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A COMPLETE CONVERGENCE FOR LINEAR PROCESS UNDER ρ-MIXING ASSUMPTION

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Abstract. For the maximum partial sum of linear process generated by a doubly infinite sequence of identically distributed ρ-mixing random variables with mean zeros, a complete convergence is obtained under suitable conditions.

1. Introduction and lemmas

A sequence $\{Y_i, -\infty < i < \infty\}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is called ρ-mixing if the correlation coefficient

$$(1.1) \qquad \rho(n) = \sup_{k\geq 1} \sup_{X\in L^2(F_1^k), Y\in L^2(F_{n+k}^{\infty})} \{\frac{|Cov(X,Y)|}{\sqrt{(VarX)(VarY)}}\} \to 0$$

as $n \to \infty$, where $F_m^n = \sigma\{Y_i, m \leq i \leq n\}, -\infty < m \leq n < \infty$.

The notion of ρ-mixing was first introduced by Kolmogorov and Rozano (1960). Since then, the limiting behaviors of ρ-mixing random variables have been extensively investigated by many authors.

Shao(1989) proved the following results of partial sum of ρ-mixing random variables:

Theroem A 1. (Shao(1989)) *Let h be a function slowly varying at infinity, $p \geq 1$, $p\alpha \geq 1$ and $\alpha > \frac{1}{2}$. Let $\{Y_n, n \geq 1\}$ be a sequence of identically distributed ρ-mixing random variables with $EY_1 = 0$ and $E|Y_1|^p h(|Y_1|^{\frac{1}{\alpha}}) < \infty$. Assume that $\sum_{n=1}^{\infty} \rho^{\frac{2}{r}}(n) < \infty$, where $r = 2$ if*

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$1 \le p < 2$ and $r > p$ if $p \ge 2$. Then,

$$\sum_{n=1}^{\infty} n^{p\alpha-2}h(n)P\{\max_{1\le k\le n}|\sum_{j=1}^{k}Y_j| \ge \epsilon n^{\alpha}\} < \infty \text{ for all } \epsilon > 0.$$

THEROEM A 2. (Shao(1989)) *Let h be a non-decreasing slowly varying function at infinity and $1 \le p \le 2$. Let $\{Y_n, n \ge 1\}$ be a sequence of identically distributed ρ-mixing random variables with $EY_1 = 0$ and $E|Y_1|^p h(|Y_1|^p) < \infty$. Assume that $\sum_{n=1}^{\infty}\rho(2^n) < \infty$. Then,*

$$\sum_{n=1}^{\infty} \frac{h(n)}{n} P\{\max_{1\le k\le n}|\sum_{j=1}^{k}Y_j| \ge \epsilon n^{\frac{1}{p}}\} < \infty \text{ for all } \epsilon > 0.$$

Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed random variables and let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers. We define the moving average process

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n},\ n \ge 1 \tag{1.2}$$

based on the a sequence $\{Y_i, -\infty < i < \infty\}$.

Ibragimov(1962) established the central limit theorem, Burton and Dehling (1990) obtained a large deviation principle, and Li et al.(1992) obtained the complete convergence result for a moving average process $\{X_n, n \ge\}$ based on independent identically distributed random variables $\{Y_i, -\infty < i < \infty\}$. Under the assumption that $\{Y_i, -\infty < i < \infty\}$ is a sequence of dependent identically distributed random variables, many limiting results for the moving average process also have been obtained. For example, Zhang(1996) investigated complete convergence of moving average processes based on φ-mixing random variables, Li and Zhang(2004) proved complete moment convergence of moving average processes based on negatively associated random variables, Kim and Ko(2008) obtained complete moment convergence of moving average process under φ-mixing assumption and Chen et al.(2009) showed limiting behavior of moving average process under φ-mixing assumption.

In this paper we extend Theorems A1 and A2 to the moving average process and prove the complete convergence for the maximum partial sums of moving average process generated by ρ-mixing random variables.

Now we close this section by stating some lemmas needed to prove the main results.

LEMMA 1.1 (Shao(1995)). *Let $\{Y_n,\ n \geq 1\}$ be a sequence of ρ-mixing random variables with $EY_i = 0$ and $E|Y_i|^r < \infty$ for some $r \geq 2$ and $S_n = \sum_{i=1}^n Y_i$. Then, there exists a positive number $K = K(r, \rho(\cdot))$ depending only r and $\rho(\cdot)$ such that for any $n \geq 1$,*

$$
(1.3)\qquad E\max_{1\leq k\leq n}|S_k|^r \leq K(n^{\frac{r}{2}}\exp(K\sum_{i=0}^{[\log n]}\rho(2^i))\max_{1\leq i\leq n}(EY_i^2)^{\frac{r}{2}}
+n\exp(K\sum_{i=0}^{[\log n]}\rho^{2/r}(2^i))\max_{1\leq i\leq n}E|Y_i|^r).
$$

LEMMA 1.2 (Shao(1995)). *Let $\{Y_n,\ n \geq 1\}$ be a sequence of ρ-mixing random variables with $EY_i = 0$ and $EY_i^2 < \infty$ and let $S_n = \sum_{i=1}^n Y_i$. Then, there exists a positive number $K = K(\rho(\cdot))$ depending only $\rho(\cdot)$ such that for any $n \geq 1$,*

$$
(1.4)\qquad E\max_{1\leq k\leq n}|S_k|^2 \leq Kn\exp(6\sum_{i=0}^{[\log n]}\rho(2^i))\max_{1\leq i\leq n}EY_i^2.
$$

2. Results

THEOREM 2.1. *Let h be a function slowly varying at infinity, $p \geq 1$, $p\alpha > 1$, $\alpha > \frac{1}{2}$. Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of identically distributed ρ-mixing random variables with $EY_1 = 0$ and $E|Y_1|^p h(|Y_1|^{\frac{1}{\alpha}})$ $< \infty$ and define the moving average process X_n as in (1.2). Assume that $\sum_{n=1}^\infty \rho^{\frac{2}{r}}(n) < \infty$, where $r = 2$ if $1 \leq p < 2$ and $r > p$ if $p \geq 2$. Then*

$$
\sum_{n=1}^\infty n^{p\alpha-2}h(n)P\{\max_{1\leq k\leq n}|\sum_{j=1}^k X_j| \geq \epsilon n^\alpha\} < \infty \text{ for all } \epsilon > 0.
$$

Proof. Note

$$
\sum_{k=1}^n X_k = \sum_{k=1}^n\sum_{i=-\infty}^\infty a_i Y_{i+k} = \sum_{i=-\infty}^\infty a_i \sum_{j=i+1}^{i+n} Y_j.
$$

Let $Y_{xy} = Y_j I[|Y_j| \leq x] - EY_j I[|Y_j| \leq x]$. For $x \geq n^\alpha$, if $\frac{1}{2} < \alpha \le 1$

$$(2.1)\quad \begin{aligned} & x^{-1}|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I[|Y_j| \le x]| \\ & \le x^{-1} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E|Y_j| I[|Y_j| > x] \\ \le \;& x^{-1} n \sum_{i=-\infty}^{\infty} |a_i| E|Y_1| I[|Y_1| > x] \\ \le \;& C x^{-1} x^{1/\alpha} E|Y_1| I[|Y_1| > x] \\ \le \;& C E|Y_1|^p I[|Y_1| > x] \to 0 \text{ as } x \to \infty \end{aligned}$$

and if $\alpha > 1$

$$(2.2)\quad \begin{aligned} & x^{-1}|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I[|Y_j| \le x]| \\ & \le x^{-1} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} E|Y_j| I[|Y_j| \le x] \\ & \le C x^{-1} n E|Y_1| I[|Y_1| \le x] \le C n^{1-\alpha} \to 0 \; as \; n \to \infty. \end{aligned}$$

Hence, from (2.1) and (2.2) for large n enough we have

$$n^{-\alpha} E|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I[|Y_j| \le n^{\alpha}]| \le \frac{\epsilon}{4}.$$

Then

$$(2.3)\quad \begin{aligned} & \sum_{n=1}^{\infty} n^{p\alpha-2} h(n) P\{\max_{1\le k\le n} |\sum_{j=i+1}^{i+k} X_j| \ge \epsilon n^{\alpha}\} \\ \le \;& C\sum_{n=1}^{\infty} n^{p\alpha-2} h(n) P\{\max_{1\le k\le n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I[|Y_j| > n^{\alpha}]| \ge \epsilon n^{\alpha}/2\} \\ & + C\sum_{n=1}^{\infty} n^{p\alpha-2} h(n) P\{\max_{1\le k\le n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}| \ge \epsilon n^{\alpha}/4\} \\ = \;& I + II, \text{ where } Y_{nj} = Y_j I[|Y_j| \le n^{\alpha}] - EY_j I[|Y_j| \le n^{\alpha}]. \end{aligned}$$

By Markov inequality we have

$$
\begin{aligned}
(2.4)\ I &\leq C\sum_{n=1}^{\infty} n^{p\alpha-2}h(n)n^{-\alpha}E\max_{1\leq k\leq n}|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+k} Y_jI[|Y_j|>n^{\alpha}]| \\
&\leq C\sum_{n=1}^{\infty} n^{p\alpha-1-\alpha}h(n)E|Y_1|I[|Y_1|>n^{\alpha}] \\
&= C\sum_{n=1}^{\infty} n^{p\alpha-1-\alpha}h(n)\sum_{m=n}^{\infty} E|Y_1|I[m^{\alpha}<|Y_1|\leq(m+1)^{\alpha}] \\
&= C\sum_{m=1}^{\infty} E|Y_1|I[m^{\alpha}<|Y_1|\leq(m+1)^{\alpha}]\sum_{n=1}^{m} n^{p\alpha-1-\alpha}h(n) \\
&\leq C\sum_{m=1}^{\infty} m^{p\alpha-\alpha}h(m)E|Y_1|I[m^{\alpha}<|Y_1|\leq(m+1)^{\alpha}] \\
&\leq CE|Y_1|^p h(|Y_1|^{\frac{1}{\alpha}})<\infty.
\end{aligned}
$$

By Markov and Holder inequalities, (1.3) and (1.4) we have that for any $r\geq 2$

$$
\begin{aligned}
(2.5)\quad II &\leq C\sum_{n=1}^{\infty} n^{p\alpha-2}h(n)n^{-r\alpha}E\max_{1\leq k\leq n}|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+k} Y_{nj}|^r \\
&\leq C\sum_{n=1}^{\infty} n^{p\alpha-2}h(n)n^{-r\alpha}E(\sum_{i=-\infty}^{\infty}|a_i|^{1-\frac{1}{r}}(|a_i|^{\frac{1}{r}}\max_{1\leq k\leq n}|\sum_{j=i+1}^{i+k} Y_{nj}|))^r \\
&\leq C\sum_{n=1}^{\infty} n^{p\alpha-2-r\alpha}h(n)(\sum_{i=-\infty}^{\infty}|a_i|)^{r-1}\sum_{i=-\infty}^{\infty}|a_i|E\max_{1\leq k\leq n}|\sum_{j=i+1}^{i+k} Y_{nj}|^r \\
&\leq C\sum_{n=1}^{\infty} n^{p\alpha-2-r\alpha}h(n)[n^{\frac{r}{2}}\exp\{K\sum_{j=1}^{[\log n]}\rho^{\frac{r}{2}}(2^i)\}\max_{i+1\leq j\leq i+n}(EY_{nj}^2)^{\frac{r}{2}} \\
&\quad +n\exp\{K\sum_{j=1}^{[\log n]}\rho^{\frac{r}{2}}(2^i)\}\max_{i+1\leq j\leq i+n}E|Y_{nj}|^r] \\
&= II_1+II_2.
\end{aligned}
$$

For II_1, we estimate

$$\begin{aligned} II_1 &\leq C\sum_{n=1}^{\infty} n^{p\alpha-2-r\alpha}h(n)n^{\frac{r}{2}}(EY_{n1}^2)^{\frac{r}{2}} \\ &\leq C\sum_{n=1}^{\infty} n^{p\alpha-2-r\alpha+\frac{r}{2}}h(n)(E|Y_1|^2 I[|Y_1|\leq n^{\alpha}])^{\frac{r}{2}}. \end{aligned}$$

Now we consider the following two case for I_2. If $1 \leq p < 2$, take $r > 2$, in this case $p\alpha - \frac{pr\alpha}{2} + \frac{r}{2} - 2 < -1$. We have

$$\begin{aligned} II_1 &\leq C\sum_{n=1}^{\infty} n^{p\alpha-2-r\alpha+\frac{r}{2}}h(n)(E|Y_1|^p|Y_1|^{2-p} I[|Y_1|\leq n^{\alpha}])^{\frac{r}{2}} \\ &\leq C\sum_{n=1}^{\infty} n^{p\alpha-2-r\alpha+\frac{r}{2}+r\alpha-\frac{pr\alpha}{2}}h(n)(E|Y_1|^p I[|Y_1|\leq n^{\alpha}])^{\frac{r}{2}} \\ &\leq C\sum_{n=1}^{\infty} n^{p\alpha-\frac{pr\alpha}{2}+\frac{r}{2}-2}h(n) < \infty. \end{aligned}$$

If $p \geq 2$, take $r > \frac{p\alpha-1}{\alpha-\frac{1}{2}}$. In this case $p\alpha - \frac{r\alpha}{2} + \frac{r}{2} - 2 < -1$ and $E(Y_1)^2 < \infty$. Hence

$$II_1 \leq C\sum_{n=1}^{\infty} n^{p\alpha-2-r\alpha+\frac{r}{2}}h(n)(E|Y_1|^2 I[|Y_1|\leq n^{\alpha}])^{\frac{r}{2}} < \infty. \tag{2.6}$$

For II_2, take $r > \max(p, 2)$, then $p\alpha - r\alpha < 0$, which yields

$$\begin{aligned} II_2 &\leq C\sum_{n=1}^{\infty} n^{p\alpha-2-r\alpha}h(n)n(E|Y_1|^r I[|Y_1|\leq n^{\alpha}]) \\ &\leq C\sum_{n=1}^{\infty} n^{p\alpha-1-r\alpha}h(n)E|Y_1|^r I[|Y_1|\leq n^{\alpha}] \\ &\leq C\sum_{n=1}^{\infty} n^{p\alpha-1-r\alpha}h(n)\sum_{m=1}^{n} E|Y_1|^r I[m-1 < |Y_1|^{\frac{1}{\alpha}} \leq m] \\ &\leq C\sum_{m=1}^{\infty} E|Y_1|^r I[m-1 < |Y_1|^{\frac{1}{\alpha}} \leq m]\sum_{n=1}^{m} m^{p\alpha-1-r\alpha}h(m) \\ &\leq C\sum_{m=1}^{\infty} m^{p\alpha-r\alpha}h(m)E|Y_1|^r I[m-1 < |Y_1|^{\frac{1}{\alpha}} \leq m] \\ &\leq CE|Y_1|^p h(|Y_1|^{\frac{1}{\alpha}}) < \infty. \end{aligned} \tag{2.7}$$

Thus by combining (2.3)-(2.7) the proof is complete.

□

THEOREM 2.2. *Let h be a non-decreasing slowly varying function at infinity and $1 \le p < 2$. Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed ρ-mixing random variables with $EY_1 = 0$ and $E|Y_1|^p h(|Y_1|^p) < \infty$ and let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers. We define the moving average process $X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}$, $n \ge 1$. Assume that $\sum_{n=1}^{\infty} \rho(2^n) < \infty$. Then*

$$\sum_{n=1}^{\infty} \frac{h(n)}{n} P\{\max_{1\le k\le n} |\sum_{j=1}^{k} X_j| > \epsilon n^{\frac{1}{p}}\} < \infty \text{ for all } \epsilon > 0.$$

Proof. As in the proof of Theorem 2.1

$$\begin{aligned} (2.8) \qquad & \sum_{n=1}^{\infty} \frac{h(n)}{n} P\{\max_{1\le k\le n} |\sum_{j=1}^{k} X_j| > \epsilon n^{\frac{1}{p}}\} \\ \le\ & C\sum_{n=1}^{\infty} \frac{h(n)}{n} P\{\max_{1\le k\le n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=1}^{k} Y_j I[|Y_j| > n^{\frac{1}{p}}]| \ge \epsilon n^{\frac{1}{p}}/2\} \\ & + C\sum_{n=1}^{\infty} \frac{h(n)}{n} P\{\max_{1\le k\le n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=1}^{k} Y_{nj}| \ge \epsilon n^{\frac{1}{p}}/4\} \\ \le\ & III + IV, \text{ where } Y_{nj} = Y_j I[|Y_j| \le n^{\frac{1}{p}}] - EY_j I[|Y_j| \le n^{\frac{1}{p}}]. \end{aligned}$$

For L, by Markov inequality

$$\begin{aligned} (2.9) \qquad III \ \le\ & C\sum_{n=1}^{\infty} \frac{h(n)}{n} n^{-\frac{1}{p}} E \max_{1\le k\le n} |\sum_{i=-\infty}^{\infty} a_i \sum_{j=1}^{k} Y_j I[|Y_j| > n^{\frac{1}{p}}]| \\ \le\ & C\sum_{n=1}^{\infty} \frac{h(n)}{n} n^{-\frac{1}{p}} n E|Y_1| I[|Y_1| > n^{\frac{1}{p}}] \\ \le\ & C\sum_{n=1}^{\infty} n^{-\frac{1}{p}} h(n) \sum_{m=n}^{\infty} E|Y_1| I[m^{\frac{1}{p}} < |Y_1| \le (m+1)^{\frac{1}{p}}] \\ \le\ & C\sum_{m=1}^{\infty} E|Y_1| I[m^{\frac{1}{p}} < |Y_1| \le (m+1)^{\frac{1}{p}}] \sum_{n=1}^{m} n^{-\frac{1}{p}} h(n) \\ \le\ & C\sum_{m=1}^{\infty} m^{-\frac{1}{p}+1} h(m) E|Y_1| I[m^{\frac{1}{p}} < |Y_1| \le (m+1)^{\frac{1}{p}}] \\ \le\ & CE|Y_1|^p h(|Y_1|^p) < \infty. \end{aligned}$$

By Markov and Hölder inequalities, and (1.4), we have

$$
\begin{aligned}
(2.10)\ IV &\leq C\sum_{n=1}^{\infty}\frac{h(n)}{n}n^{-\frac{2}{p}}E\max_{1\leq k\leq n}|\sum_{i=-\infty}^{\infty}a_i\sum_{j=1}^{k}Y_{nj}|^2\\
&\leq C\sum_{n=1}^{\infty}n^{-1-\frac{2}{p}}h(n)(\sum_{i=-\infty}^{\infty}|a_i|)^2E|\max_{1\leq k\leq n}\sum_{j=1}^{k}Y_{nj}|^2\\
&\leq C\sum_{n=1}^{\infty}n^{-1-\frac{2}{p}}h(n)n[K\exp\{6\sum_{i=1}^{[\log n]}\rho(2^i)\}\max_{1\leq j\leq n}(EY_{nj}^2)]\\
&\leq C\sum_{n=1}^{\infty}n^{-\frac{2}{p}}h(n)(EY_{n1}^2)\\
&= C\sum_{n=1}^{\infty}n^{-\frac{2}{p}}h(n)E|Y_1|^2I[|Y_1|\leq n^{\frac{1}{p}}]\\
&= C\sum_{n=1}^{\infty}n^{-\frac{2}{p}}h(n)\sum_{m=1}^{n}E|Y_1|^2I[m-1<|Y_1|^p\leq m]\\
&= C\sum_{m=1}^{\infty}E|Y_1|^2I[m-1<|Y_1|^p\leq m]\sum_{n=m}^{\infty}n^{-\frac{2}{p}}h(n)\\
&\leq C\sum_{m=1}^{\infty}m^{-\frac{2}{p}+1}h(m)E|Y_1|^2I[m-1<|Y_1|^p\leq m]\\
&\leq C\sum_{m=1}^{\infty}m^{-\frac{2}{p}+1}h(m)E(|Y_1|^p|Y_1|^{2-p}I[m-1<|Y_1|^p\leq m])\\
&\leq C\sum_{m=1}^{\infty}h(m)E|Y_1|^pI[m-1<|Y_1|^p\leq m]\\
&= CE|Y_1|^ph(|Y_1|^p)<\infty.
\end{aligned}
$$

Hence, by (2.8)-(2.10) the proof is complete. □

THEOREM 2.3. *Let h be a function slowly varying at infinity, $p\geq 1$, $p\alpha\geq 1$ and $\alpha>\frac{1}{2}$. Let $X_n=\sum_{i=-\infty}^{\infty}a_iY_{i+n}$ be a moving average process, where $\{Y_i,-\infty<i<\infty\}$ is a sequence of identically distributed ρ-mixing random variables with $EY_1=0$ and $E|Y_1|^ph(|Y_1|^{\frac{1}{\alpha}})<\infty$ and $\{a_i,-\infty<i<\infty\}$ is an absolutely summable sequence of real numbers.*

Assume that $\sum_{n=1}^{\infty} \rho^{\frac{2}{r}}(2^n) < \infty$, *for* $r \geq 2$. *Then,*

$$\sum_{n=1}^{\infty} n^{p\alpha-2}h(n)P\{\sup_{k\geq n}|k^{-\alpha}\sum_{j=1}^{k} X_j| \geq \epsilon\} < \infty \text{ for all } \epsilon > 0. \tag{2.11}$$

Proof.

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{p\alpha-2}h(n)P\{\sup_{k\geq n}|k^{-\alpha}\sum_{j=1}^{k} X_j| > \epsilon\} \\
= & \sum_{i=1}^{\infty}\sum_{n=2^{i-1}}^{2^i-1} n^{p\alpha-2}h(n)P\{\sup_{k\geq n}|k^{-\alpha}\sum_{j=1}^{k} X_j| > \epsilon\} \\
\leq & \; C\sum_{i=1}^{\infty} P\{\sup_{k\geq 2^{i-1}}|k^{-\alpha}\sum_{j=1}^{k} X_j| > \epsilon\}\sum_{n=2^{i-1}}^{2^i-1} n^{p\alpha-2}h(n) \\
\leq & \; C\sum_{i=1}^{\infty} 2^{i(p\alpha-1)}h(2^i)P\{\sup_{k\geq 2^{i-1}}|k^{-\alpha}\sum_{j=1}^{k} X_j| > \epsilon\} \\
\leq & \; C\sum_{i=1}^{\infty} 2^{i(p\alpha-1)}h(2^i)\sum_{l=i}^{\infty} P\{\max_{2^{l-1}\leq k<2^l}|k^{-\alpha}\sum_{j=1}^{k} X_j| > \epsilon\} \\
\leq & \; C\sum_{i=1}^{\infty} P\{\max_{2^{l-1}\leq k<2^l}|k^{-\alpha}\sum_{j=1}^{k} X_j| > \epsilon\}\sum_{i=1}^{l} 2^{i(p\alpha-1)}h(2^i) \\
\leq & \; C\sum_{l=1}^{\infty} 2^{l(p\alpha-1)}h(2^l)P\{\max_{2^{l-1}\leq k<2^l}|\sum_{j=1}^{k} X_j| > \epsilon 2^{(l-1)\alpha}\} \\
\leq & \; C\sum_{l=1}^{\infty}\sum_{n=2^{l-1}}^{2^l-1} n^{p\alpha-2}h(n)P\{\max_{1\leq k\leq n}|\sum_{j=1}^{k} X_j| > 2^{(l-1)}\epsilon\} \\
\leq & \; C\sum_{n=1}^{\infty} n^{p\alpha-2}h(n)P\{\max_{1\leq k<n}|\sum_{j=1}^{k} X_j| > n^{\alpha}2^{-\alpha}\epsilon\} \\
& \quad (\text{letting } \epsilon_1 = 2^{-\alpha}\epsilon) \\
\leq & \; C\sum_{n=1}^{\infty} n^{p\alpha-2}h(n)P\{\max_{1\leq k<n}|\sum_{j=1}^{k} X_j| > \epsilon_1 n^{\alpha}\} < \infty \text{ by Theorem 2.1.}
\end{aligned}$$

Hence, the proof is complete. □

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