_0)\Delta s], \ t \ge t_0. \end{aligned}$$

Hence the proof is complete. □

Corollary 3.5. *Under the same assumptions of Theorem 3.4, then*

$$u(t) \leq c + \int_{t_0}^{t} f(s)[u(s) + \int_{t_0}^{s} k(s,\tau)u(\tau)\Delta\tau]\Delta s \text{ for all } t \in \mathbb{T}_0$$

implies

$$u(t) \leq c\exp[\int_{t_0}^{t} f(s)e_p(s,t_0)\Delta s] \text{ for all } t \in \mathbb{T}_0,$$

where $p(t,t_0) = f(t) + k(\sigma(t),t) + \int_{t_0}^{t} k^{\Delta_t}(t,s)\Delta s$.

Letting $k(t,s) = h(t)g(s)$ in Theorem 3.4, we obtain the following corollary.

Corollary 3.6. *Suppose that* $u, f, h, g \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ *are nonnegative functions, and* c *is a nonnegative constant. Suppose that* $h^{\Delta}(t)$ *exists and is a nonnegative and rd-continuous function. Then*

$$u(t) \leq c + \int_{t_0}^{t} f(s)[u(s) + h(s)\int_{t_0}^{s} g(\tau)u(\tau)\Delta\tau]\Delta s \text{ for all } t \in \mathbb{T}_0 \quad (3.5)$$

implies

$$u(t) \leq c[1 + \int_{t_0}^{t} f(s)e_p(s,t_0)\Delta s] \text{ for all } t \in \mathbb{T}_0,$$

where $p(t,t_0) = f(t) + h(\sigma(t))g(t) + h^{\Delta}(t)\int_{t_0}^{t} g(s)\Delta s$.

Remark 3.7. If $\mathbb{T} = \mathbb{R}$ in Corollary 3.6, then Corollary 2.4 in [6] is immediate consequence of Corollary 3.6 by using the facts of $\sigma(t) = t$ and $h^{\Delta}(t) = h'(t)$ for $t \in \mathbb{R}$.

Corollary 3.8. *If we set* $k(t,s) = 0$ *in Theorem 3.4, then our estimate reduces to Lemma 3.2 in Section 3.*

Proof. It follows from Theorem 3.4 that we have

$$\begin{aligned} u(t) &\leq c[1 + \int_{t_0}^{t} f(s)e_p(s,t_0)\Delta s] \\ &= c[1 + e_f(s,t_0)|_{t_0}^{t}] \\ &= ce_f(t,t_0),\ t \geq t_0, \end{aligned}$$

since $e_f(t_0,t_0) = 1$ for all $t_0 \in \mathbb{T}$. □

The following results in [16, Theorem 1] and [4, Corollary 4.9] follow from Theorem 3.4.

Corollary 3.9. *Under the same assumptions of Theorem 3.4 with $k(t,s) = d(s)$ for all $t, s \in \mathbb{T}$ with $t \geq s$, then*

$$u(t) \leq c + \int_{t_0}^{t} f(s)[u(s) + \int_{t_0}^{s} d(\tau)u(\tau)\Delta\tau]\Delta s \text{ for all } t \in \mathbb{T}_0$$

implies

$$\begin{aligned} u(t) &\leq c[1 + \int_{t_0}^{t} f(s)e_{f+d}(s,t_0)\Delta s] \\ &\leq ce_{f+d}(t,t_0) \text{ for all } t \in \mathbb{T}_0. \end{aligned}$$

If $\mathbb{T} = \mathbb{R}$ in Theorem 3.4, then we obtain the following corollary as a continuous version of Theorem 3.4.

Corollary 3.10. [6, Theorem 2.2] *Let $u(t), f(t)$ be nonnegative functions in an interval $I = [a,b]$, and c be a nonnegative constant. Suppose that $k(t,s)$ and $k_t(t,s)$ are nonnegative and continuous functions for $s, t \in I$. Then*

$$u(t) \leq c + \int_{t_0}^{t} f(s)[u(s) + \int_{t_0}^{s} k(s,\tau)u(\tau)d\tau]ds, t \geq t_0 \tag{3.6}$$

implies

$$u(t) \leq c[1 + \int_{t_0}^{t} f(s)\exp(\int_{t_0}^{s} p(\tau,t_0)d\tau)ds], t \geq t_0$$

where $p(\tau,t_0) = f(\tau) + k(\tau,\tau) + \int_{t_0}^{\tau} k_\tau(\tau,\sigma)d\sigma$.

Proof. If $\mathbb{T} = \mathbb{R}$, then we have

$$\begin{aligned} \sigma(t) &= t, \\ e_p(s,t_0) &= \exp(\int_{t_0}^{s}[f(\tau) + k(\tau,\tau) + \int_{t_0}^{\tau} k_\tau(\tau,\sigma)d\sigma)]d\tau), s \geq \tau \geq t_0. \end{aligned}$$

Hence the proof is complete. □

Corollary 3.11. [8, Theorem 2.5] *Let $u(n), b(n)$ be nonnegative sequences defined on $\mathbb{Z}(n_0)$ and $k(n,m), \Delta_n k(n,m)$ be a nonnegative function for $n, m \in \mathbb{Z}(n_0)$ with $n \geq m$. Suppose that*

$$u(n) \leq c + \sum_{s=n_0}^{n-1} b(s)[u(s) + \sum_{\tau=n_0}^{s-1} k(s,\tau)u(\tau)],\ n \in \mathbb{Z}(n_0),$$

where c is a positive constant. Then we have

$$\begin{aligned} u(n) &\leq c[1+\sum_{s=n_0}^{n-1} b(s)e_p(s,n_0)] \\ &= c[1+\sum_{s=n_0}^{n-1} b(s)\prod_{\tau=n_0}^{s-1}(1+p(\tau))] \text{ for all } n\in\mathbb{Z}(n_0), \end{aligned}$$

where $p(n,n_0)=b(n)+k(n+1,n)+\sum_{\tau=t_0}^{n-1}\Delta_n k(n,\tau)$.

Proof. When $\mathbb{T}=\mathbb{Z}$, we easily see that

$$\begin{aligned} e_p(s,n_0) &= \exp(\sum_{\tau=n_0}^{s-1}\mathrm{Log}(1+p(\tau)) \\ &= \prod_{\tau=n_0}^{s-1}(1+p(\tau)),\ s\geq\tau\geq n_0. \end{aligned}$$

This completes the proof. □

The next well-known discrete Gronwall's inequality follows from Corollary 3.11 with $k(n,m)=0$.

Corollary 3.12. *Let $u(n),b(n)$ be nonnegative sequences defined on $\mathbb{Z}(n_0)$. Suppose that*

$$u(n)\leq c+\sum_{s=n_0}^{n-1} b(s)u(s),\ n\in\mathbb{Z}(n_0),$$

where c is a positive constant. Then we have

$$\begin{aligned} u(n) &\leq c\prod_{s=n_0}^{n-1}(1+b(s))] \\ &\leq c\exp(\sum_{s=n_0}^{n-1} b(s)), n\geq n_0. \end{aligned}$$

Proof. From Corollary 3.11 with $k(n,m)=0$, we have

$$\begin{aligned}
u(n) &\le c[1+\sum_{s=n_0}^{n-1} b(s)e_b(s,n_0)] \\
&= c[1+\sum_{s=n_0}^{n-1} b(s)\prod_{\tau=n_0}^{s-1}(1+b(\tau))] \\
&= c[1+\sum_{s=n_0}^{n-1} \Delta_s(\prod_{\tau=n_0}^{s-1}(1+b(\tau)))] \\
&= c\prod_{s=n_0}^{n-1}(1+b(s)) \le c\exp(\sum_{s=n_0}^{n-1} b(s)),\ n\ge n_0.
\end{aligned}$$

This completes the proof. □

We can obtain anther version of Theorem 3.4 without the differentiable condition of $k(t,s)$ with respect to the first variable.

THEOREM 3.13. *Suppose that $u, f \in C_{\rm rd}(\mathbb{T},\mathbb{R})$ are nonnegative functions, and c is a nonnegative constant. Assume that $k(t,s)$ is a nonnegative and rd-continuous function for $s,t\in\mathbb{T}$ with $s\le t$. Then*

$$u(t)\le c+\int_{t_0}^{t} f(s)[u(s)+\int_{t_0}^{s} k(s,\tau)u(\tau)\Delta\tau]\Delta s \text{ for all } t\in\mathbb{T}_0 \quad (3.7)$$

implies

$$u(t)\le ce_q(t,t_0),\ t\in\mathbb{T}_0,$$

where $q(t,t_0)=f(t)[1+\int_{t_0}^{t} k(t,s)\Delta s]$.

Proof. Put $v(t)$ by the right hand side of (3.7). Then, for all $t\in\mathbb{T}_0$, we have

$$\begin{aligned}
v^{\Delta}(t) &= f(t)u(t)+f(t)\int_{t_0}^{t} k(t,\tau)u(\tau)\Delta\tau,\ v(t_0)=c \\
&\le f(t)(v(t)+\int_{t_0}^{t} k(t,\tau)v(\tau)\Delta\tau) \\
&\le f(t)(1+\int_{t_0}^{t} k(t,\tau)\Delta\tau)v(t),
\end{aligned}$$

since $v(t)$ is nondecreasing. From Lemma 3.1, we obtain

$$v(t) \le v(t_0)e_q(t,t_0),\ t\ge t_0$$

where $q(t,t_0)=f(t)(1+\int_{t_0}^{t} k(t,\tau)\Delta\tau)$. Hence the proof is complete. □

REMARK 3.14. The following statements hold:

(i) If $\mathbb{T} = \mathbb{R}$ in Theorem 3.13, then

$$u(t) \leq c + \int_{t_0}^{t} f(s)[u(s) + \int_{t_0}^{s} k(s,\tau)u(\tau)d\tau]ds \text{ for all } t \in \mathbb{R}$$

implies

$$u(t) \leq c \exp(\int_{t_0}^{t} f(s)[1 + \int_{t_0}^{s} k(s,\tau)d\tau]ds), \ t \geq t_0.$$

(ii) If $\mathbb{T} = \mathbb{Z}$ in Theorem 3.13, then

$$u(t) \leq c + \sum_{s=t_0}^{t-1} f(s)[u(s) + \sum_{\tau=t_0}^{s-1} k(s,\tau)u(\tau)], \ t \in \mathbb{Z}(t_0)$$

implies

$$\begin{aligned} u(t) &\leq c \prod_{s=t_0}^{t-1} (1 + f(s)[1 + \sum_{\tau=t_0}^{s-1} k(s,\tau)]) \\ &\leq c \exp(\sum_{s=t_0}^{t-1} f(s)[1 + \sum_{\tau=t_0}^{s-1} k(s,\tau)]), \ t \geq t_0. \end{aligned}$$

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