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ANALOGUE OF WIENER MEASURE OVER THE SETS BOUNDED BY SECTIONALLY DIFFERENTIABLE BARRIERS

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Abstract. In this paper, we find the formula for the analogue of Wiener measure over the subset of $C[0,T]$ bounded by the sectionally differentiable functions, which is a generalization of Park and Skoug's results in [2].

1. Introduction

Let m_w be the classical Wiener measure on $C_0[0,T]$ with $T>0$, the space of all continuous functions x with $x(0)=0$. From [6] and [7], we can found the following equations ; for $b \geq 0$,

$$\begin{aligned} &m_w(\{x \text{ in } C_0[0,T] | \sup_{0\leq t\leq T} x(t) \geq b\}) \\ &= 2m_w(\{x \text{ in } C_0[0,T] | x(T) \geq b\}) \\ &= 2\int_{b/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &m_w(\{x \text{ in } C_0[0,T] | \sup_{0\leq t\leq T} (x(t)-at) \geq b\}) \\ &= \int_{(aT+b)/\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du + e^{-2ab} \int_{-\infty}^{(aT-b)/\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du. \end{aligned} \tag{1.2}$$

In [1], Park and Paranjape gave the probability of $\sup_{0\leq t\leq T}(W(t) - f(t))$ for a differentiable function f and for the standard Wiener process $\{W(t)|t\geq 0\}$.

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Recently, Ryu proved an equation which is a generalization of Park and Paranjape's result, as following theorem in [4].

THEOREM 1.1. *For $0 < t \le T$, $G(t)$ satisfies the following Volterra's integral equation of the second kind*

$$(1.3)\qquad \begin{aligned} &G(t)\\ &= 2\int_{-\infty}^{f(0)} \Big(\int_{f(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1-u_0)^2)}{2t}\} du_1 \Big) d\varphi(u_0)\\ &\quad -2\int_0^t G(s)M(t,s)ds \end{aligned}$$

where

$$M(t,s) = \begin{cases} \frac{\partial}{\partial s}\int_{-\infty}^{(f(t)-f(s))/\sqrt{t-s}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du & (0 \le s < t \le T) \\ 0 & (0 \le t \le s \le T) \end{cases} .$$

Park and Skoug established formulas, generalizations of Park and Paranjape's work, for the Wiener integral of $F(x)$ bounded by sectionally continuous functions in [2]. In 2002, the author and Ryu presented the definition and the theories of analogue of Wiener measure m_φ on $C[0,T]$, the space of all continuous functions on $[0,T]$ in [3]. This measure is a kind of generalization of standard Wiener measure. Indeed, if φ is the Dirac measure δ_0 at the origin in $\mathbb{R}$, then m_φ is the classical Wiener measure m_w.

The main result of this paper is to find the analogue of Wiener measure m_φ of $\{x \text{ in } C[0,T] | \sup_{0\le t\le T}(x(t)-f(t)) \ge 0\}$ for a sectionally differentiable function f on $[0,T]$.

Throughout in this paper, $\int_a^b f(u)du$ means the Henstock integral of f.

2. Statement of the result and proof

Let φ be a complete probability measure on $\mathbb{R}$ and let m_φ be the analogue of Wiener measure on $C[0,T]$ for a given measure φ.

From [3], we can find the following theorem.

THEOREM 2.1. (The Wiener integration formula for analogue of Wiener measure) *If $g : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then* *the following equality holds.*

$$\int_{C[a,b]} g(x(t_0), x(t_1), \cdots, x(t_n))\ d\omega_\varphi(x)$$
$$\stackrel{*}{=} \int_{\mathbb{R}^{n+1}} g(u_0, u_1, \cdots, u_n) W(n+1; \vec{t}; u_0, u_1, \cdots, u_n)$$
$$d(\prod_{j=1}^{n} m_L \times \varphi)((u_1, u_2, \cdots, u_n), u_0)$$

where $\stackrel{*}{=}$ *means that if one side exists then both sides exist and the two values are equal.*

In this note, "a function $f : [0,T] \to \mathbb{R}$ is sectionally differentiable" means that there is a partition $0 = t_0 < t_1 < \cdots < t_n = T$ such that f is differentiable on each interval (t_{i-1}, t_i) and the limits $\lim_{t\to t_i^-} f(t)$ and $\lim_{t\to t_i^+} f(t)$ exist for $i = 1, 2, \cdots, n$, and $f(0) = \lim_{t\to 0^+} f(t)$ and $f(T) = \lim_{t\to T^-} f(t)$. Let $SD[0,T]$ be the space of sectionally differentiable functions on $[0,T]$. For f in $SD[0,T]$ with a partition $0 = t_0 < t_1 < \cdots < t_n = T$, we let $f^*(t_i) = min\{f(t_i), \lim_{t\to t_i^-} f(t), \lim_{t\to t_i^+} f(t)\}$, $f^*(s) = 0$ if $s < 0$, and $f^*(s) = f(s)$ otherwise. For t in $[0,T]$, we suppose the limit $\lim_{s\to t^-} \frac{f(t)-f(s)}{\sqrt{t-s}}$ exists and equals to 0.
For x in $C[0,T]$, let $\tau(x)$ be the first hitting time of the curve f in $SD[0,T]$ by x, that is, $x(\tau(x)) = f(\tau(x))$. If x never hit the curve f, we let $\tau(x) = +\infty$.

For f in $SD[0,T]$ with a partition $0 = t_0 < t_1 < \cdots < t_n = T$, we let

$$U_1(t,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(f^*(t)-f^*(s))/\sqrt{t-s}} e^{-\frac{v^2}{2}} dv \tag{2.1}$$

for $0 \le s < t \le T$,

$$\begin{aligned} U_2(t) &= \frac{1}{\sqrt{(2\pi)^{k-1}}} \int_{-\infty}^{(f^*(t_1)-f^*(t_0))/\sqrt{t_1-t_0}} \cdots \\ &\int_{-\infty}^{(f^*(t_{k-1})-f^*(t_{k-2}))/\sqrt{t_{k-1}-t_{k-2}}} e^{-\sum_{j=1}^{k-1} \frac{v_j^2}{2}} dv_{k-1} dv_{k-2} \cdots dv_1 \end{aligned} \tag{2.2}$$

for $0 \le t_{k-1} < t < s \le T$, and

$$U(t,s) = \begin{cases} \frac{1}{2} U_2(t) & \text{for } 0 \le t_{k-1} < t < s \le T \\ U_1(t,s) U_2(t) & \text{for } 0 \le t_{k-1} < s < t \le T \end{cases}. \tag{2.3}$$

Then U_1, U_2 and U are all bounded by 1.

Let $G : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$G(t) = \begin{cases} 0 & (t < 0) \\ m_\varphi(A_t) & (0 \le t \le T) \\ m_\varphi(A_T) & (T < t) \end{cases},$$

where

$$\begin{aligned}(2.4)\quad A_t &\\ &= \{x \text{ in } C[0,T] | x(0) < f^*(0), \text{ and for some } i \text{ and for some } s_0 \\ &\qquad \text{in } (t_{i-1}, t_i] \text{ with } s_0 \le t, \quad x(s_0) \ge f^*(s_0)\}\end{aligned}$$

for t in $[0,T]$.

We have the following Lemma 2.2 by similar method of proof in [4].

LEMMA 2.2. *G is increasing continuous with $G(0) = 0$.*

LEMMA 2.3. ([4]) *If $0 \le s < t \le T$ then $\tau(x) = s$ and $x(t) - x(s)$ are independent.*

The following theorem is one of main theorems in this paper.

THEOREM 2.4. *For $0 < t \le T$, $G(t)$ satisfies the following Volterra's integral equation of the second kind*

$$\begin{aligned}(2.5)\quad G(t) &\\ &= \frac{2}{2 - U_2(t)} \int_{-\infty}^{f^*(0)} \Big(\int_{f^*(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1 - u_0)^2)}{2t}\} du_1 \Big) d\varphi(u_0) \\ &\quad - \frac{2}{2 - U_2(t)} \int_0^t G(s) \frac{\partial}{\partial s} U(t,s) ds.\end{aligned}$$

Proof. For $0 < t \le T$,

$$\begin{aligned} G(t) &= m_\varphi(A_t \cap \{x \text{ in } C[0,T] | x(t) \ge f^*(t)\}) \\ &\quad + m_\varphi(A_t \cap \{x \text{ in } C[0,T] | x(t) < f^*(t)\}) \\ &= m_\varphi(\{x \text{ in } C[0,T] | x(0) < f^*(0) \text{ and } x(t) \ge f^*(t)\}) \\ &\quad + m_\varphi(\{x \text{ in } C[0,T] | x(0) < f^*(0), x(t) < f^*(t), \text{ and} \\ &\qquad \text{for some } j \text{ and for some } s_0 \text{ in } (t_{j-1}, t_j] \text{ with } s_0 \le t, \\ &\qquad x(s_0) = f^*(s_0)\}).\end{aligned}$$

Here,

$$\begin{aligned}(2.6)\qquad & m_\varphi(\{x \text{ in } C[0,T] | x(0) < f^*(0) \text{ and } x(t) \ge f^*(t)\}) \\ &= \int_{-\infty}^{f^*(0)} \Big(\int_{f^*(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1 - u_0)^2)}{2t}\} du_1 \Big) d\varphi(u_0)\end{aligned}$$

and

(2.7) $m_\varphi(\{x \text{ in } C[0,T] | x(0) < f^*(0), x(t) < f^*(t), \text{ and}$
$\text{for some } j \text{ and for some } s_0 \text{ in } (t_{j-1}, t_j] \text{ with } s_0 \le t,$
$x(s_0) = f^*(s_0)\})$

$$\overset{(1)}{=} \int_0^t E^\varphi(x(t_j) < f^*(t_j), x(t) < f^*(t) | \tau(x) = s) dG(s)$$

$$\overset{(2)}{=} \int_0^t E^\varphi(x(t_j) - x(t_{j-1}) < f^*(t_j) - f^*(t_{j-1}),$$
$$x(t) - x(s) < f^*(t) - x(s) | \tau(x) = s) dG(s)$$

$$\overset{(3)}{=} \int_0^t E^\varphi(x(t_j) - x(t_{j-1}) < f^*(t_j) - f^*(t_{j-1}),$$
$$x(t) - x(s) < f^*(t) - x(s)) dG(s)$$

$$\overset{(4)}{=} \int_0^t \Big(\int_{C[0,T]} \prod_{j=1}^{k-1} \chi_{\{x \text{ in } C[0,T] | x(t_j) - x(t_{j-1}) < f^*(t_j) - f^*(t_{j-1})\}}(x)$$
$$\chi_{\{x \text{ in } C[0,T] | x(t) - x(s) < f^*(t) - x(s)\}}(x) dm_\varphi(x)\Big) dG(s)$$

$$= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{k+1}} \frac{1}{\sqrt{(2\pi)^{k+1}(t-s)(s-t_{k-1}) \prod_{j=1}^{k-1}(t_j - t_{j-1})}}$$
$$\prod_{j=1}^{k-1} \chi_{(-\infty, f^*(t_j) - f^*(t_{j-1}))}(u_j - u_{j-1}) \chi_{(-\infty, f^*(t) - x(s))}(u_{k+1} - u_k)$$
$$\exp\{-\frac{(u_{k+1} - u_k)^2}{2(t-s)} - \frac{(u_k - u_{k-1})^2}{2(s - t_{k-1})} - \sum_{j=1}^{k-1} \frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\}$$
$$du_{k+1} du_k \cdots du_1 d\varphi(u_0) dG(s)$$

$$\overset{(5)}{=} \int_0^t \Big(\frac{1}{\sqrt{(2\pi)^k}} \int_{-\infty}^{(f^*(t_1) - f^*(t_0))/\sqrt{t_1 - t_0}} \cdots$$
$$\int_{-\infty}^{(f^*(t_{k-1}) - f^*(t_{k-2}))/\sqrt{t_{k-1} - t_{k-2}}} \int_{-\infty}^{(f^*(t) - f^*(s))/\sqrt{t-s}}$$
$$e^{-\frac{v_{k+1}^2}{2} - \sum_{j=1}^{k-1} \frac{v_j^2}{2}} dv_{k+1} dv_{k-1} dv_{k-2} \cdots dv_1\Big) dG(s)$$

$$\overset{(6)}{=} \int_0^t U(t,s) dG(s)$$

$$\overset{(7)}{=} \lim_{s \to t^+} U(t,s) \lim_{s \to t^+} G(s) - \lim_{s \to 0^-} U(t,s) \lim_{s \to 0^-} G(s)$$
$$- \int_0^t G(s) dU(t,s)$$

$$\overset{(8)}{=} \frac{1}{2} U_2(t) G(t) - \int_0^t G(s) \frac{\partial}{\partial s} U(t,s) ds,$$

where $j = 1, 2, \cdots, k-1$ and s in $(t_{k-1}, t_k]$. Step (1) follows from the basic properties of conditional expectation. From $x(s) = f^*(s)$, we have Step (2). By Lemma 2.3, we obtain Step (3). Using the Wiener integration formula for analogue of Wiener measure, we can check Step (4). Step (5) come from the change of variables theorem. We get Step (6) by (2.3). And by the integration by part, we have Step (7). Since $\lim_{s\to 0^-} G(s) = G(0) = 0$, we obtain Step (8).

Hence, from (2.6) and (2.7), we have the equality (2.5), as desired. □

The equality (2.5) and the change of order of integration give

$$
\begin{aligned}
(2.8)\quad & G(t) \\
&= \frac{2}{2-U_2(t)} \int_{-\infty}^{f^*(0)} \Big(\int_{f^*(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1-u_0)^2}{2t}\} du_1 \Big) d\varphi(u_0) \\
&\quad - \frac{4}{2-U_2(t)} \int_0^t \Big[\int_{-\infty}^{f^*(0)} \Big(\int_{f^*(s)}^{+\infty} \frac{1}{\sqrt{2\pi s}} \exp\{-\frac{(u_1-u_0)^2}{2s}\} du_1 \Big) \\
&\quad d\varphi(u_0) \Big] \frac{\partial}{\partial s} U(t,s) ds \\
&\quad + \frac{4}{2-U_2(t)} \int_0^t \Big(\int_z^t \frac{\partial}{\partial z} U(s,z) \frac{\partial}{\partial s} U(t,s) ds \Big) G(z) dz
\end{aligned}
$$

if $\frac{\partial}{\partial z} U(s,z) \frac{\partial}{\partial s} U(t,s) G(z)$ is integrable on $\{(s,z) | 0 \le z < s \le t\}$.

By [4, 8], we have the following theorem.

THEOREM 2.5. *If $\int_z^t \frac{\partial}{\partial z} U(s,z) \frac{\partial}{\partial s} U(t,s) ds$ is square integrable on $\{ (z,t) | 0 \le z < t \le T\}$, then the equation (2.5) has one and essentially only one solution in the class L_2. This solution is given by the formula*

$$
\begin{aligned}
(2.9)\quad & G(t) \\
&= \frac{2}{2-U_2(t)} \int_{-\infty}^{f^*(0)} \Big(\int_{f^*(t)}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(u_1-u_0)^2}{2t}\} du_1 \Big) d\varphi(u_0) \\
&\quad + \sum_{n=1}^{\infty} (-1)^n \frac{2^{n+1}}{2-U_2(t)} \int_0^t \Big[\int_{-\infty}^{f^*(0)} \Big(\int_{f^*(s)}^{+\infty} \frac{1}{\sqrt{2\pi s}} \\
&\quad \exp\{-\frac{(u_1-u_0)^2}{2s}\} du_1 \Big) d\varphi(u_0) \Big] H_n(t,s) ds
\end{aligned}
$$

where $H_1(t,s) = \frac{\partial}{\partial s} U(t,s)$ and $H_{n+1}(t,s) = \int_s^t H_n(t,z) H_1(z,s) dz$.

REMARK 2.6. For f in $SD[0,T]$ with a partition $0 = t_0 < t_1 < \cdots < t_n = T$, if $\tau(x) = s$ is in (t_0, t_1), then (2.5) and (1.3) are essentially same.

Example 2.7. For f in $SD[0,T]$ with $0 < t_1 < T$, and $f(0) = \lim_{s\to 0^+} f(s)$, $f(t_1) = \lim_{s\to t_1^-} f(s)$ and $\lim_{s\to t_1^-} f(s) < \lim_{s\to t_1^+} f(s)$, let

$$A_t = \{x \text{ in } C[0,T] | f(0) < x(0),\ \alpha_- < x(t_1) < \alpha^+ \text{ and } \tau(x) = s \text{ in } [0,t]\}$$

where $\alpha_- = \lim_{s\to t_1^-} f(s)$ and $\alpha^+ = \lim_{s\to t_1^+} f(s)$. Then

$$\begin{aligned} &G(t) \\ &= \frac{2}{2-U_2(t)} \int_{f(0)}^{+\infty} \Big(\int_{\alpha_-/\sqrt{t_1}}^{\alpha^+/\sqrt{t_1}} \int_{f(t)/\sqrt{t-t_1}}^{+\infty} \frac{1}{2\pi} \exp\{-\frac{1}{2}(v_1^2+v_2^2)\} \\ &\qquad dv_2 dv_1 \Big) d\varphi(v_0) - \frac{2}{2-U_2(t)} \int_0^t G(s) \frac{\partial}{\partial s} U(t,s) ds \end{aligned}$$

where

$$U(t,s) = \begin{cases} \frac{1}{2} U_2(t) & \text{for } 0 \le t_1 < t < s \le T \\ U_1(t,s) U_2(t) & \text{for } 0 \le t_1 < s < t \le T \end{cases},$$

$$U_1(t,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(f(t)-f(s))/\sqrt{t-s}} e^{-\frac{v^2}{2}} dv$$

for $0 \le s < t \le T$ and

$$U_2(t) = \frac{1}{\sqrt{2\pi}} \int_{f(0)}^{+\infty} \int_{(\alpha_- - v_0)/\sqrt{t_1}}^{(\alpha^+ - v_0)/\sqrt{t_1}} e^{-\frac{v_1^2}{2}} dv_1 d\varphi(v_0)$$

for $0 < t_1 < t < s \le T$.

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