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THE CLASSIFICATION OF SELF-DUAL CODES OF LENGTH 6 OVER $\mathbb{Z}_m$ FOR SMALL m

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Abstract. In this article we study self-dual codes of length 6 over $\mathbb{Z}_m$. A classification of such codes for $m \leq 24$ is given. Main tool for the classification is the new double cosets decomposition method given in the recent article of the author.

1. Introduction

Let m, n be positive integers. A *modular code* C of length n over $\mathbb{Z}_m$ is a $\mathbb{Z}_m$-submodule of $\mathbb{Z}_m^n$. For the generality and definitions of modular codes and their lifts, we refer to [6, 7, 10, 12, 17]. A code C with generator matrix G will be denoted by $C : G$.

$\mathbb{Z}_m^n$ is equipped with the standard inner product defined by $x \cdot y = \sum x_i y_i$ where $x = (x_i)$, $y = (y_i)$. The dual code $C^\perp$ of a code C of length n is defined by $C^\perp = \{x \in \mathbb{Z}_m^n \mid x \cdot y = 0 \text{ for all } y \in C\}$. C is called a *self-dual* code if $C = C^\perp$. Self-dual codes are an important class of linear codes and much work has been done towards the classification of self-dual codes. The main tool for the classification is the so called *mass formula*, which counts the number of self-dual codes over $\mathbb{Z}_m$ of given length. See [10, 12] for generality or [1, 14, 15] for recent results. Self-dual codes of moderate length over prime fields are classified for small primes by the effort of many authors [2, 3, 9, 11, 13, 19].

It is known, at least theoretically, that the classification of self-dual codes over a ring $\mathbb{Z}_m$ can be done by the Chinese Remainder Theorem and the classification of self-dual codes over the rings $\mathbb{Z}_{p^e}$ where p is a prime [5]. However, it has not been pursued any further and has not been done in almost all cases. Recently, some result on classification of modular codes is obtained by a method based on a classification of m-frames of unimodular lattices [8]. In [18], the author pursued the Chinese

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Remainder Theorem to give an elementary classification method using double cosets decomposition and classified self-dual codes of length 4 and of length 8 over $\mathbb{Z}_m$ for many m. In this article, we investigate classification problem of self-dual codes of length 6 over $\mathbb{Z}_m$, where $m \leq 24$.

2. Chinese products and a classification method

The group S_n of symmetries on n letters acts on $\mathbb{Z}_m^n$ by

$$(c_1, c_2, \cdots, c_n)\sigma = (c_{\sigma(1)}, c_{\sigma(2)}, \cdots, c_{\sigma(n)}).$$

Denote by p_σ the $n \times n$ permutation matrix corresponding to this action by σ, i,e., $p_\sigma = (e_{\sigma(1)}, e_{\sigma(2)}, \cdots, e_{\sigma(n)})$, where e_j is the jth standard basis column vector. Let

$$\mathbb{D}_m^n = \{\mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_n) \mid \gamma_i \in \mathbb{Z}_m,\ \gamma_i^2 = 1\}.$$

An element $\gamma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{D}_m^n$ acts on $\mathbb{Z}_m^n$ by

$$(c_1, c_2, \cdots, c_n)\gamma = (\gamma_1 c_1, \gamma_2 c_2, \cdots, \gamma_n c_n).$$

Let $\mathbb{T}_m^n$ be the group of all *monomial transformations* on $\mathbb{Z}_m^n$ defined by

$$\mathbb{T}_m^n = \{\gamma p_\sigma \mid \gamma \in \mathbb{D}_m^n, \sigma \in S_n\}.$$

Let $\mathscr{S}_m^n$ be the set of all self-dual codes of length n over $\mathbb{Z}_m$. The group $\mathbb{T}_m^n$ acts on $\mathscr{S}_m^n$ by $Ct = \{ct \mid c \in C\}$. Two self-dual codes C and C' in $\mathscr{S}_m^n$ are *equivalent* (denoted $C \sim C'$) if there exists an element $t \in \mathbb{T}_m^n$ such that $Ct = C'$. The group of all automorphisms of C will be denoted by $\mathrm{Aut}_{\mathbb{T}_m^n}(C)$ or simply $\mathrm{Aut}(C)$ and the set of orbits or complete representatives will be denoted by $\mathscr{S}_m^n/\mathbb{T}_m^n$.

We will usually abuse the notations and simply write $\mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ by $(\gamma_1, \cdots, \gamma_n)$ and p_σ by σ. Since $\mathbb{D}_m^n \cap S_n = \{I\}$, any element $t \in \mathbb{T}_m^n$ has a unique representation $t = \gamma\sigma$ for $\gamma \in \mathbb{D}_m^n$ and $\sigma \in S_n$. γ will be called the *sign* of t, and σ will be called the *permutation part* of t. The map $p : \mathbb{T}_m^n \to S_n$, $p(\gamma\sigma) = \sigma$ is a surjective homomorphism with kernel $\mathbb{D}_m^n$. Since what is important to us is the number $k = |\mathbb{D}_m^n \cap H|$ and the group $p(H)$ of permutations of H, we usually write

$$H = k.p(H). \tag{2.1}$$

When $H = \mathrm{Aut}(C)$ for some self-dual code C, then $p(H)$ will also be denoted simply by $p(C)$.

Now we discuss the Chinese products. See [18] for detail. For any divisor r of m we denote by $[\cdot]_r$ the natural projection $[\cdot]_r : \mathbb{Z}_m \to \mathbb{Z}_r$ defined by

$$[c]_r = (c \bmod r) \in \mathbb{Z}_r.$$

Fix a decomposition $m = rs$ of m with $\gcd(r, s) = 1$. The Chinese Remainder Theorem asserts that $\mathbb{Z}_m \to \mathbb{Z}_r \times \mathbb{Z}_s, \quad c \mapsto ([c]_r, [c]_s)$ is an isomorphism. The inverse isomorphism is denoted by

$$\mathbb{Z}_r \times \mathbb{Z}_s \to \mathbb{Z}_m, \quad (a, b) \mapsto a \odot b. \tag{2.2}$$

$a \odot b$ will be called the *Chinese product* of a and b, and $c = [c]_r \odot [c]_s$ will be called the *Chinese product decomposition* of $c \in \mathbb{Z}_m$. Note that

$$a \odot b + c \odot d = (a + c) \odot (b + d), \quad (a \odot b)(c \odot d) = (ac) \odot (bd)$$

for $a, c \in \mathbb{Z}_r$ and $b, d \in \mathbb{Z}_s$. For any integer n, these isomorphisms are extended to isomorphisms between $\mathbb{Z}_m^n$ and $\mathbb{Z}_r^n \times \mathbb{Z}_s^n$ in the natural way and use the same notations.

A code C of length n over $\mathbb{Z}_m$ can be uniquely written as a Chinese product $C = [C]_r \odot [C]_s$, and conversely, two codes A of length n over $\mathbb{Z}_r$ and B of length n over $\mathbb{Z}_s$ uniquely determine a code $A \odot B$ over $\mathbb{Z}_m$.

We have two induced projections $[\cdot]_r$ mapping $t = \gamma\sigma \in \mathbb{T}_m^n$ onto $[t]_r = [\gamma]_r\sigma \in \mathbb{T}_r^n$ and $[\cdot]_s$ mapping $t = \gamma\sigma$ onto $[t]_s = [\gamma]_s\sigma \in \mathbb{T}_s^n$. Composed with these projections, $\mathbb{T}_m^n$ also acts on $\mathbb{Z}_r^n$ and $\mathbb{Z}_s^n$ naturally, i.e., for $a \in \mathbb{Z}_r^n$, $b \in \mathbb{Z}_s^n$, $at := a[t]_r$ and $bt := b[t]_s$ so that $(a \odot b)t = at \odot bt$. Therefore $\mathbb{T}_m^n$ also acts on the sets $\mathscr{S}_r^n$ and $\mathscr{S}_s^n$ naturally as $(A \odot B)t = (At) \odot (Bt)$.

The stabilizer of $A \in \mathscr{S}_r^n$ (or $B \in \mathscr{S}_s^n$) in $\mathbb{T}_m^n$ will be denoted by $\mathrm{Aut}_{\mathbb{T}_m^n}(A)$, i.e.,

$$\mathrm{Aut}_{\mathbb{T}_m^n} A = \{\gamma\sigma \mid [\gamma]_r\sigma \in \mathrm{Aut}(A)\}.$$

The automorphism group of $A \odot B$ is described in the next theorem [18].

Theorem 2.1. *Let A, B be self-dual codes over $\mathbb{Z}_r$ and $\mathbb{Z}_s$, respectively. Then*

(i) $\mathrm{Aut}_{\mathbb{T}_m^n}(A \odot B) = \mathrm{Aut}_{\mathbb{T}_m^n}(A) \cap \mathrm{Aut}_{\mathbb{T}_m^n}(B)$.
(ii) $p(A \odot B) = p(A) \cap p(B)$.

To classify self-dual codes over $\mathbb{Z}_m$ with $m = rs$, we first break $\mathscr{S}_m^n/\mathbb{T}_m^n$ into small pieces. For $A \in \mathscr{S}_r^n$, $B \in \mathscr{S}_s^n$, let

$$LE(A, B) = \{A' \odot B' \in \mathscr{S}_r^n \odot \mathscr{S}_s^n \mid A' \sim A,\ B' \sim B\}.$$

$A' \odot B'$ in $LE(A,B)$ is said to be *locally equivalent* to $A \odot B$. It is clear that $LE(A,B)$ is invariant under the action of $\mathbb{T}_m^n$ and that

$$\mathscr{S}_m^n/\mathbb{T}_m^n = \coprod_{\substack{A\in\mathscr{S}_r^n/\mathbb{T}_r \\ B\in\mathscr{S}_s^n/\mathbb{T}_s}} LE(A,B)/\mathbb{T}_m^n. \quad \text{(disjoint union)} \tag{2.3}$$

The main theorem of [18] tells us how to obtain the classification of self-dual codes over $\mathbb{Z}_m$ using the double coset decomposition.

THEOREM 2.2. *Let $A \in \mathscr{S}_r^n$, $B \in \mathscr{S}_s^n$. The inequivalent codes in $LE(A,B)$ are given by $A \odot B\sigma_i$, where σ_i runs through the double coset representatives of S_n by $p(B)$ and $p(A)$.*

Therefore, the classification of self-dual codes over $\mathbb{Z}_m$ reduces to the classification of self-dual codes over $\mathbb{Z}_r$ and $\mathbb{Z}_s$, and the double coset decompositions. For each pair A, B of codes from $\mathscr{S}_r^n/\mathbb{T}_m^n$ and $\mathscr{S}_s^n/\mathbb{T}_m^n$, respectively, we classify $LE(A,B)$ by starting with the double coset $p(B)p(A)$ and try to find distinct double cosets $p(B)\sigma_2 p(A)$, $p(B)\sigma_3 p(A)$, $\cdots$, $p(B)\sigma_k p(A)$ until we reach the identity

$$\sum_{i=1}^{k} |p(B)\sigma_i p(A)| = n!, \tag{2.4}$$

which will be called the *local mass formula*. The local mass formula can be rephrased as

$$\sum_{i=1}^{k} \frac{1}{|\sigma_i^{-1} p(B)\sigma_i \cap p(A)|} = \frac{n!}{|p(B)||p(A)|}. \tag{2.5}$$

This equality ensures that we have all of inequivalent codes $A \odot B\sigma_i$, $1 \le i \le k$, in $LE(A,B)$. There is a case when the classification is trivial.

COROLLARY 2.3. *Let $A \in \mathscr{S}_r^n$, $B \in \mathscr{S}_s^n$ with $p(B) = S_n$. Then all codes in $LE(A \odot B)$ are equivalent.*

3. Known results on classification of self-dual codes of length 6

The following theorem is well-known.

THEOREM 3.1. *Let p be an odd prime. There exists a self-dual code of length n over $\mathbb{Z}_p$ if and only if*

$$\begin{cases} 2 \mid n, & \text{if } p \equiv 1 \pmod 4 \\ 4 \mid n, & \text{if } p \equiv 3 \pmod 4 \end{cases}$$

For codes over $\mathbb{Z}_{2^m}$, we have the following.

THEOREM 3.2. (i) *Suppose $m = 2k$ is even. Then the length 1 code C_1 with generator matrix (2^k) is a self-dual code over $\mathbb{Z}_{2^m}$. Therefore there exist self-dual codes of every length over $\mathbb{Z}_{2^m}$.*

(ii) *Suppose $m = 2k + 1$ is odd. Then the code C_2 with generator matrix*

$$\begin{pmatrix} 2^k & 2^k \\ 0 & 2^{k+1} \end{pmatrix}$$

is a self-dual code of length 2 over $\mathbb{Z}_{2^m}$. Therefore, there exist self-dual codes of every even length over $\mathbb{Z}_{2^m}$.

Proof. Recall that C is self-dual code of length n over $\mathbb{Z}_{2^m}$ if and only if C is self-orthogonal and $|C| = 2^{mn/2}$. It is easy to check that C_1 and C_2 are self-orthogonal. Also $|C_1| = 2^{m-k} = 2^k = 2^{m\cdot 1/2}$ and $|C_2| = 2^{m-k} \cdot 2^{m-k-1} = 2^m = 2^{m\cdot 2/2}$. Thus C_1 and C_2 are self-orthogonal. Finally, we can get codes with desired length by taking the direct products $C_i \oplus \cdots \oplus C_i$. □

It is known that there is a unique inequivalent binary self-dual code of length 6 generated by

$$\begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \end{pmatrix}.$$

From these results and Chinese remainder theorem, we see that there is a self-dual code of length 6 over $\mathbb{Z}_m$, where $m \leq 24$ only if

$$m = 4, 5, 8, 13, 16, 17, 20.$$

For $m = 4, 5, 8, 13, 17$ we have the following known results by several authors.

THEOREM 3.3 ([3]). *There are three inequivalent self-dual codes of length 6 over $\mathbb{Z}_4$ as follows.*

$$C_{4,1} = D_6^{\oplus} : \begin{pmatrix} 1 & 1 & 1 & 3 & & \\ & & 1 & 1 & 1 & 3 \\ 1 & 3 & & & 1 & 1 \\ & & & & 2 & 2 \end{pmatrix}$$

$$C_{4,2} = A_1^2 \oplus D_4^{\oplus} : \begin{pmatrix} 1 & 1 & 1 & 1 & & \\ & 2 & & 2 & & \\ & & 2 & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix}$$

$$C_{4,3} = A_1^6 : \mathrm{diag}(2, 2, 2, 2, 2, 2)$$

Theorem 3.4 ([11]). *There are two inequivalent self-dual codes of length 6 over* $\mathbb{Z}_5$ *as follows.*

$$C_{5,1} : \begin{pmatrix} 1 & 4 & 2 & 2 & & \\ & & 1 & 4 & 2 & 2 \\ 2 & 2 & & & 1 & 4 \end{pmatrix}$$

$$C_{5,2} : \begin{pmatrix} 1 & 2 & & & & \\ & & 1 & 2 & & \\ & & & & 1 & 2 \end{pmatrix}$$

Theorem 3.5 ([4]). *There is a unique inequivalent self-dual code of length 6 over* $\mathbb{Z}_8$ *generated by*

$$\begin{pmatrix} 2 & 2 \\ & 4 \end{pmatrix} \oplus \begin{pmatrix} 2 & 2 \\ & 4 \end{pmatrix} \oplus \begin{pmatrix} 2 & 2 \\ & 4 \end{pmatrix}.$$

Theorem 3.6 ([2]). *There are five inequivalent self-dual codes of length 6 over* $\mathbb{Z}_{13}$ *and six over* $\mathbb{Z}_{17}$.

The mass formula for self-dual codes over $\mathbb{Z}_{16}$ is determined in [14]. However the classification has not been done yet. Except this case, codes over $\mathbb{Z}_{20}$ remains to be classified.

4. Classification over $\mathbb{Z}_{20}$

First we give the groups of permutations of $C_{4,i}$ and $C_{5,j}$. Let H, K, L be subgroups of S_6 as follows:

$$\begin{aligned} H &= \langle (12), (34), (3546) \rangle, \\ K &= \text{the permutation group on } \{3, 4, 5, 6\}\ , \\ L &= \langle (23564), (3546) \rangle. \end{aligned}$$

Then we have that $|H| = 16$, $|K| = 4!$ and $|L| = 20$. By a computer search we obtain the following permutation groups.

Theorem 4.1. *The groups of permutations in the automorphism groups of* $C_{4,i}$ *and* $C_{5,j}$ *are given by*

$$\begin{aligned} p(C_{4,1}) &= H \cup (13)(24)H \cup (15)(26)H, \\ p(C_{4,2}) &= K \cup (12)K, \\ p(C_{4,3}) &= S_6, \\ p(C_{5,1}) &= L \cup (12)L \cup (13)L \cup (14)L \cup (15)L \cup (16)L, \\ p(C_{5,2}) &= p(C_{4,1}) \end{aligned}$$

of order $48, 48, 720, 120, 48$ *respectively.*

To classify self-dual codes of length 6 over $\mathbb{Z}_{20}$ using Theorem 2.2, we need to compute double cosets $p(C_{4,i})\backslash S_6/p(C_{5,j})$ for each i and j. By a help of a computer, we obtain our main result.

THEOREM 4.2. *There are 10 inequivalent self-dual codes of length 6 over* $\mathbb{Z}_{20}$ *as follows:*

(i) $C_{4,1} \odot C_{5,1}$ *with generator matrix*

$$\begin{pmatrix} 1 & 9 & 17 & 7 & 0 & 0 \\ 0 & 0 & 1 & 9 & 17 & 7 \\ 17 & 7 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{pmatrix}$$

and weight distribution $\{1, 0, 3, 8, 303, 1752, 5933\}$.

(ii) $C_{4,1} \odot C_{5,1}(45)$ *with generator matrix*

$$\begin{pmatrix} 1 & 9 & 17 & 15 & 12 & 0 \\ 0 & 0 & 1 & 17 & 9 & 7 \\ 17 & 7 & 0 & 16 & 5 & 9 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{pmatrix}$$

and weight distribution $\{1, 0, 3, 8, 303, 1752, 5933\}$.

(iii) $C_{4,1} \odot C_{5,2}$ *with generator matrix*

$$\begin{pmatrix} 1 & 17 & 5 & 15 & 0 & 0 \\ 0 & 0 & 1 & 17 & 5 & 15 \\ 5 & 15 & 0 & 0 & 1 & 17 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{pmatrix}$$

and weight distribution $\{1, 0, 27, 8, 639, 984, 6341\}$

(iv) $C_{4,1} \odot C_{5,2}(45)$ *with generator matrix*

$$\begin{pmatrix} 1 & 17 & 5 & 15 & 0 & 0 \\ 0 & 0 & 1 & 5 & 17 & 15 \\ 5 & 15 & 0 & 16 & 5 & 17 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{pmatrix}$$

and weight distribution $\{1, 0, 19, 40, 431, 1336, 617\}$.

(v) $C_{4,1} \odot C_{5,2}(23)(45)$ *with generator matrix*

$$\begin{pmatrix} 1 & 5 & 17 & 15 & 0 & 0 \\ 0 & 16 & 5 & 5 & 17 & 15 \\ 5 & 15 & 0 & 16 & 5 & 17 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{pmatrix}$$

and weight distribution $\{1, 0, 15, 56, 327, 1512, 6089\}$.

(vi) $C_{4,2} \odot C_{5,1}$ *with generator matrix*

$$\begin{pmatrix} 6 & 4 & 12 & 12 & 0 & 0 \\ 0 & 10 & 16 & 4 & 12 & 12 \\ 12 & 12 & 5 & 5 & 1 & 9 \\ 0 & 0 & 0 & 10 & 0 & 10 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{pmatrix}$$

and weight distribution $\{1, 2, 7, 12, 527, 2066, 5385\}$.

(vii) $C_{4,2} \odot C_{5,2}$ *with generator matrix*

$$\begin{pmatrix} 6 & 12 & 0 & 0 & 0 & 0 \\ 0 & 10 & 16 & 12 & 0 & 0 \\ 0 & 0 & 5 & 5 & 1 & 17 \\ 0 & 0 & 0 & 10 & 0 & 10 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{pmatrix}$$

and weight distribution $\{1, 2, 39, 76, 783, 1234, 5865\}$.

(viii) $C_{4,2} \odot C_{5,2}(23)$ *with generator matrix*

$$\begin{pmatrix} 6 & 0 & 12 & 0 & 0 & 0 \\ 0 & 6 & 0 & 12 & 0 & 0 \\ 0 & 0 & 5 & 5 & 1 & 17 \\ 0 & 0 & 0 & 10 & 0 & 10 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{pmatrix}$$

and weight distribution $\{1, 2, 31, 108, 575, 1586, 5697\}$.

(ix) $C_{4,3} \odot C_{5,1}$ *with generator matrix*

$$\begin{pmatrix} 6 & 4 & 12 & 12 & 0 & 0 \\ 0 & 10 & 16 & 4 & 12 & 12 \\ 12 & 12 & 10 & 0 & 16 & 4 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{pmatrix}$$

and weight distribution $\{1, 6, 15, 20, 975, 2694, 4289\}$.

(x) $C_{4,3} \odot C_{5,2}$ *with generator matrix*

$$\begin{pmatrix} 6 & 12 & 0 & 0 & 0 & 0 \\ 0 & 10 & 16 & 12 & 0 & 0 \\ 0 & 0 & 10 & 0 & 16 & 12 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{pmatrix}$$

and weight distribution $\{1, 6, 63, 212, 1071, 1734, 4913\}$.

Proof. The generator matrices can be obtained using a method in [17] and then the weight distributions can be computed easily. For each pair (i, j), the double cosets decomposition $p(C_{4,i})\backslash S_6/p(C_{5,j})$ is computed by a computer. However, we may also prove the theorem using the local mass formula (2.5) as follows. Let $A = C_{4,1}$, $B = C_{5,1}$. We can show that $|p(A) \cap p(B)| = 24$ and $|(45)p(A)(45) \cap p(B)| = 12$. Since

$$\frac{1}{24} + \frac{1}{12} = \frac{1}{8} = \frac{720}{48 \times 12} = \frac{6!}{|p(B)||p(A)|}$$

we conclude that $LE(C_{4,1}, C_{5,1})$ has two inequivalent codes $A \odot B$ and $A \odot B(45)$. For the remaining cases, we list the intersection numbers $|\sigma^{-1}p(A)\sigma \cap p(B)|$. In all cases, the intersection numbers are all distinct, so it is easy to check the local mass formula. □

References

[1] J. M. P. Balmaceda, R. A. L. Betty and F. R. Nemenzo, *Mass formula for self-dual codes over* $\mathbb{Z}_{p^2}$, Discrete Math. **308** (2009), 2984–3002.

[2] K. Betsumiya, S. Georgiou, T. A. Gulliver, M. Harada, C. Kououvinos, *On self-dual codes over some prime fields*, Discrete Math. **262** (2009), 37–58.

[3] J. H. Conway and N. J. A. Sloane, *Self-dual codes over the integers modulo* 4, J. Comin. Theory Ser. A **62** (1993), 30–45.

[4] S. T. Dougherty, T. A. Gulliver and J. N. C. Wong, *Self-dual codes over* $\mathbb{Z}_8$ *and* $\mathbb{Z}_9$, Des Codes Crypt **41** (2006), 235–249.

A	B	$\|p(A)\|$	$\|p(B)\|$	$A \odot B\sigma$	$\|\sigma^{-1}p(B)\sigma \cap p(A)\|$
$C_{4,1}$	$C_{5,1}$	48	120	$A \odot B$	24
				$A \odot B(45)$	12
$C_{4,1}$	$C_{5,2}$	48	48	$A \odot B$	48
				$A \odot B(45)$	8
				$A \odot B(23)(45)$	6
$C_{4,2}$	$C_{5,1}$	48	120	$A \odot B$	8
$C_{4,2}$	$C_{5,2}$	48	48	$A \odot B$	16
				$A \odot B(23)$	4
$C_{4,3}$	$C_{5,1}$	720	120	$A \odot B$	120
$C_{4,3}$	$C_{5,2}$	720	48	$A \odot B$	48

[5] S. T. Dougherty, M. Harada and P. Sole, *Self-dual codes over rings and the Chinese Remainder Theorem*, Hokkaido Math Journal **28** (1999), 253-283.

[6] S. T. Dougherty, S. Y. Kim and Y. H. Park, *Lifted codes and their weight enumerators*, Discrite Math. **305** (2005), 123–135.

[7] S. T. Dougherty and Y. H. Park, *Codes over the p-adic integers*, Des. Codes. Cryptogr. **39** (2006), 65–80.

[8] M. Harada and A. Munemasa, *On the classfication of self-dual $\mathbb{Z}_k$-codes*, Lecture Notes in Computer Science **5921**, (2009) 78-90.

[9] W. C. Huffman, *On the classification and enumeration of self-dual codes*, Finite fields and their applications, **11** (2005), 451–490.

[10] W. C. Huffman and V. Pless, *Fundamentals of error-correcting codes*, Cambridge, 2003.

[11] J. S. Leon, V. Pless and N. J. A. Sloane, *Self-dual codes over $GF(5)$*, J. Combin. Theory Ser. A **32** (1982), 178–194.

[12] F. J. MacWilliams and N. J. A. Sloane, *The theory of error-correcting codes*, North-Holland, Amsterdam, 1977.

[13] C. L. Mallows, V. Pless and N. J. A. Sloane, *Self-dual codes over $GF(3)$*, Siam J. Appl. Math. **31** (1976), 649–666.

[14] K. Nagata, F. Nemenzo and H. Wada, *On self-dual codes over $\mathbb{Z}_{16}$*, Lecture Notes in Computer Science **5527** (2009), 107–116.

[15] K. Nagata, F. Nemenzo and H. Wada, *Constructive algorithm of self-dual error-correcting codes*, Eleventh international workshop on algebraic and combinatorial coding theory, June 16-22, Bulgaria, 215–220, 2008.

[16] G. Nebe, E. Rains and N. J. A. Sloane, *Self-dual codes and invariant theory*, Springer-Verlag, 2006.

[17] Y. H. Park, *Modular independence and generator matrices for codes over $\mathbb{Z}_m$*, Des. Codes. Crypt **50** (2009), 147–162.

[18] Y. H. Park, *The classification of self-dual modular codes*, submitted, 2010

[19] V. Pless and V. D. Tonchev, *Self-dual codes over $GF(7)$*, IEEE Trans. Inform. Theory **33** (1987), 723–727.

[20] E. Rains and N. J. A. Sloane, *Self-dual codes*, in the Handbook of Coding Theory, V.S. Pless and W.C. Huffman, eds., Elsevier, Amsterdam, 1998, 177-294.

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